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REGULARITY FOR ELLIPTIC EQUATIONS UNDER MINIMAL ASSUMPTIONS

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Abstract. The aim of this paper is to give a brief account on the problem of regularity for linear elliptic PDEs of the second order.

Key words and phrases: *linear elliptic PDEs, regularity problems.*

1. Introduction

This booklet is a brief exposition of my lectures given at the summer school MYSAGA 2018 at ITB - Bandung.

The aim of the lectures is to give a brief account on the problem of regularity for linear elliptic PDEs of the second order. This goal is too ambitious to be covered in a few lectures so we suggest the interested reader to study also some classical books like [9], [16] or [8]. The focus is on divergence form equations and on the minimal assumptions regarding the lower order terms. No smoothness assumption is required to the leading coefficients.

I hope this can be of some utility to young researchers and PhD students and anybody interested in the regularity theory for elliptic equations.

Any suggestion or warning about misprints and typos is welcome from everybody. You may send your remarks to me by email.

I wish to thank all the organizers for the warm hospitality, nice atmosphere and useful conversation I had with many interested students and colleagues. Special thanks to Professor Hendra Gunawan, Denny Ivanal Hakim, Ifronika, Muh. Nur, Nicky K. Tumalun and Harmanus Batkunde. Thanks to the other lecturers for important opportunity to share common knowledge and new ideas. Thanks to all the participants because without any of them and us the school could not happen.

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2. Linear Elliptic PDEs and Their Solutions

In this chapter we introduce linear elliptic partial differential equations and several notions of solution. Indeed, classical solutions very often do not exist. For this reason in the literature many different kind of solution have been introduced. In particular we will focus on weak and very weak solutions.

2.1. **Uniformly elliptic equations.** Let us consider a partial differential equation of the following kind

$$\operatorname{div}(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u + \operatorname{div}(u\,\mathbf{d}(x)) + V(x)u = f(x) + \operatorname{div}\mathbf{g}(x) \tag{1}$$

in a given domain $\Omega \subset \mathbb{R}^n$ where $\mathbf{b}$, c, $\mathbf{d}$, V, f and $\mathbf{g}$ are given measurable functions. In general we are interested in the regularity properties of generalized solutions of an equation like (1). In order to make sense to (1) we will make suitable assumptions on the coefficients that we will make precise later. The simplest equations of the kind (1) are Laplace equation $\Delta u = 0$ and Poisson equation $\Delta u = f$.

Let A be a $n \times n$ matrix with bounded and measurable entries $\{a_{ij}\}$. Although it is not really needed, we will assume that the matrix is symmetric. In what follows we also assume an important condition on the function A, the so called uniform ellipticity condition.

Definition 2.1 (Ellipticity)**.** *We say that equation* (1) *is uniformly elliptic in* Ω *if the following condition is satisfied*

$$\exists \nu > 0 \ : \ \nu|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \nu^{-1}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n \ \textit{a.e.}\ x \in \Omega. \tag{2}$$

Condition in Definition 2.1 is said *uniform* ellipticity because the ν does not depend on x. To simplify the notation we will refer to Definition 2.1 as ellipticity and the number ν will be called ellipticity constant. The prototype of elliptic equations is Laplace equation.

2.2. **Weak solutions and classical results.** Since our interest is to allow discontinuity in the functions A and B we have to define what we mean by solution of the equation (1). Indeed, under this generality, classical solutions do not exist.

Definition 2.2 (Weak Solution)**.** *Let us consider the elliptic equation* (1) *in a domain* $\Omega \subset \mathbb{R}^n$ *where* $\boldsymbol{b}$*,* c*,* $\boldsymbol{d}$*,* V*,* f *and* $\boldsymbol{g}$ *are given measurable functions. We say that a function* $u \in W^{1,2}(\Omega)$ *is a weak solution of* (1) *if and only if the following identity hold true*

$$\int_\Omega A(x)\nabla u \nabla\varphi\,dx + \int_\Omega \boldsymbol{b}(x)\varphi\nabla u\,dx + \int_\Omega \boldsymbol{d}(x)u\nabla\varphi\,dx + \int_\Omega V(x)u\varphi\,dx$$
$$= -\int_\Omega f(x)\varphi\,dx + \int_\Omega \boldsymbol{g}(x)\nabla\varphi\,dx \qquad \forall\varphi \in W_0^{1,2}(\Omega). \tag{3}$$

In the following A will be a symmetric elliptic matrix of bounded measurable functions in Ω.

Moreover - to keep the exposition simple - we will set

$$V = 0 \qquad \mathbf{b} = \mathbf{d} = \mathbf{g} = \mathbf{0}$$

so the equation will contain only the leading term A and the known term f that usually is an element of $W^{-1,2}(\Omega)$

$$Lu \equiv \operatorname{div}(A(x)\nabla u) = f \tag{4}$$

which is called variational (or divergence form) equation in principal part.

It is a very well known and classical result that if $f \in W^{-1,2}(\Omega)$ then a unique weak solution exists. Indeed it is an easy consequence of Riesz representation Theorem.

Once we know that the weak solution exists we would like to know if the solution enjoys good regularity properties. The most outstanding classical result about regularity is given by the Theorem of De Giorgi - Nash - Moser for linear homogeneous elliptic equations (see [4], [18] and [19]).

Theorem 2.3 (De Giorgi - Nash - Moser)**.** *Any weak solution of the equation $Lu = 0$ is α Hölder continuous in Ω. The exponent α is a function of n and ν only. Moreover, for any ball $B_{R_0}(y) \Subset \Omega$ there exists a constant $c = c(n, \nu)$ such that the following estimate holds true*

$$\operatorname*{osc}_{B_R(y)} u \leq c \left(\frac{R}{R_0}\right)^{\alpha} \left(\fint_{B_{R_0}(y)} |u|^2\, dx\right)^{1/2} \qquad \forall R < R_0\, .$$

Remark 2.4. *Two very important points are the following.*

1. *The constant c does not depend on the solution. It depends on the operator via its ellipticity constant ν.*
2. *We do not require any continuity assumption on the function A. Without further notice the function A will be a bounded measurable function in Ω.*

Theorem 2.3 is a very important result in the theory of elliptic PDEs and it has been extended to several different kind of elliptic equations. It is one of the fundamental tools to attack some classes of non linear problems.

If we focus on equation (4) and we want to allow unbounded and discontinuous term f we are leaded to assume some integrability conditions. This has already been done and now it is something that is considered as a classic in the theory of regularity.

Before stating some classical results we focus on the following examples (see the book [16]).

Example 2.5. *The function*

$$u(x) = \log|\log|x|| \qquad \forall 0 < |x| < R < 1/e$$

for suitably small R is a weak solution of

$$\Delta u = \frac{n-2}{|x|^2 \log|x|} - \frac{1}{|x|^2 \log^2|x|}$$

We notice that the right hand side belongs to L^p for $p < n/2$. Moreover, the solution is not locally bounded and then we cannot have any regularity on u.

We can also exhibit the bad behavior of the Dirichlet problem when the coefficients are not much integrable.

Example 2.6. *The functions* $u(x)$ *and* $\mathrm{v}(x) = \log|\log R|$ *for any* $R < 1/e$ *are both weak solutions of the Dirichlet problem*

$$\begin{cases} \Delta u + \boldsymbol{b} \cdot \nabla u = 0 & 0 < |x| < R \\ u = \log|\log R| & |x| = R \end{cases}$$

where $\boldsymbol{b} = -F(x)\log|x|x$ *and* $F(x) = \dfrac{n-2}{|x|^2 \log|x|} - \dfrac{1}{|x|^2 \log^2|x|}$ *so the Dirichlet problem does not have uniqueness.*

The following is a result that parallels De Giorgi Theorem and refers to differential equation with lower order terms. We quote here a reduced version of Stampacchia Theorem adapted to equation (4).

Theorem 2.7 (Stampacchia see [25])**.** *Let* $f \in L^p(\Omega)$ *with* $p > n/2$*. Then any weak solution of the equation* (4) *is Hölder continuous in* Ω*.*

Previous examples and Stampacchia Theorem show that there exists an integrability threshold depending on dimension such that below the threshold there is no regularity. Moreover, this phenomena does not depend on the smoothness of the function A. Looking at the examples and Theorem 2.7 it seems that we can have regularity if - and only if - the function f belongs to L^p with $p > n/2$. This is true if we consider Lebesgue spaces as the only spaces where to put the coefficients of our differential equation.

By using a different approach we will see that this can be improved very much allowing a different family of spaces to be chosen as family of spaces where to assume the coefficients of the equation.

2.3. **Weak solutions and the modern results.** The first result in a different direction is contained in the paper by Aizenman & Simon (see [1]) where they showed - by using probabilistic arguments - that the weak solutions of the equation

$$\Delta u + Vu = 0$$

are continuous. The assumption in [1] regarding V is the following

Definition 2.8 (Stummel-Kato class)**.** *Let* V *be an integrable function in* $\mathbb{R}^n$*. Let*

$$\eta(r) = \sup_{x \in \mathbb{R}^n} \int_{B_r(x)} \frac{|V(y)|}{|x-y|^{n-2}}\, dy\,.$$

We say that V *belongs to the Stummel - Kato classes* $\tilde{S}$ *or* S *if the function* η *is bounded or infinitesimal in a neighborhood of zero respectively. The function* η *is called the Stummel - Kato modulus of* V*. If* Ω *is a domain we say* $V \in S(\Omega)$ *or* $V \in \tilde{S}(\Omega)$ *means* $V(x)\chi_\Omega(x)$ *belongs to* S *or* $\tilde{S}$ *respectively. In the case* $n = 2$*,* $\log|x-y|$ *replaces* $\frac{1}{|x-y|^{n-2}}$ *in the definition.*

In general it is not easy to check if a function belongs to these classes. In the special case of radial functions we have the following result.

Theorem 2.9. *(see [1]) Let V be a radial function in $\mathbb{R}^n$. Then, V belongs to S if and only if*

$$\int_{0^+} r\,|V(r)|\,dr < +\infty\,.$$

Some years later - in 1986 - Chiarenza, Fabes & Garofalo (see [3]) gave analytic proof of Aizenman & Simon result for any uniformly elliptic equation

$$\operatorname{div}(A(x)\nabla u) + Vu = 0 \tag{5}$$

assuming V in the Stummel - Kato class $S(\Omega)$.

Theorem 2.10 (Chiarenza Fabes Garofalo)**.** *Let u be a weak solution of the equation (5) in $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Let us assume V in the Stummel - Kato class $S(\Omega)$. Then, any weak solution u is continuous in Ω.*

To better understand Stummel - Kato class we introduce a family of function spaces introduced by C.B.Morrey in [17] that will be very useful in the sequel. In order these spaces to be more handy we will make the following assumption on the boundary of the given domain Ω.

Definition 2.11 (Domain of type A)**.** *A domain $\Omega \subset \mathbb{R}^n$ is said to be of type A if for every $x \in \partial\Omega$ there exists a constant $A > 0$ such that*

$$|\Omega \cap B_r(x)| \geq A|B_r(x)|$$

for any $0 < r < \operatorname{diam}(\Omega)$ and any $x \in \partial\Omega$.

It is easy to realize that the geometric meaning of the Definition is to avoid outward cusps.

Definition 2.12 (Morrey spaces - see [12])**.** *Let f be a locally integrable function in a type (A) domain $\Omega \subset \mathbb{R}^n$, $1 \leq p < \infty$ and $0 < \lambda < n$. We say that f belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if*

$$\|f\|^p_{p,\lambda} \equiv \sup_{\substack{x\in\Omega \\ 0<r<diam\Omega}} \frac{1}{r^\lambda} \int_{\Omega\cap B(x,r)} |f(y)|^p\,dy < \infty.$$

There is a simple relation between Morrey spaces and the Stummel class.

Theorem 2.13 ([5] and [6])**.** *The following inclusions hold true*

$$L^p \subset S(\Omega) \qquad p > n/2$$

and

$$L^{1,\lambda} \subset S \qquad n-2 < \lambda < n.$$

Moreover, the inclusions are strict.

Proof. The first inclusion is a consequence of Hölder inequality

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}}\, dy \le \|f\|_p \left(\int_{B(x,r)} |x-y|^{(n-2)p'}\, dy \right)^{1/p'}$$

and the last integral is finite because $p > n/2$. In fact, evaluating it in polar coordinates we get

$$\eta(r) \le c\, \|f\|_p r^{2-n/p}.$$

To prove the second inclusion we use Hedberg trick (see [11]) and Hölder inequality to obtain

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}}\, dy = \sum_{k=0}^{+\infty} \int_{B(x,r/2^k)\setminus B(x,r/2^{k+1})} |f(y)||x-y|^{2-n}\, dy \le \|f\|_{1,\lambda} \sum_{k=0}^{+\infty} \left(\frac{r}{2^k}\right)^{\lambda-n+2}$$

where the series is convergent because $\lambda > n-2$.

Finally we get

$$\eta(r) \le c\, \|f\|_{1,\lambda} r^{\lambda-n+2}$$

□

In view of the development of the theory a natural question then is: "Can we obtain better regularity results assuming f in suitable Morrey spaces?"

Morrey spaces are not so easy to handle as Lebesgue spaces are. Here follows some negative properties.

1: Morrey spaces are not the closure of smooth functions.
2: Mollifiers do not converge in Morrey spaces.
3: The Morrey norm does not have the absolute continuity property i.e. $\|u\|_{L^{p,\lambda}(B_R)}$ is not infinitesimal with R.
4: For any $1 \le p < \infty$ there exists a function u in $L^{p,\lambda}$ such that $u \notin L^q$ for any $q > p$.

Despite of that we will see that

1: Morrey spaces imply regularity.
2: If the function f has a sign they are also necessary for regularity.

In the following sections we will show that regularity can be obtained even if the integrability of the coefficients is lower than the integrability required by the classical theory.

3. Very weak solution and Green function

We will introduce a different definition of solution. It will be called very weak solution and it is much more general than that of weak solution.

Our motivation for that is an important result due to Littman, Stampacchia & Weinberger (see [15]) stating that equation (4) has a weak solution if and only if f belongs to $W^{-1,2}(\Omega)$. This is very important for us because in general $L^{p,\lambda}$ is not contained in $W^{-1,2}$ and then weak solutions may not exist.

3.1. **Very weak solution.** We introduce the definition of very weak solution as in [25] and [15].

Definition 3.1 (Very weak solution)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and let μ be a bounded variation measure in Ω. A function $u \in L^1(\Omega)$ is a very weak solution of the Dirichlet problem*

$$\begin{cases} Lu = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

if, for any $\varphi \in W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ such that $L\varphi \in C^0(\overline{\Omega})$, we have

$$\int_\Omega uL\varphi\, dx = \int_\Omega \varphi\, d\mu.$$

Remark 3.2. *The class of test functions is non empty by De Giorgi and Stampacchia regularity Theorems.*

Indeed, let ψ be a $C^0(\overline{\Omega})$ given function. Then, there exists a Hölder continuous weak solution φ of the equation $L\varphi = \psi$ so $\varphi \in W_0^{1,2}(\Omega)$ and then $\varphi \in C^0(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$.

Remark 3.3. *Very weak solution is unique. Indeed, if u is a very weak solution of the homogeneous problem*

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then $u = 0$. To show this, let $\psi \in C^0(\overline{\Omega})$ and φ in $W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ such that $L\varphi = \psi$. Since $\int_\Omega u\, L\varphi\, dx = 0$ i.e. $\int_\Omega u\,\psi\, dx = 0$ for any continuous ψ, we have $u = 0$ in Ω.

3.2. **Green function.** In this section we introduce a very important tool for linear differential operators that is the Green function. Since our exposition follows closely the paper [15] we first define the Green operator.

Let us consider the problem

$$\begin{cases} Lu = T & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

By the definition of very weak solution, there exists a linear application

$$G : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \tag{8}$$

defined by $G(T) = u$. This is what we call the Green operator. Now, by the local boundedness and the local Hölder continuity Theorems we have that (8) hold true and for any $T \in W^{-1,2}(\Omega)$ the function

$$u = G(T)$$

is the unique weak solution in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (7).

Theorem 3.4. *For any bounded variation measure μ on $\overline{\Omega}$ there exists a unique solution of the equation $Lu = \mu$ that is zero on the boundary $\partial\Omega$. Moreover it belongs to $W_0^{1,p'}(\Omega)$ for any $p > n$.*

Moreover, the operator G maps continuously $W^{-1,p}(\Omega)$ in $C^0(\overline{\Omega})$. Indeed, if $p > n$, by De Giorgi - Nash - Moser Theorem we have

$$G : W^{-1,p}(\Omega) \to C^0(\overline{\Omega})$$

and there exists c such that

$$\max_{\overline{\Omega}} |G(\psi)| \leq c\,\|\psi\|_{-1,p} \quad \forall \psi \in W^{-1,p}(\Omega)\,.$$

Then, u is a very weak solution of $Lu = \mu$ vanishing on $\partial\Omega$ if and only if

$$\int_\Omega u\psi\,dx = \int_\Omega G(\psi)\,d\mu$$

for all $\psi \in C^0(\overline{\Omega})$ and

$$\left|\int_\Omega u\psi\,dx\right| = \left|\int_\Omega G(\psi)\,d\mu\right| \leq c\int_\Omega |d\mu|\|\psi\|_{-1,p}$$

for all $\psi \in C^0(\overline{\Omega})$.

By density we have

$$\|u\|_{W^{1,p'}} \leq c\int_\Omega |d\mu|$$

for $p > n$. The application $\mu \mapsto u$ is the adjoint of G, i.e. $u = G^*(\mu)$.

Since

$$G : W^{-1,p} \to C^0(\overline{\Omega})$$

is continuous by duality we have that G^* is also continuous from the space $\mathcal{M}$ of the measures with bounded variation in $\overline{\Omega}$ to $W_0^{1,p'}(\Omega)$. For any $\mu \in \mathcal{M}$ we have $G^*(\mu) \in W_0^{1,p'}(\Omega)$.

Now we are ready to compare the notions of weak and very weak solutions.

Theorem 3.5 (Littman - Stampacchia - Weinberger)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and μ be a bounded variation measure. Let $u \in L^1(\Omega)$ be the very weak solution of the Dirichlet problem* (6)*. Then u is a weak solution if and only if $\mu \in W^{-1,2}(\Omega)$.*

Proof. Let u be the unique weak solution in $W_0^{1,2}(\Omega)$ of equation $Lu = \mu$. We have

$$\int_\Omega A(x)\nabla u\nabla\phi\,dx = \int_\Omega \phi\,d\mu \qquad \forall \phi \in W_0^{1,2}(\Omega)$$

and then,

$$\left|\int_\Omega \phi\,d\mu\right| \leq \nu\,\|\nabla u\|_2\|\nabla\phi\|_2 \qquad \forall \phi \in W_0^{1,2}(\Omega)$$

which means that $\mu \in W^{-1,2}(\Omega)$.

Now, if $\mu \in W^{-1,2}(\Omega)$ there exist $f_0, \mathbf{f}$ whose components are $L^2(\Omega)$ functions such that $\mu = f_0 + \operatorname{div}\mathbf{f}$. If Ω is bounded (which is our case) we may assume $f_0 = 0$.

Then we can say that equation $Lu = \operatorname{div} \mathbf{f}$ has a weak solution by classical Hilbert space approach. $\square$

Definition 3.6 (Green function)**.** *If $y \in \Omega$ the Dirac mass at y, δ_y is a bounded variation measure. Then we may consider the very weak solution $g(\cdot, y)$ of the Dirichlet problem in the case $\mu \equiv \delta_y$. Such a function will be called the Green function for the operator L with respect to the domain Ω* with pole at *y.*

Remark 3.7. *The Green function satisfies*

$$\int_\Omega g(x,y) L\varphi(x) dx = \int_\Omega \varphi(x) d\delta_y(x)$$

for all $\varphi \in C^0(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$ such that $L\varphi \in C^0(\overline{\Omega})$. Then, by definition of δ_y, we have

$$\varphi(y) = \int_\Omega g(x,y) L\varphi(x) dx\,.$$

It is worth to mention that the concept of Green function has been deeply revisited in 1982 in a famous paper by Grüter & Widman. We summarize here some of their results.

Theorem 3.8 (Grüter & Widman - see [10])**.** *There exists a unique function $G : \Omega \times \Omega \setminus \{x = y\} \to \mathbb{R} \cup \{\infty\}$ such that*

1. *$G(x,y) \geq 0$ where it is defined.*
2. *For any $y \in \Omega$ and any $r > 0$ such that $B_r(y) \subset \Omega$ the function $G(\cdot, y)$ belongs to $W^{1,2}(\Omega \setminus B_r(y)) \cap W_0^{1,1}(\Omega)$.*
3. *The following relation holds true*

$$\int_\Omega A(x) \nabla_x G(x,y) \nabla\varphi \, dx = \varphi(y)$$

for all $\varphi \in C_0^\infty(\Omega)$. Moreover, if we set $G(x) \equiv G(x,y)$, the function G satisfies the following properties

4. *G belongs to the space $L_w^{n/(n-2)}(\Omega)$ with bounds depending on the ellipticity and dimension only.*
5. *$\nabla_x G$ belongs to the space $L_w^{n/(n-1)}(\Omega)$ with bounds depending on the ellipticity and dimension only.*
6. *G belongs to the space $W_0^{1,s}(\Omega)$ for any $1 \leq s < \dfrac{n}{n-1}$ with bounds depending on the ellipticity, dimension and the exponent s only.*
7. *There exists a positive constant c depending on the ellipticity and dimension only such that $G(x,y) \leq c|x-y|^{2-n}$ for all $x, y \in \Omega\,, x \neq y$.*
8. *There exists a positive constant c depending on the ellipticity and dimension only such that*

$$G(x,y) \geq c|x-y|^{2-n}$$

for all $x, y \in \Omega$ such that $0 < |x-y| \leq \dfrac{1}{2} d(y, \partial\Omega)\,, x \neq y$.

Remark 3.9. *The Green function defined in* [15] *and in* [10] *come in very different ways.*

We have the following result.

Theorem 3.10. *The Green function g defined by Stampacchia and the one G defined by Grüter & Widman are the same function.*

Proof. If $y \in \Omega$ and $\varrho > 0$ consider the family of functions

$$f_\varrho(x) \equiv |B_\varrho|^{-1} \chi_{B_\varrho(y)}(x)$$

where $\chi_I(x)$ denotes the characteristic function of the set I.

The set of solutions $G_\varrho(x, y)$ of the problems

$$\begin{cases} Lu = f_\varrho & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

is bounded in $W_0^{1,p}(\Omega)$, $1 \le p < n/(n-1)$ because of the estimate

$$\|G_\varrho(x, \cdot)\|_{1,p} \le C\|f\|_1 = C$$

due to [15]. Moreover it is easy to see that f_ϱ converges weakly* to the Dirac mass i.e.

$$\int_\Omega \varphi(x) f_\varrho(x)dx \to \int_\Omega \varphi(x)d\delta \qquad \forall\varphi \in C^0(\Omega) \cap L^\infty(\Omega).$$

Following [10] we select the sequence $\{G^{\varrho_k}(x, y)\}$ that converges weakly in $W_0^{1,p}(\Omega)$ $(1 \le p < n/(n-1))$ to $G(x, y)$ as $k \to \infty$. Moreover we may assume that $G^{\varrho_k}(x, y)$ converges to $G(x, y)$ in $L^1(\Omega)$. The function $G^{\varrho_k}(x, y)$ is the variational solution, and then the very weak one, of (12) with ϱ_k in place of ϱ so that

$$\int_\Omega G^{\varrho_k}(x, y)L\varphi(x)dx = \int_\Omega \varphi(x) f_{\varrho_k}(x)dx \quad \forall k \in \mathbb{N} \quad \forall\varphi \in T_L.$$

Here we write T_L to denote the class of test functions related to the operator L as in the Definition 3.1.

Now if $k \to +\infty$ we get

$$\int_\Omega G(x, y)L\varphi(x)dx = \int_\Omega \varphi(x)d\delta_y \quad \forall\varphi \in T_L$$

that is $G(x, y)$ is the very weak solution of

$$\begin{cases} Lu = \delta_y & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

By uniqueness of the very weak solution and the fact that $g(x, y)$ satisfies to the same problem it follows that

$$g(x, y) = G(x, y).$$

□

3.3. **Representation formula.** In this section we will prove a representation by using our knowledge about the Green function. The formula is going to be very useful to prove our regularity results.

Indeed, let us consider $\psi \in C^0(\overline{\Omega})$. Since $C^0(\overline{\Omega}) \subset W^{-1,2}(\Omega)$ the problem

$$\begin{cases} L\varphi = \psi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution that we call φ and it belongs to $W_0^{1,2}(\Omega)$. By De Giorgi Theorem we have $\varphi \in C^0(\overline{\Omega})$. The Definition of very weak solution now yields

$$\int_\Omega g(x,y)\psi(x)dx = \int_\Omega \varphi(x)d\delta_y(x)$$

i.e.

$$\varphi(y) = \int_\Omega g(x,y)\psi(x)dx\,.$$

We can represent the very weak solution of the Dirichlet problem (6).

Theorem 3.11 (Representation formula)**.** *Let μ be a bounded variation measure in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $u \in L^1(\Omega)$ be the very weak solution of the Dirichlet problem* (6)*. Then, the following representation formula holds true*

$$u(x) = \int_\Omega g(x,y)d\mu(y)$$

where $g(x,y)$ is the Green function for the operator L with respect to Ω with pole at $y \in \Omega$.

Proof. Existence and uniqueness have already been proven. We simply verify the formula. Let $\varphi \in W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ be such that $L\varphi \in C^0(\overline{\Omega})$. Then

$$\begin{aligned}\int_\Omega \varphi(y)d\mu(y) &= \int_\Omega \left(\int_\Omega g(x,y)L\varphi(x)dx\right)d\mu(y)\\ &= \int_\Omega \left(\int_\Omega g(x,y)d\mu(y)\right)L\varphi(x)dx\\ &= \int_\Omega u(x)L\varphi(x)dx\end{aligned}$$

and then $\int_\Omega u(x)L\varphi(x)dx = \int_\Omega \varphi(x)d\mu(x)$ for any $\varphi \in W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ such that $L\varphi \in C^0(\overline{\Omega})$ that is the result. □

Representation formula will give us important information about the regularity of the very weak solution.

4. Regularity

In this section we apply what we know from previous sections to deduce regularity properties of the very weak and weak solutions.

4.1. **Sufficient conditions for the regularity.** In this section we give some sufficient conditions for the regularity of the very weak solution of the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

First we recall the following Chiarenza–Frasca Theorem (see [2]) which is a generalization of the Hardy – Littlewood maximal Theorem to the case of Morrey spaces.

Theorem 4.1 (Chiarenza – Frasca)**.** *Let $1 < p < +\infty$ and $0 < \lambda < n$. Then, there exists a constant c which depend on n, p and λ such that*

$$\|Mf\|_{L^{p,\lambda}} \leq c\|f\|_{L^{p,\lambda}}\,.$$

If $p = 1$ we have the following weak type estimate

$$t|\,\{y \in B_r(x)\ :\ Mf(y) > t\}\,| \leq c\,r^{\lambda}\|f\|_{L^{1,\lambda}}\,.$$

For $1 \leq p \leq +\infty$, $0 < \lambda < n$ the function Mf is finite for almost all $x \in \mathbb{R}^n$. Here M is the maximal Hardy – Littlewood operator.

It is convenient to give here the following Definition.

Definition 4.2 (Weak Morrey space)**.** *Let $\Omega \subset \mathbb{R}^n$, $1 \leq p < \infty$, $0 < \lambda < n$. We say that a locally integrable function f belongs to the weak Morrey space $L^{p,\lambda}_w(\Omega)$ if there exists a constant c such that*

$$t^p|\{x \in \Omega\ :\ |f(x)| > t\} \cap B_r(x)| \leq cr^{\lambda} \qquad \forall t > 0\,.$$

Remark 4.3. *It is quite simple to check that*

$$L^{p,\lambda}_w(\Omega) \subset L^{q,\lambda}(\Omega) \qquad \textit{for any } q \textit{ such that } 1 \leq q < p\,.$$

Theorem 4.4 (Extra integrability of very weak solutions)**.** *Let $0 < \lambda < n-2$, $f \in L^{1,\lambda}(\Omega)$ and let u be the very weak solution of the Dirichlet problem* (10)*. Then, $u \in L^{p_\lambda,\lambda}_w(\Omega)$ where the exponent p_λ is given by the relation*

$$\frac{1}{p_\lambda} = 1 - \frac{2}{n-\lambda}\,.$$

As a consequence, $u \in L^p(\Omega)$ for any $1 \leq p < p_\lambda$. Moreover, there exists $c \geq 0$ such that $\|u\|_{L^p} \leq c\,\|f\|_{L^{1,\lambda}}$ where c does not depend on u and f.

Proof. By the representation formula we have

$$|u(x)| \leq \int_\Omega g(x,y)\,|f(y)|dy \leq c\int_\Omega |x-y|^{2-n}|f(y)|dy$$

a.e. in Ω where $g(x,y)$ is the Green function of L with respect to Ω with pole at $y \in \Omega$ and c is a constant which depends on n and ν. Since $B_r(x) \subset B_{2r}(x) \subset \Omega$ we have

$$\int_\Omega \frac{|f(y)|}{|x-y|^{n-2}}\,dy = \int_{B_{2r}(x)} \frac{|f(y)|}{|x-y|^{n-2}}\,dy + \int_{\Omega\setminus B_{2r}(x)} \frac{|f(y)|}{|x-y|^{n-2}}\,dy \equiv I + II.$$

We estimate the two integrals by using Hedberg trick (see [11]) and so we have

$$I=\sum_{k=0}^{\infty}\int_{B(x;r/2^{k-1})\setminus B(x;r/2^k)}\frac{|f(y)|}{|x-y|^{n-2}}\,dy\le c\sum_{k=0}^{\infty}\left(\frac{r}{2^k}\right)^2 Mf(x)=c\,r^2Mf(x)$$

where $Mf(x)$ is the Hardy-Littlewood maximal function of f at $x\in\Omega$. Now estimate II. We have

$$II=\sum_{k=1}^{\infty}\int_{r2^{k+1}\le|x-y|<r2^k}\frac{|f(y)|}{|x-y|^{n-2}}\,dy\le c\sum_{k=0}^{\infty}\left(2^k r\right)^{\lambda-n+2}\|f\|_{1,\lambda}=c\,r^{\lambda-n+2}\|f\|_{1,\lambda}\,.$$

Then, for any $r>0$,

$$\int_\Omega\frac{|f(y)|}{|x-y|^{n-2}}\,dy\le c\left(r^2Mf(x)+r^{\lambda-n+2}\|f\|_{1,\lambda-n+2}\right)$$

for $r>0$. By taking the minimum of the right hand side we get

$$\int_\Omega\frac{|f(y)|}{|x-y|^{n-2}}\,dy\le c\,(Mf(x))^{1/p_\lambda}\,\|f\|_{1,\lambda}^{\frac{2}{n-\lambda}}$$

so that

$$|u(x)|\le c\,\|f\|_{1,\lambda}^{\frac{2}{n-\lambda}}\,(Mf(x))^{1/p_\lambda}$$

a.e. in Ω and the result follows by Chiarenza - Frasca Theorem 4.1. □

We would like to point out that if $\lambda\to n-2$ then $p_\lambda\to\infty$. Because of that we would expect that $f\in L^{1,n-2}(\Omega)$ should imply $u\in L^\infty(\Omega)$. Unfortunately this is not true as the following example shows.

Example 4.5. *Let $\Omega=\{x\in\mathbb{R}^n\ :\ 0<|x|<1\}$, $n\ge 3$. Then the very weak solution of the Dirichlet problem*

$$\begin{cases}\Delta u=\dfrac{n-2}{|x|^2} & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\end{cases}$$

is the function $u(x)=\log|x|$.

The example suggests that $f\in L^{1,n-2}$ implies $u\in BMO$. This is indeed the case as the following Theorem shows.

Theorem 4.6. *If $f\in L^{1,n-2}(\Omega)$ the very weak solution u of the Dirichlet problem* (10) *belongs to* BMO *locally in the following sense. Let $\Omega'\Subset\Omega$ and $d=\operatorname{dist}(\Omega',\partial\Omega)$. Then, there exists a constant $C\equiv C(n,\nu,d)>0$ such that*

$$⨍_{B_r(x)}\left|u(y)-u_{B_r(x)}\right|dy\le C\|f\|_{1,n-2}\,.$$

for all $0<r<\dfrac{d}{2}$ and $x\in\Omega'$.

Proof. For any $x_0 \in \Omega'$ and $0 < r < d/2$ let us consider $B \equiv B_r(x_0)$, $B^* \equiv B_{2r}(x_0)$ and $f_1 = f\chi_{B^*}$, $f_2 = f(1-\chi_{B^*})$. Let us denote by u_1 and u_2 the very weak solutions of the following Dirichlet problems

$$\begin{cases} Lu = f_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} Lu = f_2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

respectively. By representation formula we have

$$u_1(x) = \int_\Omega g(x,y) f_1(y)dy, \qquad u_2(x) = \int_\Omega g(x,y) f_2(y)dy$$

and then

$$u_1(x) + u_2(x) = \int_\Omega g(x,y) f(y)dy$$

so by uniqueness we get

$$u_1 + u_2 = u\,.$$

We perform separate estimates for u_1 and u_2. We first estimate u_1.

$$\begin{aligned}
\fint_B |u_1(x) - u_{1_B}|\, dx \le 2|u_1|_B &= 2\fint_B \left| \int_\Omega f_1(y) g(x,y)dy \right| dx \\
&\le c\fint_B \int_{B^*} \frac{|f(y)|}{|x-y|^{n-2}} dydx \\
&= c\int_{B^*} |f(y)| \left(\fint_B |x-y|^{2-n} dx \right) dy \\
&\le \frac{c}{|B|} \int_{B^*} |f(y)| \left(\int_{B_{2r}(y)} |x-y|^{2-n}\, dx \right) dy \\
&= cr^{2-n} \int_{B^*} |f(y)|dy \le c\, \|f\|_{1,n-2}
\end{aligned}$$

that is

$$\fint_B |u_1 - u_{1_B}| dx \le c\|f\|_{1,n-2}\,.$$

We now estimate u_2 by using the representation formula. Then we get

$$\begin{aligned}
\fint_B |u_2(x) - u_{2_B}| dx = \fint_B \Bigg| \int_{\Omega\setminus B^*} & g(x,y) f(y) dy - \\
&- \fint_B \int_{\Omega\setminus B^*} g(z,y) f(y) dy dz \Bigg| dx \\
&= \fint_B \left| \int_{\Omega\setminus B^*} f(y) \left[g(x,y) - \fint_B g(z,y)dz \right] dy \right| dx \\
&\le \int_{\Omega\setminus B^*} |f(y)| \left(\fint_B \left| g(x,y) - \fint_B g(z,y)dz \right| dx \right) dy \equiv I + II
\end{aligned}$$

where

$$I = \int_{(\Omega\setminus B^*)\cap\{y:|x_0-y|\le d\}} |f(y)| \left(\fint_B \left| g(x,y) - \fint_B g(z,y)dz \right| dx \right) dy$$

$$II = \int_{(\Omega\setminus B^*)\cap\{y:|x_0-y|> d\}} |f(y)| \left(\fint_B \left| g(x,y) - \fint_B g(z,y)dz \right| dx \right) dy\,.$$

To estimate I we note that $g(\cdot,y)$ is a weak solution of $Lu = 0$ in B and by De Giorgi Theorem it is α-Hölder continuous for some $0 < \alpha < 1$. Let x^* be a point in B such that $g(x^*,y) = \fint_B g(z,y)\,dz$.

Then

$$\begin{aligned}\fint_B \left| g(x,y) - \fint_B g(z,y)dz \right| dx &= \fint_B |g(x,y) - g(x^*,y)|\,dx \\ &\le c \left(\fint_{B_{|x_0-y|/2}(x_0)} g^2(x,y)dx \right)^{\frac{1}{2}} \left(\frac{2r}{|x_0-y|} \right)^{\alpha}.\end{aligned}$$

Since $g(\cdot,y)$ is a positive weak solution we may apply Harnack inequality in $B_{|x_0-y|/2}(x_0)$ and then

$$\begin{aligned}g(x,y) &\le \max_{B_{|x_0-y|/2}(x_0)} g(x,y) \le C \min_{B_{|x_0-y|/2}(x_0)} g(x,y) \\ &\le Cg(x_0,y) \le C|x_0-y|^{2-n}\,.\end{aligned}$$

By using the previous estimates we can bound the integral involving the Green function

$$\fint_B \left| g(x,y) - \fint_B g(z,y)dz \right| dx \le c_{n,\nu} |x_0-y|^{2-n-\alpha}\, r^{\alpha}$$

so we get

$$\begin{aligned}I &\le c\,r^{\alpha} \int_{(\Omega\setminus B^*)\cap\{|x_0-y|\le d\}} |f(y)||x_0-y|^{2-n-\alpha}dy \\ &= Cr^{\alpha} \sum_{k=1}^{\infty} \int_{\{2^k r<|x_0-y|\le 2^{k+1}r\}\cap\{|x_0-y|\le d\}} |f(y)||x_0-y|^{2-n-\alpha}dy \le C\|f\|_{1,n-2}\,.\end{aligned}$$

Now we estimate II.

$$\begin{aligned}II &\le \int_{(\Omega\setminus B^*)\cap\{|x_0-y|>d\}} |f(y)| \left(\fint_B \left| g(x,y) - \fint_B g(z,y)dz \right| dx \right) dy \\ &\le Cd^{2-n} \int_{(\Omega\setminus B^*)\cap\{|x_0-y|>d\}} |f(y)|dy \le C\|f\|_{1,n-2}\,.\end{aligned}$$

Indeed, by triangle inequality we have

$$|x_0-y| \le |x_0-x| + |x-y| < r + |x-y|$$

and then

$$|x-y| \geq |x_0-y|-r > \frac{d}{2} \qquad \forall x \in B.$$

□

Up to now we have shown regularity properties for the very weak solution because in this generality weak solutions do not exist. If we want to obtain more regularity we need to make the assumptions on f stronger.

So now we show a very useful inclusion relation to prove that, from now on, the very weak solution is indeed the weak solution.

Theorem 4.7. *We have $\tilde{S}(\Omega) \subset W^{-1,2}(\Omega)$.*

Proof. We show that there exists a positive constant c such that

$$|\langle f,\varphi\rangle| \leq c\,\eta(f)\,\|\nabla\varphi\|_2 \quad \forall\varphi \in C_0^\infty(\Omega).$$

Indeed, by using the Fefferman inequality we have

$$|\langle f,\varphi\rangle| \leq \left(\int_\Omega |f|\,dx\right)^{1/2}\left(\int_\Omega |f|\varphi^2\,dx\right)^{1/2} \leq c\left(\int_\Omega |f|\,dx\right)^{1/2} \sup_{r>0}\eta(r)\,\|\nabla\varphi\|_2.$$

□

We remark that the previous was so quick because we have used the following inequality that is very useful for many results in PDEs and it is of independent interest.

Theorem 4.8. *(see [27]) Let $1<p<n$, Ω be a bounded domain in $\mathbb{R}^n$ and V belongs to the class $\tilde{S}(\Omega)$. Then there exists a constant c such that*

$$\left(\int_B |V(x)||u(x)|^2\,dx\right)^{1/2} \leq c\,\eta(2R)\int_B |\nabla u(x)|^2\,dx \quad \forall u \in C_0^\infty(\Omega) \tag{11}$$

where R is the radius of a ball $B \equiv B_R$, containing the support of u.

Proof. Since u is a smooth compactly supported function we use the following elementary sub representation formula

$$|u(x)| \leq \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}}dy \equiv \int_{\mathbb{R}^n} |\nabla u(y)|k(x,y)dy \tag{12}$$

from which it follows that

$$\begin{aligned}
\int_B |V(x)||u(x)|^2\,dx &\leq c\int_B |V(x)||u(x)|\left(\int_B |\nabla u(\xi)|k(x,\xi)d\xi\right)dx \\
&= c\int_B |\nabla u(\xi)|\left(\int_B |V(x)||u(x)|k(x,\xi)\,dx\right)d\xi \\
&\leq c\left(\int_B |\nabla u(\xi)|^2 d\xi\right)^{1/2}\left(\int_B\left(\int_B |V(x)||u(x)|k(x,\xi)\,dx\right)^2 d\xi\right)^{1/2}
\end{aligned}$$

By considering the last integral we can write

$$\left(\int_B |V(x)||u(x)|k(x,\xi)\,dx\right)^2 \le \left(\int_B |V(x)|k(x,\xi)\,dx\right) \times \\ \times \left(\int_B |V(x)||u(x)|^2 k(x,\xi)\,dx\right)$$

and then, we obtain

$$\begin{aligned}
\int_B |V(x)||u(x)|^2\,dx &\le \left(\int_B |\nabla u(\xi)|^2\,d\xi\right)^{1/2} \times \\
&\times \left(\int_B \left(\int_B |V(z)|k(z,\xi)\,dz\right) \int_B |V(x)|k(x,\xi)|u(x)|^2\,dx\,d\xi\right)^{1/2} \\
&= \left(\int_B |\nabla u(\xi)|^2\,d\xi\right)^{1/2} \times \\
&\times \int_B |V(x)||u(x)|^2 \int_B k(x,\xi)\,\left(\int_B |V(z)|k(z,\xi)\,dz\right)\,dx \\
&\le \left(\int_B |\nabla u(\xi)|^2\,d\xi\right)^{1/2} \times \left(\int_B \eta(R)|V(x)||u(x)|^2\,dx\,\right)^{1/2} \\
&= \eta^{1/2}(R)\left(\int_B |\nabla u(\xi)|^2\,d\xi\right)^{1/2} \left(\int_B |V(x)||u(x)|^2\,dx\,\right)^{1/2}
\end{aligned}$$

from which (11) easily follows. □

Now we prove a regularity result concerning boundedness of the weak solution. Indeed, because of Theorem 4.7 the very weak solution is the weak one.

Theorem 4.9. *If $f \in \tilde{S}(\Omega)$ then the weak solution u of (10) is bounded in Ω.*

Proof. For any $x \in \Omega$ we have

$$\begin{aligned}
|u(x)| &\le \int_\Omega g(x,y)|f(y)|dy \le c\int_\Omega |f(y)||x-y|^{2-n}dy \\
&\le c\sup_{r>0}\int_{\Omega\cap B_r(x)} |f(y)||x-y|^{2-n}dy\,.
\end{aligned}$$

□

Now we restrict our assumption on f and prove the result about the continuity of the weak solution. The proof is inspired by the Theorem of Chiarenza – Fabes and Garofalo in [3].

Theorem 4.10. *If $f \in S(\Omega)$, then any weak solution u of equation $Lu = f$ is continuous in Ω.*

Before giving the proof of Theorem 4.10 we recall two important results in the form we need for our argument.

Theorem 4.11 (Caccioppoli)**.** *Let u be a weak solution of the equation $Lu = 0$. Then there exists a constant $c = c(n, \nu)$ such that*

$$\int_\Omega |\nabla u|^2 \varphi^2 \, dx \le c \int_\Omega u^2 |\nabla \varphi|^2 \, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Theorem 4.12 (Harnack)**.** *Let u be a non negative weak solution of equation $Lu = 0$. Then there exists a constant $c = c(n, \nu)$ such that, for any ball B such that $2B \Subset \Omega$ we have*

$$\sup_B u \le c \inf_B u$$

Now we can prove Theorem 4.10 about the continuity of weak solutions.

Proof. Let η be the Stummel modulus of f. By the embedding the solution is weak and by inclusion of S in $\tilde{S}$ it is also bounded.

We have

$$\int_\Omega A(x) \nabla u \nabla \psi dx = \int_\Omega f(x) \psi(x) dx$$

for all $\psi \in C_0^\infty(\Omega)$. If B_r is a ball such that $B_{4r} \Subset \Omega$ let ϕ be a cut-off function $C_0^\infty(\Omega)$ such that $0 \le \phi \le 1$ in Ω, $\phi \equiv 1$ in $B_{3r/2}$, $\phi \equiv 0$ out of B_{2r}. Then $u\phi$ is a weak solutions of

$$L(u\phi) = f\phi - \operatorname{div}(A(x) u \nabla \phi) - A(x) \nabla u \nabla \phi \, .$$

By representation formula we get

$$\begin{aligned} u(x)\phi(x) = & \int_\Omega f(y)\phi(y) g(x,y) dy + \\ & + \int_\Omega \nabla_y g(x,y) A(y) u(y) \nabla \phi(y) dy \\ & - \int_\Omega \nabla u(y) A(y) \nabla \phi(y) g(x,y) dy \, . \end{aligned}$$

For any $x \in B_{r/2}(x_0)$

$$\begin{aligned} u(x) - u(x_0) = & \int_\Omega f(y)\phi(y) \left(g(x,y) - g(x_0,y) \right) dy \\ & - \int_\Omega A(y) \nabla u \nabla \phi \left(g(x,y) - g(x_0,y) \right) dy \\ & + \int_\Omega \left(\nabla g_y(x,y) - \nabla g_y(x_0,y) \right) A(y) u(y) \nabla \phi dy \\ & \equiv I + II + III \end{aligned}$$

First estimate I. Let $N > 1$ to be chosen later.

$$
\begin{aligned}
|I| \le & \int\limits_{\{y\in\Omega: |x_0 - y| > N|x-x_0|\}} |f(y)\phi(y)\,(g(x,y) - g(x_0,y))|\,dy \\
& + \int\limits_{\{y\in\Omega: |x_0 - y| \le N|x-x_0|\}} |f(y)\phi(y)\,(g(x,y) - g(x_0,y))|\,dy \\
& \equiv A + B.
\end{aligned}
$$

To estimate A we use the remarkable information that the Green function is a weak solution outside the pole. This means that $g(\cdot, y)$ is α-Hölder continuous out of the pole because of the De Giorgi Theorem and moreover it satisfies Harnack inequality because it is a positive solution. Then the following inequality hold true

$$
\begin{aligned}
|g(x,y) - g(x_0,y)| &\le C\left(\frac{|x_0 - x|}{r}\right)^{\alpha}\left(\fint_{B_r} |g(x,y)|^2 dx\right)^{\frac{1}{2}} \\
&\le CN^{-\alpha}\max_{x\in B_r} g(x,y) \le CN^{-\alpha}\min_{x\in B_r} g(x,y) \\
&\le CN^{-\alpha} g(x_0,y) \le CN^{-\alpha}|x_0 - y|^{2-n}
\end{aligned}
$$

and then

$$
A \le CN^{-\alpha}\int_{B_{2r}(x_0)} |f(y)|\phi(y)|x_0 - y|^{2-n} dy \le CN^{-\alpha}\eta(2r).
$$

Now estimate B by using Theorem 3.8.

$$
|g(x,y) - g(x_0,y)| \le C\left(\frac{1}{|x-y|^{n-2}} + \frac{1}{|x_0 - y|^{n-2}}\right)
$$

Then, due to the domain of integration,

$$
\begin{aligned}
B &\le C \int\limits_{|x_0 - y| \le N|x-x_0|} \frac{|f(y)|}{|x-y|^{n-2}} dy + \int\limits_{|x_0 - y| \le N|x-x_0|} \frac{|f(y)|}{|x_0 - y|^{n-2}} dy \\
&\le C \int\limits_{|x-y| \le (N+1)|x-x_0|} \frac{|f(y)|}{|x-y|^{n-2}} dy + \eta(N|x - x_0|) \le \\
&\le C\eta((N+1)|x_0 - x|) + \eta(N|x - x_0|)
\end{aligned}
$$

By choosing now $N = \left(\frac{r}{|x - x_0|}\right)^{\frac{1}{2}}$ we get

$$
\begin{aligned}
|I| \le & \left(\frac{|x - x_0|}{r}\right)^{\alpha/2} \eta(2r) + \\
& + \eta(\sqrt{r|x - x_0|}) + \eta(\sqrt{r|x - x_0|} + |x - x_0|)\,.
\end{aligned}
$$

Now we estimate II and III.

$$
II = \int_{B_{2r}\setminus B_{3r/2}} (g(x,y) - g(x_0,y))A(y)\nabla u \nabla\varphi dy
$$

By De Giorgi Theorem there exists $\alpha \equiv \alpha(n,\nu) > 0$ such that

$$|g(x,y) - g(x_0,y)| \le c \left(\frac{|x-x_0|}{r}\right)^\alpha \frac{1}{|x_0-y|^{n-2}}$$

if $y \in B_{2r} \setminus B_{3r/2}$, so that

$$\begin{aligned}|II| &\le \frac{c}{r}\left(\frac{|x-x_0|}{r}\right)^\alpha \int_{B_{2r}\setminus B_{3r/2}} \frac{|\nabla u|}{|x_0-y|^{n-2}}dy\\ &\le cr^{1-n}\left(\frac{|x-x_0|}{r}\right)^\alpha \int_{B_{2r}} |\nabla u|dy\end{aligned}$$

and then

$$|II| \le c(n,\nu)\left(\frac{|x-x_0|}{r}\right)^\alpha r\left(\fint_{B_{2r}} |\nabla u|^2 dy\right)^{\frac12}.$$

Then, by Caccioppoli inequality

$$|II| \le c(n,\nu)\left(\frac{|x-x_0|}{r}\right)^\alpha \left(\fint_{B_{4r}} |u|^2 dy\right)^{\frac12}.$$

Finally we estimate III.

$$III = \int_{B_{2r}\setminus B_{3r/2}} \left(\nabla g_y(x,y) - \nabla g_y(x_0,y)\right) A(y)\nabla\varphi u(y)dy$$

By Cauchy Schwarz inequality and Caccioppoli inequality we have

$$\begin{aligned}|III| &\le \frac{c}{r}\int_{B_{2r}\setminus B_{3r/2}} |\nabla g_y(x,y) - \nabla g_y(x_0,y)||u|dy\\ &\le \frac{c}{r}\left(\int_{B_{2r}} |u|^2dy\right)^{\frac12}\left(\int_{B_{2r}\setminus B_{3r/2}} |\nabla g_y(x,y) - \nabla g_y(x_0,y)|^2dy\right)^{\frac12}\\ &= \frac{c}{r^2}\left(\int_{B_{2r}} |u|^2dy\right)^{\frac12}\left(\int_{\frac34 r<|x_0-y|<\frac94 r} |g(x,y) - g(x_0,y)|^2dy\right)^{\frac12}\end{aligned}$$

De Giorgi Theorem and pointwise estimates of Green function yield

$$|III| \le c\left(\frac{|x-x_0|}{r}\right)^\alpha \left(\fint_{B_{2r}} |u|^2dy\right)^{\frac12}$$

Merging previous estimates we get

$$\begin{aligned}|u(x) - u(x_0)| \le c\Bigg[&\eta(2r)\left(\frac{|x-x_0|}{r}\right)^{\alpha/2} + \eta(\sqrt{r|x-x_0|})\\ &+ \eta(\sqrt{r|x-x_0|} + |x-x_0|)\\ &+ \left(\frac{|x-x_0|}{r}\right)^\alpha\left(\fint_{B_{2r}} |u|^2dy\right)^{\frac12}\Bigg] \to 0\end{aligned}$$

as $x \to x_0$ because $f \in S(\Omega)$. $\square$

As $L^{1,\lambda}$ is contained in S we can hope better regularity if we assume $f \in L^{1,\lambda}$, $\lambda > n-2$.

Theorem 4.13. *If $f \in L^{1,\lambda}(\Omega)$, $n-2 < \lambda < n$, then any weak solution u of $Lu = f$ belong to $C^{0,\alpha}(\Omega)$ where $\alpha \equiv \alpha(n, \lambda, \nu, \|f\|_{1,\lambda})$.*

Proof. It is a refinement of the previous result because $L^{1,\lambda}$ is contained in S. In order to show the result we use the fact that the function f belongs to the Morrey space $L^{1,\lambda}$ with $n-2 < \lambda < n$ that implies the following estimate

$$\eta(r) \leq c\,\|f\|_{1,\lambda} r^{\lambda-n+2}\,.$$

By using the estimate we finally get

$$\begin{aligned} |u(x) - u(x_0)| \leq c\,\|f\|_{1,\lambda} &\left[\left(\frac{|x-x_0|}{r} \right)^{\alpha/2} r^{\lambda-n+2} + \right. \\ &\left. + \left(\sqrt{r|x-x_0|} + |x-x_0| \right)^{\lambda-n+2} \right] \\ &+ c \left(\frac{|x-x_0|}{r} \right)^{\alpha} \left(\fint_{B_{2r}} |u|^2 dy \right)^{\frac{1}{2}} \end{aligned}$$

and then

$$|u(x) - u(x_0)| \leq c|x-x_0|^{\beta}$$

where

$$\beta = \frac{1}{2} \min\left(\lambda - n + 2, \alpha\right)$$

where α is the Hölder exponent of the elliptic operator L arising from De Giorgi Theorem. $\square$

4.2. **Necessary conditions for regularity.** In this section we reverse the implication proven before under the additional assumption $f \geq 0$ in Ω (see [7]).

We start with the result concerning L^p regularity.

Definition 4.14. *Let $1 \leq p \leq \infty$. The Schechter spaces M^p is the set of all functions $f \in L^1(\Omega)$ for which there exists $\delta > 0$ such that*

$$M_{p,\delta}(f) \equiv \begin{cases} \left(\int_\Omega \left(\int_{\Omega \cap B_\delta(x)} \frac{|f(y)|}{|x-y|^{n-2}}\, dy \right)^p dx \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \sup\limits_{x \in \Omega} \int_{\Omega \cap B_\delta(x)} \frac{|f(y)|}{|x-y|^{n-2}}\, dy, & \text{if } p = \infty \end{cases}$$

is finite. Local versions can be defined as usual. We say that f belongs to the local Schechter space $M^p_{\mathrm{loc}}(\Omega)$ if $f\chi_K \in M^p(\Omega)$ for any $K \Subset \Omega$.

Theorem 4.15. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ and $1 \leq p < q \leq \infty$ Then, $M^\infty \subseteq M^q \subseteq M^p$*

Theorem 4.16. *Let $f \in L^1(\Omega)$, $f \geq 0$, $1 < p < \infty$ and $u \in L^1(\Omega)$ be the very weak solution of the problem* (10)*. Then, $u \in L^p_{\mathrm{loc}}(\Omega)$ if and only if $f \in M^p_{\mathrm{loc}}(\Omega)$.*

Proof. Let K be a compact subset of Ω. Then, if $u \in L^p(K)$ by representation formula and the positivity of f we have

$$\begin{aligned}\int_K |u(x)|^p dx &= \int_K \left(\int_\Omega g(x,y)f(y)dy\right)^p dx \\ &\geq \int_K \left(\int_{\{y\in\Omega:|x-y|\geq\delta\}} g(x,y)f(y)dy\right)^p dx \\ &\equiv \int_K |u_\delta(x)|^p dx\end{aligned}$$

where

$$u_\delta(x) \equiv \int_{\{y\in\Omega:|x-y|\geq\delta\}} g(x,y)f(y)dy\,, \quad \delta > 0\,.$$

We have $0 \leq u_\delta(x) \leq u(x)$ for any $\delta > 0$ and almost all $x \in K$. Thus, $0 \leq u(x) - u_\delta(x) \leq u(x) \in L^p(K)$ and then

$$\lim_{\delta\to 0^+} \int_K |u_\delta(x) - u(x)|^p dx = 0$$

that is

$$\lim_{\delta\to 0^+} \int_K \left(\int_{\{y\in\Omega:|x-y|<\delta\}} g(x,y)f(y)dy\right)^p dx = 0\,.$$

Now choose $\delta < \dfrac{1}{2}\text{dist}(K, \partial\Omega)$ and apply Green function estimates to obtain

$$\lim_{\delta\to 0^+} \int_K \left(\int_{\{y\in\Omega:|x-y|<\delta\}} |x-y|^{2-n} f(y)dy\right)^p dx = 0\,.$$

Thus, $f \in M^p_{\text{loc}}(\Omega)$. Let us assume now that $f \in M^p_{\text{loc}}(\Omega)$ and show that $u \in L^p_{\text{loc}}(\Omega)$. By representation formula we have

$$u(x) = \int_{\{y\in\Omega:|x-y|<\delta\}} \frac{f(y)}{|x-y|^{n-2}} dy$$

if we choose $\delta \geq \text{diam}\Omega$. Meanwhile the integral belongs to $L^p(K)$ for any K compact subset in Ω by the definition of Scheter class $M^p_{\text{loc}}(\Omega)$. Thus $u \in L^p(K)$ for all compact sets K. □

Theorem 4.17. *Let $f \in L^1(\Omega)$, $f \geq 0$ and $u \in L^1(\Omega)$ be the very weak solution of the problem* (10). *If $f \in \tilde{S}_{loc}(\Omega)$ then $u \in L^\infty_{loc}(\Omega)$.*

Proof. We assume u to be locally bounded in Ω. Let K be a compact subset in Ω, let $x \in K$ and $y \in \Omega$ be such that $|x-y| < \delta \leq \frac{1}{2}\text{dist}(K, \partial\Omega)$. By estimates on the Green function we have

$$\int_{\{y\in\Omega:|x-y|<\delta\}} f(y)|x-y|^{2-n} dy \leq c \int_{\{y\in\Omega:|x-y|<\delta\}} f(y)g(x,y)dy \leq c\|u\|_{L^\infty(K)}\,.$$

□

Now we reverse the result about continuity of the weak solutions.

Theorem 4.18. *Let $f \in L^1(\Omega)$, $f \geq 0$ and $u \in L^1(\Omega)$ be the very weak solution of the problem* (10). *Then, $u \in C^0(\Omega)$ if and only if $f \in S(\Omega)$.*

Proof. We already know that S implies continuity of solution.

We show that if $u \in C^0(\Omega)$ then $f \in S(\Omega)$. We apply Dini Theorem about uniform convergence of sequences of continuous functions. If

$$u_\delta(x) = \int_{\{y\in\Omega: |x-y|\geq\delta\}} f(y)g(x,y)dy$$

and $x_0 \in \Omega$, by estimates for the Green function we have

$$|f(y)||g(x,y)\chi_{B^c_\delta(x)}(y) - g(x_0,y)\chi_{B^c_\delta(x_0)}| \leq \frac{c}{\delta^{n-2}}|f(y)| \in L^1(\Omega)$$

By Lebesgue dominated convergence Theorem we have

$$u_\delta(x) \to u_\delta(x_0)$$

for $\delta > 0$. Moreover, $0 \leq u_\delta(x) \leq u(x)$ and $u_\delta \to u$ everywhere in Ω.

Then, for K compact subset of Ω, since u is continuous the convergence is uniform in K by Dini Theorem. Then

$$\sup_{x\in K}(u(x) - u_\delta(x)) = \sup_{x\in K}\int_{\{y\in\Omega:|x-y|<\delta\}} f(y)g(x,y)dy \to 0$$

and, if $0 < \delta < \dfrac{1}{2}\mathrm{dist}(K, \partial\Omega)$, we have

$$\sup_{x\in K}\int_{\{y\in\Omega:|x-y|<\delta\}} \frac{f(y)}{|x-y|^{n-2}}dy \leq c\sup_{x\in K}\int_{\{y\in\Omega:|x-y|<\delta\}} f(y)g(x,y)dy \to 0.$$

□

Our next step is to show the necessary condition for the local Hölder continuity. The result will be achieved by a suitable Caccioppoli type inequality. First we recall that the solution is the weak one. Indeed, it is locally bounded and then $f \in \tilde{S}(\Omega) \subset W^{-1,2}(\Omega)$. For any $\Omega' \Subset \Omega$ the function u is the weak solution of the equation $Lu = f$ and $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Indeed, if f_k denote the truncation of f at level k we have $0 \leq f_k \leq f_{k+1} \leq f$. Denote by u_k the weak solution of the Dirichlet problem

$$\begin{cases} Lu_k = f_k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\int_\Omega A\nabla u_k \nabla\varphi\, dx = \int_\Omega f_k\varphi\, dx \qquad \forall\varphi \in W^{1,2}_0(\Omega)$$

Now simply take $\varphi = u_k$ and get

$$\|\nabla u_k\|_2 \leq \nu\, \|f_k\|_1 \|u_k\|_\infty \leq \nu\, \|f\|_1 \|u\|_\infty\, .$$

Since u_k weakly converges in $W^{1,p}_0(\Omega)$ for $1 < p \leq \frac{n}{n-1}$ then a subsequence will converge in $W^{1,2}_0$ so it converges to u in $W^{1,2}_0$.

Theorem 4.19 (Caccioppoli type inequality)**.** *Let $f \in L^1(\Omega)$, $f \geq 0$, and $u \in L^1(\Omega)$ be the weak solution of the problem* (10)*. Then, for any ball $B_r \subseteq B_{2r} \Subset \Omega$, we have*

$$\int_{B_r} |\nabla u(x)|^2 \, dx \leq C \left\{ r^\alpha \int_{B_{2r}} f(x) dx + r^{n-2+2\alpha} \right\} .$$

Proof. Let η be a cut-off function $\eta(x) = 1$ in B_r, $0 \leq \eta(x) \leq 1$, $|\nabla \eta| \leq \dfrac{c}{r}$. We can use $\varphi \equiv \eta^2(x)(u(x) - u_{2r})$ as a test function in the definition so we obtain

$$\int_\Omega A \nabla u \nabla \varphi dx = \int_\Omega f(x) \varphi(x) dx \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

We have

$$\int_\Omega A \nabla u \left(2\eta \nabla \eta (u - u_{2r}) + \eta^2 \nabla u\right) dx = \int_\Omega f(x) \eta^2(x)(u - u_{2r}) dx$$

and by ellipticity assumption

$$\begin{aligned} \nu^{-1} \int_{B_{2r}} |\nabla u(x)|^2 \eta^2(x) dx \leq 2 &\int_{B_{2r}} a_{ij} u_{x_i} \eta_{x_j} \eta(x) |u(x) - u_{2r}| dx \\ &+ \int_{B_{2r}} f(x) \eta^2(x) |u(x) - u_{2r}| dx \, . \end{aligned}$$

Using the elementary inequality

$$0 \leq 2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \qquad \forall \varepsilon, a, b > 0$$

taking $\varepsilon = \dfrac{1}{4\nu^2}$ we get

$$\begin{aligned} 2 \int_{B_{2r}} A \nabla u \nabla \eta \eta(x) |u - u_{2r}| dx &\leq \nu \int_{B_{2r}} 2 |\nabla u| |\nabla \eta| \eta(x) |u - u_{2r}| dx \\ &\leq \nu \varepsilon \int_{B_{2r}} |\nabla u|^2 \eta^2(x) dx + \nu \frac{1}{\varepsilon} \int_{B_{2r}} |\nabla \eta|^2 |u - u_{2r}|^2 dx \\ &= \frac{1}{4\nu} \int_{B_{2r}} |\nabla u|^2 \eta^2(x) dx + 4\nu^3 \int_{B_{2r}} |\nabla \eta|^2 |u - u_{2r}|^2 dx \end{aligned}$$

we have

$$\begin{aligned} \nu^{-1} \int_{B_{2r}} |\nabla u|^2 \eta^2(x) dx \leq &\frac{1}{4\nu} \int_{B_{2r}} |\nabla u|^2 \eta^2(x) dx \\ &+ 4\nu^3 \int_{B_{2r}} |\nabla \eta|^2 |u(x) - u_{2r}|^2 dx \\ &+ \int_{B_{2r}} f(x) \eta^2(x) |u(x) - u_{2r}| dx \end{aligned}$$

that is

$$\int_{B_{2r}} |\nabla u|^2 \eta^2(x)dx \leq C \left(\int_{B_{2r}} |\nabla \eta|^2 |u(x) - u_{2r}|^2 dx \right.$$
$$\left. + \int_{B_{2r}} f(x)\eta^2(x)|u(x) - u_{2r}|dx \right)$$

and by Hölder continuity of the solution u we get the inequality. □

The following lemma will be very useful at the end of the proof.

Lemma 4.20 (Stampacchia)**.** *Let $\omega : [0, R] \to \mathbb{R}$ be an increasing function. Let us assume that there exist $0 < \eta, \alpha < 1$ and $H > 0$ such that*

$$\omega(r) \leq \eta\, \omega(4r) + H\, r^{\alpha} \qquad \forall r \leq R\,.$$

Then, there exist $0 < \lambda < 1$ and $c \geq 0$ such that

$$\omega(r) \leq c\, r^{\lambda} \qquad \forall r \leq R\,.$$

Proof. Let $a = \dfrac{\eta + 1}{2}$ and β such that $\eta 4^{\beta} = a < 1$. We show that $\lambda = \min(\alpha, \beta)$.

Let $M = \sup\limits_{]R/4,R[} \dfrac{\omega(\varrho)}{\varrho^{\lambda}}$ i.e. $\omega(\varrho) \leq M\varrho^{\lambda}$ for any $\varrho \in]R/4, R[$ and $n \in \mathbb{N}$ be such that $\dfrac{R}{4^{n+1}} \leq \varrho < \dfrac{R}{4^n}$.

By iteration we have

$$\begin{aligned}
\omega(\varrho) &\leq \eta\omega(4\varrho) + H\varrho^{\alpha} \leq \eta\omega(4\varrho) + H\varrho^{\lambda} \\
&\leq \eta \left\{ \eta\omega(4^2\varrho) + H(4\varrho)^{\lambda} \right\} + H\varrho^{\lambda} \\
&= \eta^2\omega(4^2\varrho) + H\varrho^{\lambda}(1 + 4^{\lambda}\eta) \leq \\
&\leq \eta^n\omega(4^n\varrho) + H\varrho^{\lambda}(1 + 4^{\lambda}\eta + (4^{\lambda}\eta)^2 + \cdots + (4^{\lambda}\eta)^{n-1}) \leq \\
&\leq \eta^n M(4^n\varrho)^{\lambda} + H\varrho^{\lambda}\frac{1}{1 - 4^{\lambda}\eta} \\
&= \varrho^{\lambda}\left((4^{\lambda}\eta)^n M + \frac{H}{1 - 4^{\lambda}\eta} \right) \\
&\leq \varrho^{\lambda}\left(M + \frac{H}{1 - a} \right).
\end{aligned}$$

□

Remark 4.21. *Since $\beta = \log_4 \dfrac{a}{\eta} \to \infty$, if $\eta \to 0^+$ by suitable choose of β we may assume that $\lambda = \alpha$.*

Now we are ready to show the necessary condition for Hölder continuity.

Theorem 4.22. *Let $f \in L^1(\Omega)$, $f \geq 0$, and let $0 < \alpha < 1$ be such that the weak solution u of the problem* (10) *belongs to $C^{0,\alpha}(\Omega)$. Then $f \in L^{1,n-2+\alpha}(\Omega)$.*

Proof. As in the previous result, the solution is the weak one because it is locally bounded. Indeed, if the solution is locally bounded, the function f must belong to $\tilde{S}$ and then to $W^{-1,2}(\Omega)$. Now let ε be a positive number to be chosen later and let $B_r \subseteq B_{4r} \Subset \Omega$.

If $\varphi \in C_0^\infty(B_{2r})$ $0 \le \varphi \le 1$, $\varphi \equiv 1$ in B_r, using the elementary inequality

$$2ab \le \varepsilon r^{-\alpha} a^2 + \frac{1}{\varepsilon} r^\alpha b^2 \,,$$

we easily get

$$\begin{aligned}\int_{B_r} f(x)dx &\le \int_{B_{2r}} f(x)\varphi(x)dx = \int_{B_{2r}} a_{ij}(x) u_{x_i} \varphi_{x_j} dx \\ &\le \varepsilon C r^{-\alpha} \int_{B_{2r}} |\nabla u|^2 dx + \frac{1}{\varepsilon} r^\alpha \int_{B_{4r}} |\nabla \varphi|^2 dx \,.\end{aligned}$$

Now, by Caccioppoli inequality we obtain

$$\int_{B_r} f(x)dx \le c \left\{ \varepsilon \int_{B_{4r}} f(x)dx + \varepsilon r^{n-2+\alpha} + \frac{1}{\varepsilon} r^{n-2+\alpha} \right\}$$

that means

$$\int_{B_r} f(x)dx \le c\,\varepsilon \int_{B_{4r}} f(x)dx + c_\varepsilon r^{n-2+\alpha}.$$

We can choose $\varepsilon = \dfrac{1}{2c}$ and then by Stampacchia Lemma 4.20 there exists c such that

$$\int_{B_r} f(x)dx \le c\, r^{n-2+\alpha}$$

□

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