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Edge Balanced 3-Uniform Hypergraph Designs

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Abstract: In this paper, we completely determine the spectrum of edge balanced H-designs, where H is a 3-uniform hypergraph with 2 or 3 edges, such that H has strong chromatic number $\chi_s(H) = 3$.

Keywords: hypergraph design; edge-balance

1. Introduction

Let $K_v^{(3)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank 3, defined on a vertex set $X = \{x_1, \ldots, x_v\}$, so that $\mathcal{E}$ is the set of all triples of X. Let $H = (V, \mathcal{F})$ be a sub-hypergraph of $K_v^{(3)}$. We call 3-*edges* the triples of V contained in the family $\mathcal{F}$ and *edges* the pairs of V contained in the 3-edges of $\mathcal{F}$. Such pairs will be denoted by $[x, y]$.

An H-*decomposition* of $K_v^{(3)}$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a collection of hypergraphs all isomorphic to H that partition the edge set of $K_v^{(3)}$. An H-decomposition is also called a H-*design of order* v and the elements of $\mathcal{B}$ are called *blocks*

If $\Sigma = (X, \mathcal{B})$ is a H-design, for any $x \in X$, we call *degree of the vertex* x the number $d(x)$ of blocks of $\mathcal{B}$ containing x; for any $x, y \in X$, $x \neq y$, we call *degree of the edge* $[x, y]$ (see [1]) the number $d(x, y)$ of blocks of $\mathcal{B}$ containing the edge $[x, y]$.

Given a hypergraph $H = (X, \mathcal{F})$, there exists an induced action of the automorphism group $\mathrm{Aut}(H)$ of H on the set of the 2-subsets of the triples of $\mathcal{F}$. We call *edge orbits* the orbits of $\mathrm{Aut}(H)$ on this set.

Following the classical definition of balanced designs, it is possible to define *balanced H-designs*.

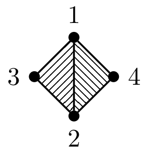
Definition 1. *An H-design Σ is said to be* balanced *if the degree $d(x)$ of each vertex $x \in X$ is a constant.*

In [2], generalizing this idea, the concept of *edge balanced* designs has been introduced:

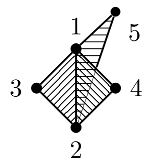
Definition 2. *An H-design is called* edge balanced *if for any $x, y \in X$, $x \neq y$, the degree $d(x, y)$ is constant.*

We will call a balanced hypergraph design *vertex balanced*, in order to make a distinction with edge balanced hypergraph designs. The concept of balanced G-design, in the case that G is a graph, was introduced by Hell and Rosa in [3]. Later, a lot of work has been done in this field (see, for example, [2–14]) both for graph designs and hypergraph designs.

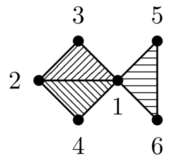
A hypergraph is called *linear* if any two 3-edges have at most one vertex in common. It is trivial to see that any H-design, with H a linear hypergraph, is edge balanced of constant degree $v - 2$. In this paper, continuing the problem introduced in [2], we study edge balanced H-designs, where H is one of the following hypergraphs:



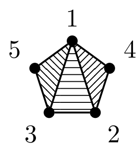
$P^{(3)}(2,4)$



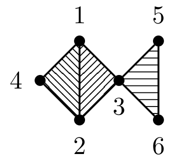
$S^{(3)}(2,5)$



$S^{(3)}(2,4)+E$



$P^{(3)}(2,5)$



$P^{(3)}(2,4)+E$

We denote the above hypergraphs $P^{(3)}(2,4)$ by $[1,2,3,4]_{P^{(3)}(2,4)}$, $S^{(3)}(2,5)$ by $[1,2,3,4,5]_{S^{(3)}(2,5)}$, $S^{(3)}(2,4)+E$ by $[1,2,3,4,5,6]_{S^{(3)}(2,4)+E}$, $P^{(3)}(2,5)$ by $[1,2,3,4,5]_{P^{(3)}(2,5)}$ and $P^{(3)}(2,4)+E$ by $[1,2,3,4,5,6]_{P^{(3)}(2,4)+E}$. From now on by H we will denote one of these hypergraphs. We will denote by $m(H)$ be the number of triples in H, so that:

$$m(H) = \begin{cases} 2 & \text{for } H = P^{(3)}(2,4) \\ 3 & \text{for } H \in \{S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}. \end{cases}$$

For the hypergraphs $P^{(3)}(2,4)$, $S^{(3)}(2,5)$, $S^{(3)}(2,4)+E$ and $P^{(3)}(2,4)+E$ we denote by A the edge orbit that corresponds to the edge $[1,2]$; for $P^{(3)}(2,5)$ by A we denote the edge orbit corresponding to the edges $[1,2]$ and $[1,3]$. Note that, for $P^{(3)}(2,4)$, $S^{(3)}(2,4)+E$ and $P^{(3)}(2,4)+E$, the edge $[1,2]$ has degree 2 and all the others degree 1; for $S^{(3)}(2,5)$, the edge $[1,2]$ has degree 3 and all the others degree 1; for $P^{(3)}(2,5)$, the edges $[1,2]$ and $[1,3]$ have degree 2 and all of the others degree 1.

Given a hypergraph H, a *strong vertex coloring* of H assigns distinct colors to the vertices of a 3-edge of H. The minimum number k such that there exists a strong vertex coloring of H with k colors is called *strong chromatic number* of H and denoted by $\chi_s(H)$ (see [1,15,16]). In this paper, we consider H-designs, with H hypergraph with 2 or 3 edges, such that $\chi_s(H) = 3$.

In the constructions we will use the following remarks:

- if X and Y are disjoint sets, with $|X| = 2k+1$, for some $k \in \mathbb{N}$, $X = \{x_1, \dots, x_{2k+1}\}$, the triples $\{x_1, x_2, y\}$, with $x_1, x_2 \in X$, $x_1 \neq x_2$, and $y \in Y$, are all of the type $\{x_i, x_{i+r}, y\}$, with $i = 1, \dots, 2k+1$, $r = 1, \dots, k$ and $y \in Y$, where the indices are taken mod $2k+1$;
- if X and Y are disjoint sets, with $|X| = 2k$, for some $k \in \mathbb{N}$, $X = \{x_1, \dots, x_{2k}\}$, the triples $\{x_1, x_2, y\}$, with $x_1, x_2 \in X$, $x_1 \neq x_2$, and $y \in Y$, are all of the type either $\{x_i, x_{i+r}, y\}$, with $i = 1, \dots, 2k$, $r = 1, \dots, k-1$, and $y \in Y$ or $\{x_i, x_{i+k}, y\}$, with $i = 1, \dots, k$, where the indices are taken mod $2k$;
- if X, Y and Z are pairwise disjoint sets, such that $|X| = |Y| = v$, $X = \{x_1, \dots, x_v\}$ and $Y = \{y_1, \dots, y_v\}$, the triples $\{x, y, z\}$, with $x \in X$, $y \in Y$ and $z \in Z$, are of type $\{x_i, y_{i+r}, z\}$, with $i = 1, \dots, v$, $r = 0, \dots, v-1$ and $z \in Z$.

2. Necessary Conditions

Let H be one of the hypergraphs listed before and let $\Sigma = (X, \mathcal{B})$ be an H-design. Using the previous notation, for any $x, y \in X$, $x \neq y$:

- we denote by $C(x,y)$ the number of blocks containing $[x,y]$ as an element of the edge orbit A; and,

- we denote by $C'(x,y)$ the number of blocks containing $[x,y]$ as an element of all the other edge orbits.

Proposition 1. *Let $\Sigma = (X, \mathcal{B})$ be an edge balanced H-design of order v and let $m = m(H)$. Subsequently, the following conditions hold:*

(1) *for any $x, y \in X$, $x \neq y$:*

$$d(x,y) = \begin{cases} \dfrac{5(v-2)}{6} & \textit{for } H = P^{(3)}(2,4) \\ \dfrac{7(v-2)}{9} & \textit{for } H \in \{S^{(3)}(2,5), P^{(3)}(2,5)\} \\ \dfrac{8(v-2)}{9} & \textit{for } H \in \{S^{(3)}(2,4)+E, P^{(3)}(2,4)+E\}; \end{cases}$$

(2) $v \equiv 2 \mod 3m$, $v > 2$;

(3) *for any $x, y \in X$, $x \neq y$*

$$C(x,y) = \begin{cases} \dfrac{v-2}{6} & \textit{for } H = P^{(3)}(2,4) \\ \dfrac{v-2}{9} & \textit{for } H \in \{S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,4)+E\} \\ \dfrac{2(v-2)}{9} & \textit{for } H = P^{(3)}(2,5); \end{cases}$$

and

$$C'(x,y) = \begin{cases} \dfrac{2(v-2)}{3} & \textit{for } H \in \{P^{(3)}(2,4), S^{(3)}(2,5)\} \\ \dfrac{7(v-2)}{9} & \textit{for } H \in \{S^{(3)}(2,4)+E, P^{(3)}(2,4)+E\} \\ \dfrac{5(v-2)}{9} & \textit{for } H = P^{(3)}(2,5). \end{cases}$$

Proof. For any H let r be the number of its edges. Let:

$$p = \begin{cases} 2 & \text{for } H \in \{P^{(3)}(2,4), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\} \\ 3 & \text{for } H = S^{(3)}(2,5). \end{cases}$$

If any edge $[x,y]$, for any $x, y \in X$, $x \neq y$, is contained in exactly d blocks of $\mathcal{B}$, because $|\mathcal{B}| = \frac{\binom{v}{3}}{3}$ we have:

$$d\binom{v}{2} = r|B| \Rightarrow d = \frac{r(v-2)}{3m}.$$

By the fact that r and m are always coprime, we immediately get that $v \equiv 2 \mod 9$, $v \geq 11$, if $m = 3$ and $v \equiv 2 \mod 6$, $v \geq 8$, if $m = 2$. For any $x, y \in X$, $x \neq y$, we also have:

$$\begin{cases} C(x,y) + C'(x,y) = d \\ pC(x,y) + C'(x,y) = v-2. \end{cases}$$

This leads us easily to the statement. □

Remark 1. *Note that, keeping the previous notation, in order to prove that an H-design of order v is edge balanced, it is sufficient to show that there exists $c \in \mathbb{N}$, such that $C(x,y) = c$ for any $x, y \in X$, $x \neq y$.*

In this paper, we want to prove that the necessary conditions for the existence of an edge balanced H-design are also sufficient:

Theorem 1. *There exists an edge balanced H-design of order v if and only if either* $v \equiv 2 \mod 6$, $v \geq 8$, *if* $H = P^{(3)}(2,4)$ *or* $v \equiv 2 \mod 9$, $v \geq 11$, *if* $H \in \{S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}$.

3. Decompositions of Multipartite Hypergraphs

In this section, and for all the rest of the paper, we denote by H an hypergraph in the set $\{P^{(3)}(2,4), S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}$. Now, we want to provide decompositions of multipartite hypergraphs that will be used in the proof of the main result. Note that this type of decomposition is possible, because the hypergraphs considered here have $\chi_s(H) = 3$.

If $X_1,\ldots,X_s$, with $s \leq r$, are pairwise disjoint sets, we denote by $K^{(r)}_{X_1,\ldots,X_s}$ the *multipartite hypergraph* having $X_1 \cup \cdots \cup X_s$ as vertex set and edge set the set of all r-subsets of $X_1 \cup \cdots \cup X_s$ containing at least one vertex from every X_i, for $i = 1,\ldots,s$.

We prove the following:

Proposition 2. *Let* $m = m(H)$ *and let X, Y and Z be three disjoint sets such that* $|X| = |Y| = |Z| = m$. *Subsequently, there exists a decomposition of* $K^{(3)}_{X,Y,Z}$ *in copies of H in such a way that:*

- *if* $H \in \{P^{(3)}(2,4), S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,4)+E\}$ *then* $\forall x \in X$, $y \in Y$, $z \in Z$:

$$C(x,y) = 1$$
$$C(x,z) = C(y,z) = 0;$$

- *if* $H = P^{(3)}(2,5)$ *then* $\forall x \in X$, $y \in Y$, $z \in Z$:

$$C(x,y) = C(x,z) = 1$$
$$C(y,z) = 0.$$

Proof. Let $X = \{x_1,\ldots,x_m\}$, $Y = \{y_1,\ldots,y_m\}$ and $Z = \{z_1,\ldots,z_m\}$. Subsequently, we get the statement by taking the following blocks:

- if $H = P^{(3)}(2,4)$, then take $[x_i, y_j, z_1, z_2]_{P^{(3)}(2,4)}$ for any $i,j = 1,2$;
- if $H = S^{(3)}(2,5)$, then take $[x_i, y_j, z_1, z_2, z_3]_{S^{(3)}(2,5)}$ for any $i,j = 1,2,3$;
- if $H = S^{(3)}(2,4)+E$, then take $[y_j, x_i, z_{j+1}, z_{j+2}, x_{i+1}, z_j]_{S^{(3)}(2,4)+E}$ for any $i,j = 1,2,3$;
- if $H = P^{(3)}(2,5)$, then take $[x_i, y_j, z_j, z_{j+1}, y_{j+1}]_{P^{(3)}(2,5)}$ for any $i,j = 1,2,3$;
- if $H = P^{(3)}(2,4)+E$, then take $[x_i, y_j, z_j, z_{j+1}, x_{i+1}, y_{j+1}]_{P^{(3)}(2,4)+E}$ for any $i,j = 1,2,3$,

where the indices are takenmod m. □

Now we can prove the following:

Proposition 3. *Let* $m = m(H)$ *and let* X_1, X_2 *and* X_3 *be three disjoint sets such that* $|X_1| = |X_2| = |X_3| = 3m$. *Subsequently, there exists a decomposition of* $K^{(3)}_{X_1,X_2,X_3}$ *in copies of H in such a way that:*

$$C(x,x') = \begin{cases} 1 & \text{if } H \in \{P^{(3)}(2,4), S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,4)+E\} \\ 2 & \text{if } H = P^{(3)}(2,5) \end{cases}$$

for any $x \in X_i$, $x' \in X_j$, *with* $i \neq j$.

Proof. Let $X_1 = X_{1,1} \cup X_{1,2} \cup X_{1,3}$, $X_2 = X_{2,1} \cup X_{2,2} \cup X_{2,3}$ and $X_3 = X_{3,1} \cup X_{3,2} \cup X_{3,3}$, where $|X_{i,j}| = m$ for any $i, j = 1, 2, 3$.

Consider now $X'_1 = \{x'_{1,1}, x'_{1,2}, x'_{1,3}\}$, $X'_2 = \{x'_{2,1}, x'_{2,2}, x'_{2,3}\}$ and $X'_3 = \{x'_{3,1}, x'_{3,2}, x'_{3,3}\}$ pairwise disjoint sets. When considering the following family $\mathcal{F}$ of paths:

$$[x'_{2,i}, x'_{1,i}, x'_{3,i}],\ [x'_{1,i}, x'_{2,i}, x'_{3,i+1}],\ [x'_{1,i}, x'_{3,i+2}, x'_{2,i}],$$
$$[x'_{2,i+1}, x'_{1,i}, x'_{3,i+2}],\ [x'_{1,i}, x'_{2,i+1}, x'_{3,i}],\ [x'_{1,i}, x'_{3,i}, x'_{2,i+2}],$$
$$[x'_{3,i}, x'_{2,i}, x'_{1,i+1}],\ [x'_{2,i}, x'_{3,i}, x'_{1,i+2}],\ [x'_{2,i+2}, x'_{1,i}, x'_{3,i+1}],$$

where $i = 1, 2, 3$ and the indices are takenmod 3. Note that the set:

$$\{\{x'_{i_1,j_1}, x'_{i_2,j_2}, x'_{i_3,j_3}\} \mid [x'_{i_1,j_1}, x'_{i_2,j_2}, x'_{i_3,j_3}] \in \mathcal{F}\}$$

is the edge set of $K^{(3)}_{X'_1,X'_2,X'_3}$.

Let $H = P^{(3)}(2,5)$. For any path $P_r = [x'_{i_1,j_1}, x'_{i_2,j_2}, x'_{i_3,j_3}] \in \mathcal{F}$, for $r = 1, \ldots, 27$, by Proposition 2 we can consider a family $\mathcal{B}_r$ of copies of H decomposing $K^{(3)}_{X_{i_1,j_1},X_{i_2,j_2},X_{i_3,j_3}}$ such that:

- $C(x, x') = 1$ for any $x \in X_{i_2,j_2}$ and $x' \in X_{i_1,j_1} \cup X_{i_3,j_3}$;
- $C(x, x') = 0$ for any $x \in X_{i_1,j_1}$ and $x' \in X_{i_3,j_3}$.

Let $\mathcal{B} = \bigcup_{r=1}^{27} \mathcal{B}_r$. Then the blocks of $\mathcal{B}$ provide the required decomposition of $K^{(3)}_{X_1,X_2,X_3}$.

Let $H \in \{P^{(3)}(2,4), S^{(3)}(2,5), S^{(3)}(2,4) + E, P^{(3)}(2,4) + E\}$. For any path $P_r = [x'_{i_1,j_1}, x'_{i_2,j_2}, x'_{i_3,j_3}] \in \mathcal{F}$, for $r = 1, \ldots, 27$, by Proposition 2, we can consider a family $\mathcal{B}_r$ of copies of H decomposing $K^{(3)}_{X_{i_1,j_1},X_{i_2,j_2},X_{i_3,j_3}}$ such that:

- $C(x, x') = 1$ for any $x \in X_{i_1,j_1}$ and $x' \in X_{i_3,j_3}$;
- $C(x, x') = 0$ for any $x \in X_{i_2,j_2}$ and $x' \in X_{i_1,j_1} \cup X_{i_3,j_3}$.

Let $\mathcal{B} = \bigcup_{r=1}^{27} \mathcal{B}_r$. Subsequently, the blocks of $\mathcal{B}$ provide the required decomposition of $K^{(3)}_{X_1,X_2,X_3}$. □

4. Proof of the Main Result

Before proving Theorem 1, we need to decompose multipartite hypergraphs, as in the following result:

Proposition 4. *Let $m = m(H)$ and let X, Y and Z be three disjoint sets, such that $|X| = |Y| = 3m$ and $|Z| = 2$. Given*

$$s = \begin{cases} 2 & \text{if } H = P^{(3)}(2,5) \\ 1 & \text{if } H \in \{P^{(3)}(2,4), S^{(3)}(2,5), S^{(3)}(2,4) + E, P^{(3)}(2,4) + E\}, \end{cases}$$

there exists a decomposition of $K^{(3)}_{X,Y,Z} \cup K^{(3)}_{X,Y}$ in copies of H in such a way that for any $x, x' \in X$, $x \neq x'$, $y, y' \in Y$, $y \neq y'$, and $z \in Z$:

- *$C(x, x') = C(y, y') = C(x, z) = C(y, z) = s$;*
- *$C(x, y) = 2s$.*

Proof. Case 1. Let $H = P^{(3)}(2,4)$. Let $X = \{x_1, \ldots, x_6\}$, $Y = \{y_1, \ldots, y_6\}$ and $Z = \{z_1, z_2\}$. In this case we get the statement by taking the following family of blocks:

- for $i = 1, \ldots, 6$ and $r = 1, 2$

$$[x_i, z_r, y_i, y_{i+1}]_{P^{(3)}(2,4)},\ [y_i, z_r, x_{i+3}, x_{i+1}]_{P^{(3)}(2,4)},$$
$$[x_i, x_{i+r}, y_{i+r}, y_{i+2r}]_{P^{(3)}(2,4)},\ [y_i, y_{i+r}, x_{i+r}, x_{i+2r}]_{P^{(3)}(2,4)}$$

- $[x_i, x_{i+3}, y_i, y_{i+3}]_{P^{(3)}(2,4)}$ and $[y_i, y_{i+3}, x_i, x_{i+3}]_{P^{(3)}(2,4)}$ for $i = 1, 2, 3$;
- $[x_i, y_{i+r}, x_{i+1}, x_{i+2}]_{P^{(3)}(2,4)}$ for $i = 1, \ldots, 6$ and $r = 0, 3, 5$;
- $[x_i, y_{i+r}, y_{i+r+1}, y_{i+r+2}]_{P^{(3)}(2,4)}$ for $i = 1, \ldots, 6$ and $r = 0, 1, 3$;
- for $i = 1, \ldots, 6$

$$[x_i, y_{i+1}, x_{i+2}, x_{i+3}]_{P^{(3)}(2,4)},\ [x_i, y_{i+5}, y_{i+1}, y_{i+2}]_{P^{(3)}(2,4)},$$
$$[x_i, y_{i+4}, x_{i+1}, y_{i+1}]_{P^{(3)}(2,4)},\ [x_i, y_{i+2}, x_{i+3}, y_{i+3}]_{P^{(3)}(2,4)}$$

- $[x_i, y_{i+r}, z_1, z_2]_{P^{(3)}(2,4)}$ for $i = 1, \ldots, 6$ and $r = 2, 4$.

Case 2. Let $H = S^{(3)}(2,5)$. Let $X = \{x_1, \ldots, x_9\}$, $Y = \{y_1, \ldots, y_9\}$ and $Z = \{z_1, z_2\}$. Let $A_r = \{i \in \{1,2,3,4\} \mid r \not\equiv i \mod 4\}$ for $r = 0, \ldots, 8$. If $A_r = \{j_1, j_2, j_3\}$, we get the statement by taking the following family of blocks for $i = 1, \ldots, 9$:

- for $r = 3, \ldots, 8$

$$[x_i, y_{i+r}, x_{i+j_1}, x_{i+j_2}, x_{i+j_3}]_{S^{(3)}(2,5)}$$
$$[y_i, x_{i+r}, y_{i+j_1}, y_{i+j_2}, y_{i+j_3}]_{S^{(3)}(2,5)};$$

- for $r = 0, 1, 2$, where $j_1, j_2 \neq r+1$ (and so $j_3 = r+1$)

$$[x_i, y_{i+r}, x_{i+j_1}, x_{i+j_2}, z_1]_{S^{(3)}(2,5)}$$
$$[y_i, x_{i+r}, y_{i+j_1}, y_{i+j_2}, z_2]_{S^{(3)}(2,5)};$$

- for $r = 1, 2, 3$

$$[x_i, x_{i+r}, y_{i+r-1}, y_{i+r}, y_{i+r+4}]_{S^{(3)}(2,5)}$$
$$[y_i, y_{i+r}, x_{i+r-1}, x_{i+r}, x_{i+r+4}]_{S^{(3)}(2,5)};$$

- $[x_i, x_{i+4}, y_i, y_{i+4}, y_{i+8}]_{S^{(3)}(2,5)}$ and $[y_i, y_{i+4}, x_i, x_{i+4}, x_{i+8}]_{S^{(3)}(2,5)}$;
- $[x_i, z_1, y_{i+3}, y_{i+7}, y_{i+8}]_{S^{(3)}(2,5)}$ and $[x_i, z_2, y_{i+1}, y_{i+2}, y_{i+3}]_{S^{(3)}(2,5)}$;
- $[y_i, z_r, x_{i+3}, x_{i+4}, x_{i+5}]_{S^{(3)}(2,5)}$ for $r = 1, 2$.

Case 3. Let $H = S^{(3)}(2,4) + E$. Let $X = \{x_1, \ldots, x_9\}$, $Y = \{y_1, \ldots, y_9\}$ and $Z = \{z_1, z_2\}$. Consider for any $r = 1, 2, 3, 4$:

$$A_1 = \{(0,7), (2,8)\}$$
$$A_2 = \{(1,1), (4,2)\}$$
$$A_3 = \{(6,2), (7,3)\}$$
$$A_4 = \{(3,1), (8,2)\}.$$

We get the statement by taking the following family of blocks for $i = 1, \ldots, 9$:

- for $r = 1,2,3,4$ and $(a,b) \in A_r$

$$[x_i, y_{i+a}, x_{i+r}, x_{i+5-r}, y_{i+b-2r}, x_{i-r}]_{S^{(3)}(2,4)+E}$$
$$[y_i, x_{i+a}, y_{i+r}, y_{i+5-r}, x_{i+b-2r}, x_{i-r}]_{S^{(3)}(2,4)+E};$$

- for $r = 2,3,4$

$$[x_{i+r}, x_i, y_{i+5-r}, y_{i+r}, x_{i+5}, y_{i+5+r}]_{S^{(3)}(2,4)+E}$$
$$[y_{i+r}, y_i, x_{i+5-r}, x_{i+r}, y_{i+5}, x_{i+5+r}]_{S^{(3)}(2,4)+E};$$

- $[x_i, z_r, y_{i+1}, y_i, y_{i+2}, z_s]_{S^{(3)}(2,4)+E}$ and $[y_i, z_r, x_{i+6}, x_{i+2}, x_{i+3}, z_s]_{S^{(3)}(2,4)+E}$, for $(r,s) \in \{(1,2),(2,1)\}$;
- and

$$[x_{i+1}, x_i, y_{i+1}, y_{i+4}, z_1, y_i]_{S^{(3)}(2,4)+E}$$
$$[y_{i+1}, y_i, x_{i+1}, x_{i+4}, z_2, x_{i+2}]_{S^{(3)}(2,4)+E}$$
$$[x_i, y_{i+4}, z_1, z_2, x_{i+4}, y_{i+5}]_{S^{(3)}(2,4)+E}$$
$$[y_{i+5}, x_i, z_1, z_2, y_{i+1}, x_{i+6}]_{S^{(3)}(2,4)+E}.$$

Case 4. Let $H = P^{(3)}(2,5)$. Let $X = \{x_1, \dots, x_9\}$, $Y = \{y_1, \dots, y_9\}$ and $Z = \{z_1, z_2\}$. Consider for any $r = 1,2,3,4$:

$$A_1 = \{(2,0),(3,8)\}$$
$$A_2 = \{(1,6),(4,7)\}$$
$$A_3 = \{(5,6),(8,7)\}$$
$$A_4 = \{(4,0),(5,8)\}$$

and let $b_1 = 7$, $b_2 = 2$, $b_3 = 3$, and $b_4 = 1$. We get the statement by taking the following family of blocks for $i = 1, \dots, 9$:

- for $r = 1,2,3,4$ and $(a,b) \in A_r$

$$[x_i, x_{i+r}, y_{i+a}, y_{i+b}, x_{i+5-r}]_{P^{(3)}(2,5)}$$
$$[y_i, y_{i+r}, x_{i+a}, x_{i+b}, y_{i+5-r}]_{P^{(3)}(2,5)}$$

- for $r = 1,2,3,4$

$$[y_{i+b_r}, x_i, x_{i+r}, x_{i+5-r}, x_{i+2r}]_{P^{(3)}(2,5)}$$
$$[x_{i+b_r}, y_i, y_{i+r}, y_{i+5-r}, y_{i+2r}]_{P^{(3)}(2,5)}$$

- $[z_r, x_i, y_{i+4}, y_{i+5}, x_{i-2}]_{P^{(3)}(2,5)}$ for $r = 1,2$ and

$$[y_{i+1}, x_i, z_1, z_2, x_{i+1}]_{P^{(3)}(2,5)}, [y_{i+2}, x_i, z_2, z_1, x_{i+2}]_{P^{(3)}(2,5)},$$
$$[x_i, y_{i+7}, z_1, z_2, y_{i+3}]_{P^{(3)}(2,5)}, [x_i, y_{i+8}, z_2, z_1, y_{i+3}]_{P^{(3)}(2,5)}.$$

Case 5. Let $H = P^{(3)}(2,4) + E$. Let $X = \{x_1, \dots, x_9\}$, $Y = \{y_1, \dots, y_9\}$ and $Z = \{z_1, z_2\}$. Consider for any $r = 1,2,3,4$:

$$A_1 = \{(0,7),(2,8)\}$$
$$A_2 = \{(1,2),(4,1)\}$$
$$A_3 = \{(6,2),(7,3)\}$$
$$A_4 = \{(3,1),(8,2)\}.$$

We get the statement by taking the following family of blocks for $i = 1,\dots,9$:

- for $r = 1,2,3,4$ and $(a,b) \in A_r$

$$[x_i, y_{i+a}, x_{i+r}, x_{i+5-r}, y_{i+b}, x_{i+2r}]_{P^{(3)}(2,4)+E}$$
$$[y_i, x_{i+a}, y_{i+r}, y_{i+5-r}, x_{i+b}, y_{i+2r}]_{P^{(3)}(2,4)+E}$$

- for $r = 1,2,3,4$

$$[x_i, x_{i+r}, y_{i+5-r}, y_{i+r}, x_{i-r}, x_{i+5-2r}]_{P^{(3)}(2,4)+E}$$
$$[y_i, y_{i+r}, x_{i+5-r}, x_{i+r}, y_{i-r}, y_{i+5-2r}]_{P^{(3)}(2,4)+E}$$

- $[x_i, y_{i+4}, z_2, z_1, x_{i+1}, y_i]_{P^{(3)}(2,4)+E}$ and $[x_i, y_{i+5}, z_1, z_2, x_{i+1}, y_i]_{P^{(3)}(2,4)+E}$;
- $[x_i, z_r, y_{i+1}, y_i, x_{i-1}, z_s]_{P^{(3)}(2,4)+E}$ and $[y_i, z_r, x_{i+6}, x_{i+2}, y_{i+3}, z_s]_{P^{(3)}(2,4)+E}$, for $(r,s) \in \{(1,2),(2,1)\}$.

□

Now we are ready to prove Theorem 1.

Proof. By Proposition 1, we just need to prove that there exists an edge balanced H-design of order v if:

- $H \in \{S^{(3)}(2,5), S^{(3)}(2,4)+E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}$ and $v \equiv 2 \mod 9$, $v \geq 11$;
- $H = P^{(3)}(2,4)$ and $v \equiv 2 \mod 6$, $v \geq 8$.

Let us first prove it in the case $v = 8$, if $H = P^{(3)}(2,4)$, and $v = 11$, if $H \in \{S^{(3)}(2,5), S^{(3)}(2,4) + E, P^{(3)}(2,5), P^{(3)}(2,4)+E\}$. We will use repeatedly (Theorem 3.3, [17]).

- Let $H = P^{(3)}(2,4)$ and $v = 8$. Subsequently, this case follows by (Theorem 4.4, [2]).
- Let $H = S^{(3)}(2,5)$ and $v = 11$. The statement follows by taking the H-design of order 11 on $X = \{0,1,\dots,10\}$ having as base blocks the following ones:

$$[0,1,2,3,4]_{S^{(3)}(2,5)},\ [0,2,4,5,6]_{S^{(3)}(2,5)},\ [0,3,2,6,7]_{S^{(3)}(2,5)},$$
$$[0,4,3,5,6]_{S^{(3)}(2,5)},\ [0,5,1,3,10]_{S^{(3)}(2,5)}.$$

- Let $H = S^{(3)}(2,4)+E$ and $v = 11$. The statement follows by taking the H-design of order 11 on $X = \{0,1,\dots,10\}$ having as base blocks the following ones:

$$[2,1,0,8,5,9]_{S^{(3)}(2,4)+E},\ [1,3,0,10,5,7]_{S^{(3)}(2,4)+E},\ [1,4,0,5,3,9]_{S^{(3)}(2,4)+E},$$
$$[5,1,0,10,2,4]_{S^{(3)}(2,4)+E},\ [1,6,0,3,4,7]_{S^{(3)}(2,4)+E}.$$

- Let $H = P^{(3)}(2,5)$ and $v = 11$. The statement follows by taking the H-design of order 11 on $X = \{0,1,\dots,10\}$ having as base blocks the following ones:

$$[1,0,2,9,4]_{P^{(3)}(2,5)},\ [2,0,4,7,8]_{P^{(3)}(2,5)},\ [3,0,6,5,1]_{P^{(3)}(2,5)},$$
$$[4,0,8,3,5]_{P^{(3)}(2,5)},\ [5,0,10,1,9]_{P^{(3)}(2,5)}.$$

- Let $H = P^{(3)}(2,4) + E$ and $v = 11$. The statement follows by taking the H-design of order 11 on $X = \{0, 1, \ldots, 10\}$ having as base blocks the following ones:

$$[1,2,0,8,3,7]_{P^{(3)}(2,4)+E}, \ [1,3,0,10,2,7]_{P^{(3)}(2,4)+E}, \ [1,4,0,5,2,8]_{P^{(3)}(2,4)+E}, \\ [1,5,0,10,3,6]_{P^{(3)}(2,4)+E}, \ [1,6,0,3,8,10]_{P^{(3)}(2,4)+E}.$$

Now, let $v = 3rh + 2$, for some $h \in \mathbb{N}$, $h \geq 2$, where:

$$r = \begin{cases} 2 & \text{if } H = P^{(3)}(2,4) \\ 3 & \text{if } H \in \{S^{(3)}(2,5), S^{(3)}(2,4) + E, P^{(3)}(2,5), P^{(3)}(2,4) + E\}. \end{cases}$$

Let us consider $X_1, \ldots, X_h$, Y, pairwise disjoint sets such that $|X_i| = 3r$ for $i = 1, \ldots, h$ and $|Y| = 2$, in such a way that $|\bigcup X_i \cup Y| = v$. We can consider the following families of blocks:

- for $i = 1, \ldots, h$ take an edge balanced H-design $\Sigma_i = (X_i \cup Y, \mathcal{B}_i)$ of order $3r + 2$, by what we just proved;
- for any edge $\{i, j, k\} \in K_h^{(3)}$ take a family of blocks $\mathcal{C}_{i,j,k}$ decomposing $K^{(3)}_{X_i,X_j,X_k}$ and satisfying the conditions of Proposition 3; and,
- for any $i, j = 1, \ldots, h$, $i \neq j$, take a family $\mathcal{D}_{i,j}$ decomposing $K^{(3)}_{X_i,X_j} \cup K^{(3)}_{X,Y,Z}$ and satisfying the conditions of Proposition 4.

Let $\mathcal{F} = \cup_{i=1}^{h} \mathcal{B}_i \cup \bigcup \mathcal{C}_{i,j,k} \cup \bigcup \mathcal{D}_{i,j}$. Subsequently, it is easy to see that $\Sigma = (X_1 \cup \cdots \cup X_h \cup Y, \mathcal{F})$ is an edge balanced H-design of order v. □

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References

1. Gionfriddo, M.; Milazzo, L.; Voloshin, V. *Hypergraphs and Designs*; Nova Science Publishers Inc.: New York, NY, USA, 2015.
2. Gionfriddo, M. Edge-Balanced $H^{(3)}$-designs. *Appl. Math. Sci.* **2015**, *9*, 3629–3641. [CrossRef]
3. Hell, P.; Rosa, A. Graph decompositions, handcuffed prisoners and balanced P–designs. *Discrete Math.* **1972**, *2*, 229–252. [CrossRef]
4. Berardi, L.; Gionfriddo, M.; Rota, R. Balanced and Strongly Balanced P_k-Designs. *Discrete Math.* **2012**, *312*, 633–636. [CrossRef]
5. Bonacini, P.; Di Giovanni, M.; Gionfriddo, M.; Marino, L.; Tripodi, A. The spectrum of balanced $P^{(3)}(1,5)$-designs. *Contrib. Discrete Math.* **2017**, *12*, 74–82.
6. Bonacini, P.; Gionfriddo, M.; Marino, L. Balanced house-systems and nestings. *Ars Combin.* **2015**, *121*, 429–436.
7. Bonacini, P.; Gionfriddo, M.; Marino, L. Nonagon quadruple systems: Existence, balance, embeddings. *Australas. J. Combin.* **2016**, *66*, 393–406.
8. Bonisoli, A.; Ruini, B. Tree-designs with balanced-type conditions. *Discrete Math.* **2013**, *313*, 1197–1205. [CrossRef]
9. Bonvicini, S. Degree- and orbit-balanced Γ-designs when Γ has five vertices. *J. Combin. Des.* **2013**, *21*, 359–389. [CrossRef]
10. Gionfriddo, M.; Kucukcifci, S.; Milazzo, L. Balanced and strongly balanced 4-kite designs. *Util. Math.* **2013**, *91*, 121–129.

11. Gionfriddo, M.; Milazzo, L.; Rota, R. Strongly balanced 4-kite designs nested into OQ-systems. *Appl. Math.* **2013**, *4*, 703–706. [CrossRef]
12. Gionfriddo, M.; Milici, S. Balanced $P^{(3)}(2,4)$-designs. *Util. Math.* **2016**, *99*, 81–88.
13. Gionfriddo, M.; Quattrocchi, G. Embedding balanced P_3-designs into balanced P_4-designs. *Discrete Math.* **2008**, *308*, 155–160. [CrossRef]
14. Huang, C.; Rosa, A. On the existence of balanced bipartite designs. *Util. Math.* **1973**, *4*, 55–75. [CrossRef]
15. Demidovich, Y.A.; Raĭgorodskiĭ, A.M. 2-colorings of uniform hypergraphs. *Math. Notes* **2016**, *100*, 629–632. [CrossRef]
16. Raĭgorodskiĭ, A.M.; Shabanov, D.A. The Erdős-Hajnal problem of hypergraph colorings, its generalizations, and related problems. *Russ. Math. Surv.* **2011**, *66*, 933–1002. [CrossRef]
17. Bonacini, P. On a 3-uniform Path-Hypergraph on 5 vertices. *Appl. Math. Sci.* **2016**, *10*, 1489–1500. [CrossRef]