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A disjointly tight irresolvable space

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Abstract

In this short note we prove the existence (in ZFC) of a completely regular countable disjointly tight irresolvable space by showing that every sub-maximal countable dense subset of $2^{\mathfrak{c}}$ is disjointly tight.

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In [2] the first author and V. Malykhin constructed using the Continuum Hypothesis a completely regular countable disjointly tight irresolvable space, and asked if it exists in ZFC. In this note we show how to combine their ideas with those of Juhász, Soukup and Szentmiklóssy [6] to obtain such a space.

Our notation is standard and follows [7, 4]. All topological spaces considered are completely regular.

Recall that a topological space X is *irresolvable* if it has no isolated points and no disjoint dense sets. Following [2] we say that a space X is *disjointly tight* (or it has *disjoint tightness*) if whenever $x \in X$ is an accumulation point of a set A there are disjoint $A_0, A_1 \subseteq A$ each containing x in its closure. A space X is *sub-maximal* if it has no isolated points and every dense subset of X is open, or equivalently (see [1]), if every open subspace of X is irresolvable and every nowhere dense subset of X is closed. These notions were introduced by Hewitt [5] and have bee extensively studied in the past fifty years.

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Answering a question in [1], Juhász, Soukup and Szentmiklóssy [6] showed that

Theorem 1.1 ([6])**.** $2^{\mathfrak{c}}$ *contains a countable dense subspace which is sub-maximal.*

The content of the paper is essentially reduced to proving the following:

Lemma 1.2. *Let X be a countable dense subspace of 2^κ, for some infinite cardinal κ. If $U \subseteq X$ is open, and $x \in X$ is an accumulation point of U, then there are disjoint open $U_0, U_1 \subseteq U$ such that x is an accumulation point of both U_0 and U_1.*

Proof. As 2^κ is c.c.c., we can write U as a countable union of disjoint basic clopen sets

$$U = \bigcup_{n\in\omega} W_n \cap X.$$

For each $n \in \omega$ there is a finite $F_n \subseteq \kappa$ and $\sigma_n : F_n \to 2$ such that

$$W_n = \{y \in 2^\kappa : \sigma_n \subseteq y\}.$$

Let $J \supseteq \bigcup_{n\in\omega} F_n$ be a countable subset of κ such that for every $y \neq y' \in X$ there is an $\alpha \in J$ such that $y(\alpha) \neq y'(\alpha)$. Then the projection function $H : 2^\kappa \to 2^J$ defined by $H(y) = y \restriction J$ is one-to-one on X. Let $Z = H[X]$. Then Z is a dense subset of 2^J, and the sets $H[W_n]$ are disjoint clopen subsets of 2^J having $H(x) = x \restriction J$ as an accumulation point of their union. As J is countable, 2^J is a compact metrizable space, hence there are disjoint infinite sets $A_0, A_1 \subseteq \omega$ such that every neighborhood of $H(x)$ in 2^J contains all but finitely many of the sets $H[W_n]$, $n \in A_0 \cup A_1$.

Let, for $i = 0, 1$

$$U_i = \bigcup_{n\in A_i} W_n \cap X.$$

Both U_0 and U_1 are then open in X and we claim that x is in their closure.

To see this let $W \subseteq 2^\kappa$ be a basic open neighborhood of x, i.e. there is a finite set $F \subseteq \kappa$ such that

$$W = \{y \in 2^\kappa : y \restriction F = x \restriction F\}.$$

Then there is a $k \in \omega$ such that $H[W] \cap H[W_n] \neq \varnothing$, for every $n \in (A_0 \cup A_1) \setminus k$. This implies that $W \cap W_n$ is a non-empty clopen subset of 2^κ for every $n \in (A_0 \cup A_1) \setminus k$. By density of X, $W \cap W_n \cap X \neq \varnothing$ for $n \in (A_0 \cup A_1) \setminus k$ and $W_n \cap X \subseteq U_i$, for $n \in A_i \setminus k$ and $i = 0, 1$. Hence, $W \cap U_0 \neq \varnothing$ and $W \cap U_1 \neq \varnothing$. □

Proposition 1.3. *Every countable dense sub-maximal subset of 2^κ is disjointly tight.*

Proof. Note (or recall) that a subset with empty interior of a sub-maximal space is closed discrete. Consequently, if x is an accumulation point of a subset A of a submaximal space, then x is an accumulation point of $\operatorname{int}(A)$. Now, if

X is a countable dense sub-maximal subset of 2^κ, then by Lemma 1.2 there are disjoint open $U_0, U_1 \subseteq \operatorname{int}(A)$ such that x is an accumulation point of both U_0 and U_1. $\square$

The main result of the note now easily follows.

Corollary 1.4. *There is a countable irresolvable space which is disjointly tight.*

Proof. Follows immediately from Proposition 1.3, Theorem 1.1 and the trivial fact that every sub-maximal space is irresolvable. $\square$

Another corollary of Proposition 1.3 is the result of [1] that no countable dense subset of 2^κ is maximal (i.e. every strictly stronger topology has an isolated point).

Proposition 1.3 actually shows that the countable sub-maximal spaces which are densely embedable in some Cantor cube are quite far from being maximal (see [3], Theorems 2.1b and 2.2c).

However, every countable maximal space is homeomorphic to a subspace of $2^\mathfrak{c}$ and, consistently, there are maximal spaces which are embedable into 2^κ for some $\kappa < \mathfrak{c}$.

Problem 1.5. *Is it consistent that there is a countable irresolvable disjointly tight space of weight strictly less than $\mathfrak{c}$?*

Another variation on tightness was also considered in [2].

Definition 1.6. A space X has empty interior tightness if for any set $A \subseteq X$ and any point $x \in \overline{A}$ there exists a set $B \subseteq A$, with empty interior in X, such that $x \in \overline{B}$.

This notion is relevant here because it fully characterizes resolvability of a countable space.

Theorem 1.7 ([2])**.** *A countable space X is resolvable if and only if it has empty interior tightness.*

Theorem 1.7 and Corollary 1.4 provide a regular space with disjoint tightness but not empty interior tightness. The next example shows that these two notions are actually mutually independent.

Example 1.8. A countable regular space with empty interior tightness but not disjoint tightness.

Proof. Let $X = (\mathbb{Q} \times \omega) \cup \{p\}$, where $\mathbb{Q}$ is the set of rationals and p a free ultrafilter on ω. We topologize X by taking the collection

$$\{U \times \{n\} : U \text{ open in the Euclidean topology on } \mathbb{Q}, n \in \omega\} \cup \{\{p\} \cup (\mathbb{Q} \times A) : A \in p\}$$

as a basis. It is easily seen that X is a countable regular resolvable space. Hence, Theorem 1.7 shows that X has empty interior tightness. On the other hand, the set $A = \{0\} \times \omega$ and $p \in \overline{A}$ witness that X does not have disjoint tightness. $\square$

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