A Traffic Equilibrium Nonlinear Programming Model for Optimizing Road Maintenance Investments

Mauro Passacantando and Fabio Raciti

Abstract We consider a traffic network in which some of the roads need maintenance jobs. Due to budget constraints not all of the roads can be maintained and a central authority has to choose which of them are to be improved. We propose a nonlinear programming model where this choice is made according to its effects on the relative variation of the total cost, assuming that users behave according to Wardrop equilibrium principle. To assess the network improvement after maintenance we use the Bureau of Public Road link cost functions.

Keywords Traffic network · Wardrop equilibrium · Investment optimization · Braess paradox

1 Introduction

Let us consider a traffic network where some of the roads need maintenance, or improvement jobs. However, the available money to be invested in the improvement of the road network is not sufficient for all the roads and a central authority has to decide which of them is better to maintain. In this regard, it is important to assess the impact that the improvement of a single road, or of a group of roads, has on the overall network efficiency. The efficiency index that we use is the relative variation of total travel time on the network under the assumption that flows are distributed according to Wardrop equilibrium principle. This means that travelers choose the roads so as to minimize their journey time, and all of the paths actually used to reach a certain destination from a given origin give rise to the same travel time. The total

M. Passacantando
Department of Computer Science, University of Pisa, Pisa, Italy
e-mail: mauro.passacantando@unipi.it

F. Raciti (✉)
Department of Mathematics and Computer Science, University of Catania, Catania, Italy
e-mail: fraciti@dmi.unict.it

M. Paolucci et al. (eds.), *Advances in Optimization and Decision Science for Society, Services and Enterprises*, AIRO Springer Series 3,
https://doi.org/10.1007/978-3-030-34960-8_24

travel time is often considered as a "social cost" because it represents the sum of the time spent by all the travelers in the network, and moreover is obviously connected with the pollution released by all the vehicles. In our analysis we make use of the link cost functions in the form given by the Bureau of Public Roads (BPR) [2] which explicitly contain the flow capacity u_i of each link a_i. We assume that after maintenance the capacity of each link varies from u_i to $\gamma_i u_i$, where $\gamma_i > 1$ is referred to as the *enhancement ratio* of link a_i. The case of a uniform γ, for all links, was considered in [6], where the authors mainly focused on the case $0 < \gamma < 1$ to describe the degradation of the network links. Let I_i be the investment required to enhance the capacity of link a_i by the ratio γ_i, and let I be the amount of money available for the network maintenance. For each set of links that can be upgraded we update the total travel time mentioned above. It has to be noted that the improvement of a link can result in a worsening of the network efficiency. This counterintuitive fact is related to the well known Braess paradox [1] and will be discussed in detail by means of a small test network for which we can compute all the relevant quantities in closed form. Once that the efficiency of the network has been assessed for all the improvements that satisfy the budget constraint, the central authority can make a decision.

The paper is structured as follows. In the following Sect. 2 we provide the main definitions regarding traffic networks and recall the concept of a Wardrop equilibrium and the network efficiency measure that will be used. In Sect. 3 we present the investment optimization model, which is then illustrated in Sect. 4 by means of a small test problem (Braess network) and a realistic traffic network. Further research perspectives are touched upon in Sect. 5. The paper ends with an appendix where we provide some analytical computations related to the small test problem treated in Sect. 4.

2 Traffic Network Equilibrium and Efficiency Measure

For a comprehensive treatment of all the mathematical aspects of the traffic equilibrium problem, we refer the interested reader to the classical book of Patriksson [9]. Here, we focus on the basic definitions and on the variational inequality formulation of a network equilibrium flow (see, e.g. [3, 11]). In what follows, we denote with $a^\top b$ the scalar product between vectors a and b, and with $M^\top$ the transpose of a given matrix M. A traffic network consists of a triple $G = (N, A, W)$, where $N = \{N_1, \dots, N_p\}$, $p \in \mathbb{N}$, is the set of nodes, $A = \{a_1, \dots, a_n\}$, $n \in \mathbb{N}$ represents the set of direct arcs (also called links) connecting pairs of nodes and $W = \{W_1, \dots, W_m\} \subset N \times N$, $m \in \mathbb{N}$ is the set of the origin-destination (O-D) pairs. The flow on the link a_i is denoted by f_i, and we group all the link flows in a vector $f = (f_1, \dots, f_n)$. A path (or route) is defined as a set of consecutive links and we assume that each O-D pair W_j is connected by r_j, $r_j \in \mathbb{N}$, paths whose set is denoted by P_j, $j = 1, \dots, m$. All the paths in the network are grouped into a vector $(R_1, \dots, R_k)$, $k \in \mathbb{N}$. The link structure of the paths can be described by using the link-path incidence matrix $\Delta = (\delta_{ir})$, $i = 1, \dots, n$, $r = 1, \dots, k$ with entries $\delta_{ir} = 1$ if $a_i \in R_r$ and 0 otherwise. To each path R_r it is associated a flow F_r. The path flows are grouped into a vector $(F_1, \ldots, F_k)$ which is called the network path-flow (or simply, the network flow if it is clear that we refer to paths). The flow f_i on the link a_i is equal to the sum of the path flows on the paths which contain a_i, so that $f = \Delta F$. We now introduce the unit cost of traveling through a_i as a real function $c_i(f) \geq 0$ of the flows on the network, so that $c(f) = (c_1(f), \ldots, c_n(f))$ denotes the link cost vector on the network. The meaning of the cost is usually that of travel time and, in the simplest case, the generic component c_i only depends on f_i. In our model we use the BPR form of the link cost function which explicitly take into account the link capacities. More precisely, the travel cost for link a_i is given by:

$$c_i(f_i) = t_i^0 \left[1 + k \left(\frac{f_i}{u_i} \right)^{\beta} \right], \tag{1}$$

where u_i describes the capacity of link a_i, t_i^0 is the free flow travel time or cost on link a_i, while k and β are model parameters which take on positive values. In many applications $k = 0.15$ and $\beta = 4$. Analogously, one can define a cost on the paths as $C(F) = (C_1(F), \ldots, C_k(F))$. Usually, $C_r(F)$ is just the sum of the costs on the links which build that path:

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f),$$

or in compact form $C(F) = \Delta^{\top} c(\Delta F)$. For each pair W_j, there is a given traffic demand $D_j \geq 0$, so that $D = (D_1, \ldots, D_m)$ is the demand vector of the network. Feasible path flows are nonnegative and satisfy the demands, i.e., belong to the set

$$K = \{F \in \mathbb{R}^k : F_r \geq 0, \text{ for any } r = 1, \ldots, k \text{ and } \Phi F = D\}, \tag{2}$$

where Φ is the pair-path incidence matrix whose entries, for $j = 1, \ldots, m$, $r = 1, \ldots, k$ are

$$\varphi_{jr} = \begin{cases} 1, & \text{if the path } R_r \text{ connects the pair } W_j, \\ 0, & \text{elsewhere.} \end{cases}$$

The notion of a user traffic equilibrium is given by the following definition.

Definition 1 A network flow $H \in K$ is a user equilibrium, if for each O-D pair W_j, and for each pair of paths R_r, R_s which connect W_j

$$C_r(H) > C_s(H) \Longrightarrow H_r = 0;$$

that is, if traveling along the path R_r takes more time than traveling along R_s, then the flow along R_r vanishes.

Remark 1 Among the various paths which connect a given O-D pair W_j some will carry a positive flow and others zero flow. It follows from the previous definition that, for a given O-D pair, the travel cost is the same for all nonzero flow paths, otherwise users would choose a path with a lower cost. Hence, as an equivalent definition of Wardrop equilibrium we can write that for each W_j,

$$C_r(H) \begin{cases} = \lambda_j & \text{if } H_r > 0, \\ \geq \lambda_j & \text{if } H_r = 0. \end{cases} \tag{3}$$

Hence, with the notation λ_j we denote the equilibrium cost shared by all the used paths connecting W_j. The variational inequality formulation of the user equilibrium is given by the following result (see, e.g., [9]).

Theorem 1 *A network flow $H \in K$ is a user equilibrium iff it satisfies the variational inequality*

$$C(H)^\top (F - H) \geq 0, \quad \forall F \in K. \tag{4}$$

Sometimes it is useful to decompose the scalar product in (4) according to the various origin-destination pairs W_j:

$$\sum_{j=1}^{m} \sum_{r \in P_j} C_r(H)\,(F_r - H_r) \geq 0, \quad \forall F \in K.$$

The network efficiency measure we consider in this paper is the total travel time when a Wardrop equilibrium is reached:

$$TC = \sum_{j=1}^{m} \sum_{r \in P_j} C_r(H)\, H_r = \sum_{j=1}^{m} \lambda_j D_j. \tag{5}$$

3 Investment Optimization Model

We consider a central authority with an amount of money I available for the network maintenance. Only a subset of links $\{a_i : i \in \mathcal{L}\}$, where $\mathcal{L} \subset \{1, \ldots, n\}$, are involved in the improvement process. Let I_i be the investment required to enhance the capacity of link a_i by a given ratio γ_i.

The central authority aims to find in which subset of links to invest in order to improve as much as possible the total cost (5), while satisfying the budget constraint. This problem can be formulated within the framework of integer nonlinear optimization.

Let x_i be a binary variable which takes on the value 1 if the investment is actually carried out on link a_i, and 0 otherwise. A vector $x = (x_i)_{i \in \mathcal{L}}$ is feasible if the budget constraint $\sum_{i \in \mathcal{L}} I_i x_i \leq I$ is satisfied. Given a feasible vector x, the capacity of each link a_i, with $i \in \mathcal{L}$, is equal to $u_i(x) := \gamma_i u_i x_i + (1 - x_i) u_i$, i.e., $u_i(x) = \gamma_i u_i$ when $x_i = 1$ and $u_i(x) = u_i$ when $x_i = 0$. We aim to maximize the relative variation of the total cost defined as

$$f(x) = 100 \cdot \frac{TC - TC(x)}{TC},$$

where TC is the total cost (5) before the maintenance job and $TC(x)$ is the total cost corresponding to the improved network. Therefore, the optimization model we propose is

$$\begin{cases} \max \ f(x) \\ \text{s.t. } \sum_{i \in \mathcal{L}} I_i x_i \leq I \\ \qquad x_i \in \{0, 1\} \quad i \in \mathcal{L}. \end{cases} \tag{6}$$

Let us notice that the computation of the nonlinear function f at a given x requires to find a Wardrop equilibrium for both the original and the improved network. Thus, model (6) can be considered as a generalized knapsack problem.

4 Numerical Experiments

This section is devoted to the numerical solution of the proposed model for two networks: the first is a small size network, while the second is the well known Sioux Falls network. The numerical computation of the Wardrop equilibrium was performed by implementing in Matlab 2018a the algorithm designed in [7].

Example 1 We consider the Braess network shown in Fig. 1. There are four nodes, five links labeled by $\{a, b, c, d, e\}$, and one origin-destination pair, from node 1 to node 4 with demand $D = 30$, which can be connected by 3 paths: $R_1 = (a, c)$, $R_2 = (b, d)$, $R_3 = (a, e, d)$. The link cost functions are given by:

$$c_a = 1 + \frac{f_a}{1/2},\ c_b(f_b) = 50\left(1 + \frac{f_b}{50}\right),\ c_c(f_c) = 50\left(1 + \frac{f_c}{50}\right),$$
$$c_d = 1 + \frac{f_d}{1/2},\ c_e(f_e) = 10\left(1 + \frac{f_e}{10}\right).$$

Fig. 1 Braess network

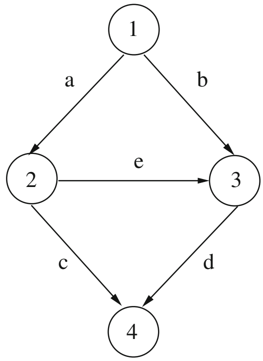

Table 1 Numerical results for Braess network

x	$f(x)$	$I(x)$
(1,0,1,1,0)	6.85	13
(1,1,0,1,0)	6.08	13
(1,0,0,1,0)	5.61	5
(0,1,0,0,1)	−0.19	13
(0,0,0,0,1)	−0.73	5

We assume that the available budget $I = 15\ k€$, the subset of links to be maintained is $\mathcal{L} = \{a, b, c, d, e\}$,

$$\gamma_a = 1.2,\ \gamma_b = 1.1,\ \gamma_c = 1.3,\ \gamma_d = 1.2,\ \gamma_e = 1.5,$$
$$I_a = 2, \quad I_b = 8, \quad I_c = 8, \quad I_d = 3, \quad I_e = 5.$$

Table 1 shows the three best feasible solutions and the two worst ones together with the percentage of total cost improvement and the corresponding investment $I(x) = \sum_{i\in\mathcal{L}} I_i x_i$. It is interesting noting that the third best solution needs a much lower investment than the one required by the optimal solution, but the corresponding objective function values are close. Moreover, the two worst solutions reflect the Braess paradox since the values of the objective function are negative. In the Appendix we analyze in more details the Braess paradox for any value of the demand D and of the enhancement factor γ_e.

Example 2 The Sioux Falls network consists of 24 nodes, 76 links and 528 O-D pairs (see Fig. 2). The complete data can be found on [8]. We assume that the available budget $I = 30\ k€$ and the subset of links to be maintained is $\mathcal{L} = \{4, 10, 21, 22, 29, 30, 31, 49, 75, 76\}$. We consider two different scenarios: in the first one the average enhancement ratio is around 1.3 (low quality maintenance), while in the second one is 1.55 (high quality maintenance). The values of γ_i and I_i of the two scenarios are shown in Table 2.

Table 3 reports the ten best feasible solutions for the two scenarios. Let us note that the value of the ten best solutions in scenario 1 varies between around 10 and

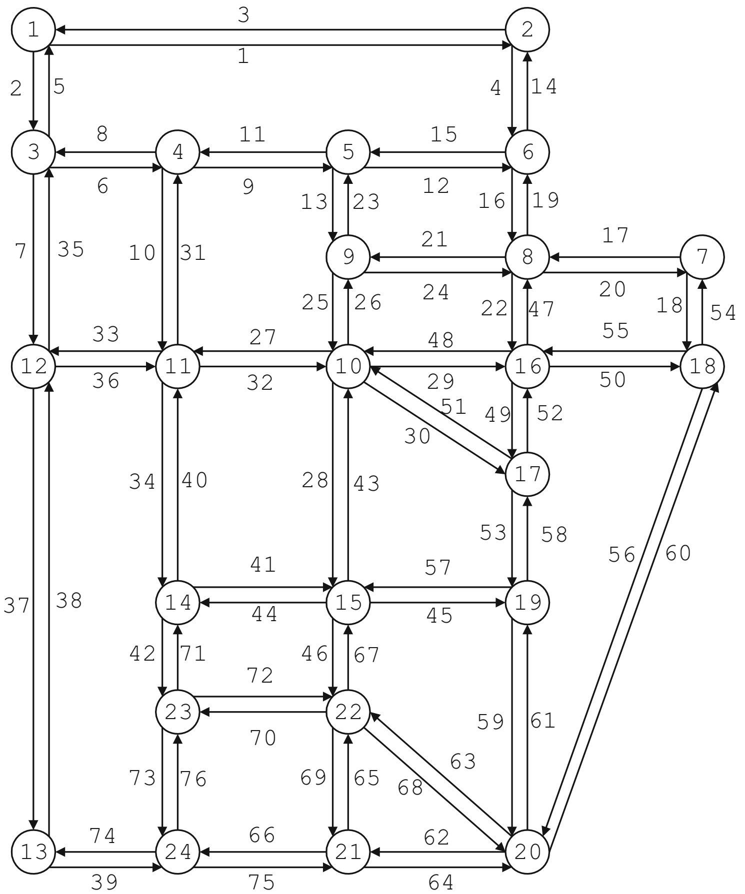

Fig. 2 Sioux Falls network

Table 2 Link capacities and investments for Sioux Falls network

	Scenario 1		Scenario 2	
Links	γ_i	I_i	γ_i	I_i
4	1.2	5	1.4	6
10	1.5	6	1.8	7
21	1.1	10	1.3	12
22	1.3	5	1.5	6
29	1.4	4	1.7	5
30	1.2	8	1.4	10
31	1.1	6	1.3	7
49	1.5	2	1.8	2.5
75	1.4	3	1.7	3.5
76	1.3	2	1.5	2.5

Table 3 Numerical results for Sioux Falls network

Scenario 1			Scenario 2		
x	$f(x)$	$I(x)$	x	$f(x)$	$I(x)$
(0,1,0,1,1,1,0,1,1,1)	10.97	30	(0,1,1,0,1,0,0,1,1,0)	14.30	30
(0,1,0,1,1,1,0,1,1,0)	10.89	28	(0,0,1,1,1,0,0,1,1,0)	14.26	29
(1,0,0,1,1,1,0,1,1,1)	10.64	29	(0,1,0,0,1,1,0,1,1,0)	14.05	28
(1,1,0,0,1,1,0,1,1,1)	10.58	30	(1,1,0,1,1,0,0,1,1,0)	14.00	30
(1,0,0,1,1,1,0,1,1,0)	10.55	27	(0,0,0,1,1,1,0,1,1,1)	13.92	30
(1,1,0,0,1,1,0,1,1,0)	10.50	28	(0,0,0,1,1,1,0,1,1,0)	13.84	27
(0,0,0,1,1,1,1,1,1,1)	10.46	30	(1,0,1,0,1,0,0,1,1,0)	13.80	29
(0,1,1,1,1,0,0,1,1,0)	10.46	30	(0,0,1,0,1,0,1,1,1,0)	13.80	30
(0,0,0,1,1,1,1,1,1,0)	10.38	28	(1,0,0,0,1,1,0,1,1,1)	13.62	30
(0,1,0,0,1,1,1,1,1,0)	10.37	29	(1,0,0,1,1,0,1,1,1,0)	13.54	30

11%, while that in scenario 2 between around 13 and 14%. Therefore, as opposite to Example 1, an improvement of the quality of maintenance implies an improvement of the total cost.

5 Conclusions and Further Perspectives

In this paper we consider the problem of maintaining a road network in an optimal manner. The decision makers are endowed with a given budget and have to decide which roads is better to improve. They make their choice by computing, for each set of possible investments, the corresponding relative improvement of total travel time. The problem is modeled as an integer nonlinear optimization program and some numerical experiments on small and medium scale networks are performed. Such an approach can help the decision makers to select the best possible investments and also displays the counterintuitive fact that some investments can produce a worsening of the traffic.

In the case of large scale networks, further methods have to be developed (e.g., Branch and Bound techniques) to cope the combinatorial nature of the problem. Moreover, since the optimal choice of the decision makers heavily depends on the traffic demand, it would be interesting to consider the realistic case of a randomly perturbed demand (see, e.g., [4, 5]).

Acknowledgements The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA—National Group for Mathematical Analysis, Probability and their Applications) of the Istituto Nazionale di Alta Matematica (INdAM—National Institute of Higher Mathematics). The work of Fabio Raciti has been partially supported by University of Catania ("Piano della Ricerca 2016/2018 Linea di intervento 2").

Appendix

We consider the Braess network shown in Fig. 1, where the traffic demand from node 1 to node 4 is D and the link cost functions are defined as follows:

$$c_a = 1 + \frac{f_a}{1/2},\ c_b(f_b) = 50\left(1 + \frac{f_b}{50}\right),\quad c_c(f_c) = 50\left(1 + \frac{f_c}{50}\right),$$
$$c_d = 1 + \frac{f_d}{1/2},\ c_e(f_e) = 10\left(1 + \frac{f_e}{10\gamma}\right),$$

where γ is the enhancement factor of arc e. We can find the exact Wardrop equilibrium for any value of parameters D and γ (see e.g. [10]). The path cost functions are then given by:

$$C_1(F) = 3F_1 + 2F_3 + 51,$$
$$C_2(F) = 3F_2 + 2F_3 + 51,$$
$$C_3(F) = 2F_1 + 2F_2 + (4 + \gamma^{-1})F_3 + 12.$$

The Wardrop equilibrium is

$$H = \begin{cases} (D/2,\ D/2,\ 0) & \text{if } D > 78, \\ \left(\frac{(2+\gamma^{-1})D-39}{3+2\gamma^{-1}},\ \frac{(2+\gamma^{-1})D-39}{3+2\gamma^{-1}},\ \frac{78-D}{3+2\gamma^{-1}}\right) & \text{if } \frac{39}{2+\gamma^{-1}} \le D \le 78, \\ (0,\ 0,\ D) & \text{if } 0 \le D \le \frac{39}{2+\gamma^{-1}}, \end{cases}$$

and the corresponding equilibrium cost is

$$\lambda = \begin{cases} \frac{3}{2}D + 51 & \text{if } D > 78, \\ \frac{(4+3\gamma^{-1})}{3+2\gamma^{-1}}D + \frac{39}{3+2\gamma^{-1}} + 51 & \text{if } \frac{39}{2+\gamma^{-1}} \le D \le 78, \\ (4+\gamma^{-1})D + 12 & \text{if } 0 \le D \le \frac{39}{2+\gamma^{-1}}. \end{cases}$$

We remark that when the demand D varies between 13 and 78, the total cost

$$TC = \frac{(4+3\gamma^{-1})D^2 + 39D}{3+2\gamma^{-1}} + 51D$$

is an increasing function with respect to $\gamma \in [1, 2]$. As a consequence, in this demand range investing for improving the link e capacity results in a growth of

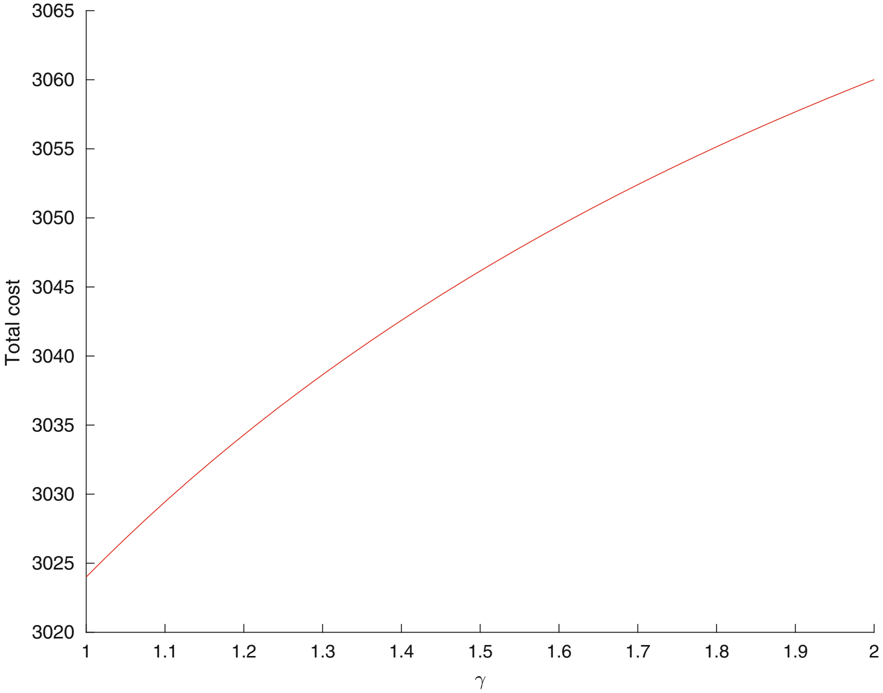

Fig. 3 Total cost vs. enhancement ratio γ for link e

the social cost and pollution. Figure 3 shows the graph of TC as a function of γ for $D = 30$.

References

1. Braess, D.: Über ein Paradoxon aus der Verkehrsplanung. Unternehmenforschung **12**, 258–268 (1968)
2. Bureau of Public Roads: Traffic Assignment Manual. U.S. Department of Commerce, Urban Planning Division, Washington DC (1964)
3. Dafermos, S.: Traffic equilibrium and variational inequalities. Transp. Sci. **14**, 42–54 (1980)
4. Gwinner, J., Raciti, F.: Some equilibrium problems under uncertainty and random variational inequalities. Ann. Oper. Res. **200**, 299–319 (2012)
5. Jadamba, B., Pappalardo, M., Raciti, F.: Efficiency and vulnerability analysis for congested networks with random data. J. Optim. Theory Appl. **177**, 563–583 (2018)
6. Nagurney, A., Qiang, Q.: Robustness of transportation networks subject to degradable links. Europhys. Lett. EPL **80**, Art. 68001, 6 (2007)
7. Panicucci, B., Pappalardo, M., Passacantando, M.: A path-based double projection method for solving the asymmetric traffic network equilibrium problem. Optim. Lett. **1**, 171–185 (2007)
8. Passacantando, M.: Personal web page, Transportation Network Test Problems. http://pages.di.unipi.it/passacantando/test_networks.html. Accessed 29 April 2019

9. Patriksson, M.: The traffic assignment problem. VSP BV, The Netherlands (1994)
10. Raciti, F., Falsaperla, P.: Improved, non iterative algorithm for the calculation of the equilibrium in the traffic network problem. J. Optim. Theory Appl. **133**, 401–411 (2007)
11. Smith, M.J.: The existence, uniqueness and stability of traffic equilibria. Transp. Res. **13B**, 295–304 (1979)

Opinion Dynamics in Multi-Agent Systems Under Proportional Updating and Any-to-Any Influence

Loretta Mastroeni, Maurizio Naldi, and Pierluigi Vellucci

Abstract We study an agent-based model to describe the formation of opinions within a group, where agents belong to classes. In the model any agent influences all the other agents at the same time, and the influence is proportional to the difference of opinions through interaction coefficients. We find that the interaction coefficients must lie within a tetrahedron for the internal consistency of the model. We show that the system of agents reaches a steady state. The long-term opinion of each agents depends anyway on its initial opinion.

Keywords Agent-based models · Opinion dynamics · Social networks

1 Introduction

Agent-based models (ABM) allow us to study all kinds of influence phenomena, in particular to analyze the formation of opinions within a group of people, which is a subject of interest in many areas, e.g. sociology and psychology. Through ABMs we can understand if and how the individuals reach a final consensus or the people polarize around a small number of different opinions [2, 12, 15, 20], by going from the description of the behaviour of individuals at the micro level to the prediction of the macro behaviour of the group.

L. Mastroeni · P. Vellucci
Department of Economics, Roma Tre University, Rome, Italy
e-mail: loretta.mastroeni@uniroma3.it; pierluigi.vellucci@uniroma3.it

M. Naldi (✉)
Department of Civil Engineering and Computer Science, University of Rome Tor Vergata, Rome, Italy

Department of Law, Economics, Politics and Modern Languages, LUMSA University, Rome, Italy
e-mail: maurizio.naldi@uniroma2.it

M. Paolucci et al. (eds.), *Advances in Optimization and Decision Science for Society, Services and Enterprises*, AIRO Springer Series 3,
https://doi.org/10.1007/978-3-030-34960-8_25

In many cases, the individuals in the group exhibit significant differences, e.g. in their religion, ethnicity, political convictions. It is natural to assume that individuals with similar characteristics behave similarly, so that we can consider a partition of the original group into classes and devise multi-class agent-based models. Examples of such multi-class models are shown in [7, 8, 13, 14, 22, 25] for two classes, in [1, 4, 26] for the classification based on the leader\follower role, and in [17], where a classification into three classes based on political convictions (leftist, centrist and rightists) is considered. A generic multi-class model has recently been studied by Monica and Bergenti [19], but the model considers agents interacting in pairs: at each time step, a single agents influences another single agent, who in turn changes its opinion due to that influence.

What happens when an agent influences many agents at the same time? Or when many agents act at the same time on all other agents? This is the most frequent situation occurring in many contexts. For example, that routinely happens in an online social network (like *Facebook* or *Twitter*), where an agent submitting its post influences all the followers at the same time. It also happens on more traditional media (like TV or the radio) where agents involved in a debate may broadcast their opinions to a large audience.

Our paper answers that question, by extending Monica and Bergenti model to the case of any-to-any interaction, where each agent influences all the other agents at once, by providing the following contributions:

- since the opinion is represented by a variable in the $[-1, 1]$ range, we identify the conditions that allow the opinion of agents to stay within the prescribed range, i.e. the *closure property* (Sect. 2);
- we show that the typical state updating equation may be set in the form of a dynamical linear system (Sect. 2);
- we show that the opinion of each agent converges in the long term, and provide the steady-state solution of the dynamical system, i.e. the steady-state opinion of all agents, provided their initial values (Sect. 3).

2 The Any-to-Any Interaction Model

As stated in the Introduction, we introduce a major difference with the model proposed in [19], by assuming that at any time step each agent influences all the other agents at once, i.e. an *any-to-any* interaction. This happens, e.g., in online forums, where each agent (each individual) is exposed in real time to the opinions of all the other agents and may change its opinion accordingly (see, e.g. [21]). In this section, we describe the resulting model and formulate the interaction described by the state updating equation as a linear dynamical system.

We consider a population of n agents, who belong to one of c classes. The differences among classes may be related to a number of factors, e.g. religion, ethnicity, political convictions and we assume that each class is homogeneous—i.e. in them people are similar to each other or are of the same type (same religion,