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A Class of Equations with Three Solutions

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Abstract: Here is one of the results obtained in this paper: Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain, let $q > 1$, with $q < \frac{n+2}{n-2}$ if $n \geq 3$ and let λ_1 be the first eigenvalue of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$. Then, for every $\lambda > \lambda_1$ and for every convex set $S \subseteq L^\infty(\Omega)$ dense in $L^2(\Omega)$, there exists $\alpha \in S$ such that the problem $-\Delta u = \lambda(u^+ - (u^+)^q) + \alpha(x)$ in Ω, $u = 0$ on $\partial\Omega$, has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional $u \to \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \lambda \int_\Omega \left(\frac{1}{2}|u^+(x)|^2 - \frac{1}{q+1}|u^+(x)|^{q+1}\right) dx - \int_\Omega \alpha(x)u(x)dx$ where $u^+ = \max\{u, 0\}$.

Keywords: minimax; multiplicity; global minima

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1. Introduction

There is no doubt that the study of nonlinear PDEs lies in the core of Nonlinear Analysis. In turn, one of the most studied topics concerning nonlinear PDEs is the multiplicity of solutions. On the other hand, the study of the global minima of integral functionals is essentially the central subject of the Calculus of Variations. In the light of these facts, it is hardly understable why the number of the known results on multiple global minima of integral functionals is extremely low. Certainly, this is not due to a lack of intrinsic mathematical interest. Probably, the reason could reside in the fact that there is not an abstract tool which has the same popularity as the one that, for instance, the Lyusternik–Schnirelmann theory and the Morse theory have in dealing with multiple solutions for nonlinear PDEs.

Abstract results on the multiplicity of global minima, however, are already present in the literature. We allude to the result first obtained in [1] and then extended in [2,3] which ensures the existence of at least two global minima provided that a strict minimax inequality holds. We already have obtained a variety of applications upon different ways of checking the required strict inequality ([4–6]).

The aim of the present paper is to establish an application of Theorem 1 of [7] which is itself an application of the main result in [3]. Precisely, we first establish a general result which ensures the existence of three solutions for a certain equation provided that another related one has no non-zero solutions (Theorem 1). Then, we present an application to nonlinear elliptic equations (Theorem 2).

2. Results

In the sequel, $(X, \|\cdot\|_X)$ is a reflexive real Banach space, $(Y, \langle\cdot,\cdot\rangle_Y)$ is a real Hilbert space, $I, \psi : X \to \mathbf{R}$ are two C^1 functionals, with $I(0) = \psi(0) = 0$ and $\sup_{\mathbf{R}} \psi > 0$, $\varphi : X \to Y$ is a C^1 operator, with $\varphi(0) = 0$. For each fixed $y \in Y$, we denote by $\partial_x\langle\varphi(\cdot), y\rangle$ the derivative of the functional $x \to \langle\varphi(x), y\rangle$. Clearly, one has

$$\partial_x\langle\varphi(x), y\rangle(u) = \langle\varphi'(x)(u), y\rangle$$

for all $x, u \in X$.

We say that I is coercive if $\lim_{\|x\|_X \to +\infty} I(x) = +\infty$. We also say that I' admits a continuous inverse on X^* if there exists a continuous operator $T : X^* \to X$ such that $T(I'(x)) = x$ for all $x \in X$.

Here is our abstract result:

Theorem 1. *Let I be weakly lower semicontinuous and coercive, and let I' admit a continuous inverse on X^*. Moreover, assume that the operators φ' and ψ' are compact and that*

$$\lim_{\|x\|_X \to +\infty} \frac{\langle \varphi(x), y \rangle_Y}{I(x)} = 0 \tag{1}$$

for all y in a convex and dense set $V \subseteq Y$.

Set

$$\theta^* := \inf_{x \in \psi^{-1}(]0,+\infty[)} \frac{I(x)}{\psi(x)},$$

$$\tilde{\theta} := \begin{cases} \liminf_{x \in \psi^{-1}(]0,+\infty[), \|x\|_X \to +\infty} \frac{I(x)}{\psi(x)} & \text{if } \psi^{-1}(]0,+\infty[) \text{ is unbounded} \\ +\infty & \text{otherwise} \end{cases}$$

and assume that

$$\theta^* < \tilde{\theta}\,.$$

Then, for each $\lambda \in]\theta^*, \tilde{\theta}[$, with $\lambda \geq 0$, either the equation

$$I'(x) = -\partial_x \langle \varphi(x), \varphi(x) \rangle + \lambda \psi'(x)$$

has a non-zero solution, or, for each convex set $S \subseteq V$ dense in Y, there exists $\tilde{y} \in S$ such that the equation

$$I'(x) = \partial_x \langle \varphi(x), \tilde{y} \rangle_Y + \lambda \psi'(x)$$

has at least three solutions, two of which are global minima in X of the functional

$$x \to I(x) - \langle \varphi(x), \tilde{y} \rangle_Y - \lambda \psi(x)\,.$$

As it was said in the Introduction, the main tool to prove Theorem 1 is a result recently obtained in [7]. For reader's convenience, we now recall its statement:

Theorem 2. *([7], Theorem 1). - Let X, E be two real reflexive Banach spaces and let $\Phi : X \times E \to \mathbf{R}$ be a C^1 functional satisfying the following conditions:*

(a) the functional $\Phi(x, \cdot)$ is quasi-concave for all $x \in X$ and the functional $-\Phi(x_0, \cdot)$ is coercive for some $x_0 \in X$;

(b) there exists a convex set $S \subseteq E$ dense in E, such that, for each $y \in S$, the functional $\Phi(\cdot, y)$ is weakly lower semicontinuous, coercive and satisfies the Palais-Smale condition.

Then, either the system

$$\begin{cases} \Phi'_x(x,y) = 0 \\ \Phi'_y(x,y) = 0 \end{cases}$$

has a solution (x^*, y^*) such that

$$\Phi(x^*, y^*) = \inf_{x \in X} \Phi(x, y^*) = \sup_{y \in E} \Phi(x^*, y)\,,$$

or, for every convex set $T \subseteq S$ dense in E, there exists $\tilde{y} \in T$ such that equation

$$\Phi'_x(x, \tilde{y}) = 0$$

has at least three solutions, two of which are global minima in X of the functional $\Phi(\cdot, \tilde{y})$.

Proof of Theorem 1. Fix $\lambda \in]\theta^*, \tilde{\theta}[$, with $\lambda \geq 0$. Assume that the equation

$$I'(x) = -\partial_x \langle \varphi(x), \varphi(x) \rangle + \lambda \psi'(x)$$

has no non-zero solution. Fix a convex set $S \subseteq Y$ dense in Y. We have to show that there exists $\tilde{y} \in S$ such that the equation

$$I'(x) = \partial_x \langle \varphi(x), \tilde{y} \rangle_Y + \lambda \psi'(x)$$

has at least three solutions, two of which are global minima in X of the functional $x \to I(x) - \langle \varphi(x), \tilde{y} \rangle_Y - \lambda \psi(x)$. To this end, let us apply Theorem 2. Consider the functional $\Phi : X \times Y \to \mathbf{R}$ defined by

$$\Phi(x, y) = I(x) - \frac{1}{2}\|y\|_Y^2 - \langle \varphi(x), y \rangle - \lambda \psi(x)$$

for all $(x, y) \in X \times Y$. Of course, Φ is C^1 and, for each $x \in X$, $\Phi(x, \cdot)$ is concave and $-\Phi(x, \cdot)$ is coercive. Fix $y \in Y$. Let us show that the operator $\partial_x \langle \varphi(\cdot), y \rangle$ is compact. To this end, let $\{x_n\}$ be a bounded sequence in X. Since φ' is compact, up to a subsequence, $\{\varphi'(x_n)\}$ converges in $\mathcal{L}(X, Y)$ to some η. That is

$$\lim_{n \to \infty} \sup_{\|u\|_X = 1} \|\varphi'(x_n)(u) - \eta(u)\|_Y = 0\,.$$

On the other hand, we have

$$\sup_{\|u\|_X=1} |\partial_x \langle \varphi(x_n), y \rangle (u) - \langle \eta(u), y \rangle| = \sup_{\|u\|_X=1} |\langle \varphi'(x_n)(u), y \rangle - \langle \eta(u), y \rangle|$$

$$\leq \sup_{\|u\|_X=1} \|\varphi'(x_n)(u) - \eta(u)\|_Y \|y\|_Y$$

and so the sequence $\{\partial_x \langle \varphi(x_n), y \rangle (\cdot)\}$ converges in X^* to $\eta(\cdot)(y)$. Then, since ψ' is compact, the operator $\partial_x \langle \varphi(\cdot), y \rangle + \lambda \psi'(\cdot)$ is compact too. From this, it follows that $\langle \varphi(\cdot), y \rangle + \lambda \psi(\cdot)$ is sequentially weakly continuous ([8], Corollary 41.9). If $\|x\|_X$ is large enough, we have $I(x) > 0$ and so we can write

$$\Phi(x, y) = I(x) \left(1 - \frac{\frac{1}{2}\|y\|_Y^2 + \langle \varphi(x), y \rangle + \lambda \psi(x)}{I(x)} \right)\,. \tag{2}$$

In view of (1), we also have

$$\liminf_{\|x\|_X \to +\infty} \left(1 - \frac{\frac{1}{2}\|y\|_Y^2 + \langle \varphi(x), y \rangle + \lambda \psi(x)}{I(x)} \right) = 1 - \limsup_{\|x\|_X \to +\infty} \frac{\lambda \psi(x)}{I(x)}\,. \tag{3}$$

We claim that

$$\limsup_{\|x\| \to +\infty} \frac{\lambda \psi(x)}{I(x)} < 1\,. \tag{4}$$

This is clear if either $\lambda = 0$ or $\limsup_{\|x\|_X \to +\infty} \frac{\psi(x)}{I(x)} \leq 0$. If $\lambda > 0$ and $\limsup_{\|x\|_X \to +\infty} \frac{\psi(x)}{I(x)} > 0$, then (4) is equivalent to

$$\limsup_{\|x\|_X \to +\infty} \frac{\psi(x)}{I(x)} < +\infty$$

and

$$\lambda < \frac{1}{\limsup_{\|x\|_X \to +\infty} \frac{\psi(x)}{I(x)}}\,. \tag{5}$$

But

$$\frac{1}{\limsup_{\|x\|_X \to +\infty} \frac{\psi(x)}{I(x)}} = \liminf_{x \in \psi^{-1}(]0,+\infty[), \|x\|_X \to +\infty} \frac{I(x)}{\psi(x)}\,,$$

and so (5) is satisfied just since $\lambda < \tilde{\theta}$. Since I is coercive and weakly lower semicontinuous, the functional $\Phi(\cdot, y)$ turns out to be coercive, in view of (2), (3), (4), and weakly lower semicontinuous, in view of the Eberlein-Smulyan theorem. Finally, since I' admits a continuous inverse on X^*, $\Phi(\cdot, y)$ satisfies the Palais-Smale condition in view of Example 38.25 of [8]. Hence, Φ satisfies the assumptions of Theorem 2. Now, we claim that there is no solution (x^*, y^*) of the system

$$\begin{cases} \Phi'_x(x,y) = 0 \\ \Phi'_y(x,y) = 0 \end{cases}$$

such that

$$\Phi(x^*, y^*) = \inf_{x \in X} \Phi(x, y^*) .$$

Arguing by contradiction, assume that such a (x^*, y^*) does exist. This amounts to say that

$$\begin{cases} I'(x^*) = \partial_x \langle \varphi(x^*), y^* \rangle + \lambda \psi'(x^*) \\ y^* = -\varphi(x^*) \end{cases}$$

and

$$I(x^*) - \langle \varphi(x^*), y^* \rangle - \lambda \psi(x^*) = \inf_{x \in X} (I(x) - \langle \varphi(x), y^* \rangle - \lambda \psi(x)) . \tag{6}$$

Therefore

$$I'(x^*) = -\partial_x \langle \varphi(x^*), \varphi(x^*) \rangle + \lambda \psi'(x^*) .$$

So, by the initial assumption, we have $x^* = 0$ and hence $y^* = 0$ (recall that $\varphi(0) = 0$). As a consequence, since $I(0) = \psi(0) = 0$, (6) becomes

$$\inf_{x \in X} (I(x) - \lambda \psi(x)) = 0 . \tag{7}$$

Now, notice that (7) contradicts the fact that $\lambda > \theta^*$. Hence, *a fortiori*, the system

$$\begin{cases} \Phi'_x(x,y) = 0 \\ \Phi'_y(x,y) = 0 \end{cases}$$

has no solution (x^*, y^*) such that

$$\Phi(x^*, y^*) = \inf_{x \in X} \Phi(x, y^*) = \sup_{y \in Y} \Phi(x^*, y)$$

and then the existence of $\tilde{y} \in S$ is directly ensured by Theorem 2. $\square$

We now present an application of Theorem 1 to a class of nonlinear elliptic equations. Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain. We denote by $\mathcal{A}$ the class of all Carathéodory's functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that, for each $u, v \in H^1_0(\Omega)$, the function $x \to f(x, u(x))v(x)$ lies in $L^1(\Omega)$. For $f \in \mathcal{A}$, we consider the Dirichlet problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

As usual, a weak solution of the problem is any $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u(x) \nabla v(x) dx = \int_\Omega f(x, u(x)) v(x) dx$$

for all $v \in H^1_0(\Omega)$.

Also, we denote by λ_1 the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \,. \end{cases}$$

For any continuous function $f : \mathbf{R} \to \mathbf{R}$, we set $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbf{R}$.

Theorem 3. *Let $f, g : \mathbf{R} \to \mathbf{R}$ be two continuous functions satisfying the following growth conditions:*

(a) *if $n \leq 3$, one has*

$$\lim_{|\xi| \to +\infty} \frac{|F(\xi)|}{\xi^2} = 0\,;$$

(b) *if $n \geq 2$, there exist $p, q > 0$, with $p < \frac{2}{n-2}$, $q < \frac{n+2}{n-2}$ if $n \geq 3$, such that*

$$\sup_{\xi \in \mathbf{R}} \frac{|f(\xi)|}{1 + |\xi|^p} < +\infty\,,$$

$$\sup_{\xi \in \mathbf{R}} \frac{|g(\xi)|}{1 + |\xi|^q} < +\infty\,.$$

Set

$$\rho := \limsup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2}\,,$$

$$\sigma := \max\left\{ \liminf_{\xi \to 0^+} \frac{G(\xi)}{\xi^2}, \liminf_{\xi \to 0^-} \frac{G(\xi)}{\xi^2} \right\}$$

and assume that

$$\max\{\rho, 0\} < \sigma\,.$$

Then, for every $\lambda \in \left] \frac{\lambda_1}{2\sigma}, \frac{\lambda_1}{2\max\{\rho,0\}} \right[$ (with the conventions $\frac{\lambda_1}{+\infty} = 0$, $\frac{\lambda_1}{0} = +\infty$), either the problem

$$\begin{cases} -\Delta u = -F(u)f(u) + \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

has a non-zero weak solution, or, for every convex set $S \subseteq L^\infty(\Omega)$ dense in $L^2(\Omega)$, there exists $\alpha \in S$ such that the problem

$$\begin{cases} -\Delta u = \alpha(x)f(u) + \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional

$$u \to \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \alpha(x)F(u(x))dx - \lambda \int_\Omega G(u(x))dx\,.$$

Proof. We are going to apply Theorem 1 taking $X = H_0^1(\Omega)$, $Y = L^2(\Omega)$, with their usual scalar products (that is, $\langle u, v \rangle_X = \int_\Omega \nabla u(x) \nabla v(x) dx$ and $\langle u, v \rangle_Y = \int_\Omega u(x)v(x)dx$), $V = L^\infty(\Omega)$ and

$$I(u) = \frac{1}{2}\|u\|_X^2\,,$$

$$\varphi(u) = F \circ u\,,$$

$$\psi(u) = \int_\Omega G(u(x))dx$$

for all $u \in X$. In view of (b), thanks to the Sobolev embedding theorem, the operator φ and the functional ψ are C^1, with compact derivative. Moreover, the solutions of the equation

$$I'(u) = -\partial_u \langle \varphi(u), \varphi(u) \rangle_Y + \lambda \psi'(u)$$

are weak solutions of (8) and, for each $\alpha \in Y$, the solutions of the equation

$$I'(u) = \partial_u \langle \varphi(u), \alpha \rangle_Y + \lambda \psi'(u)$$

are weak solutions of (9). Moreover, condition (1) follows readily from (a) which is automatically satisfied when $n \geq 4$ since $p < \frac{2}{n-2}$. We claim that

$$\limsup_{\|u\|_X \to +\infty} \frac{\psi(u)}{\|u\|_X^2} \leq \frac{\rho}{\lambda_1} \,. \tag{10}$$

Indeed, fix $\nu > \rho$. Then, there exists $\delta > 0$ such that

$$G(\xi) \leq \nu \xi^2 \tag{11}$$

for all $x \in \mathbf{R} \setminus [-\delta, \delta]$. Fix $u \in X \setminus \{0\}$. From (11) we clearly obtain

$$\psi(u) \leq \nu \|u\|_Y^2 + \operatorname{meas}(\Omega) \sup_{[-\delta,\delta]} G \leq \nu \frac{\|u\|_X^2}{\lambda_1} + \operatorname{meas}(\Omega) \sup_{[-\delta,\delta]} G$$

and so

$$\limsup_{\|u\|_X \to +\infty} \frac{\psi(u)}{\|u\|_X^2} \leq \frac{\nu}{\lambda_1} \,. \tag{12}$$

Now, we get (10) passing in (12) to the limit for ν tending to ρ. We also claim that

$$\frac{\sigma}{\lambda_1} \leq \sup_{u \in X \setminus \{0\}} \frac{\psi(u)}{\|u\|_X^2} \,. \tag{13}$$

Indeed, fix $\eta < \sigma$. For instance, let $\sigma = \liminf_{\xi \to 0^+} \frac{G(\xi)}{\xi^2}$. Then, there exists $\eta > 0$ such that

$$G(\xi) \geq \eta \xi^2 \tag{14}$$

for all $\xi \in [0, \eta]$. Fix any $v \in H_0^1(\Omega)$ such that $\|v\|_X^2 = \lambda_1 \|v\|_Y^2$ and $v(\Omega) \subseteq [0, \eta]$. From (14) we obtain

$$\psi(v) \geq \eta \|v\|_Y^2$$

and so

$$\sup_{u \in X \setminus \{0\}} \frac{\psi(u)}{\|u\|_X^2} \geq \frac{\psi(v)}{\|v\|_X^2} \geq \frac{\eta}{\lambda_1} \,. \tag{15}$$

Now, (13) is obtained from (15) passing to the limit for η tending to σ. Now, fix $\lambda \in \left] \frac{\lambda_1}{2\sigma}, \frac{\lambda_1}{2\max\{\rho,0\}} \right[$. Then, from (10) and (13), we obtain

$$\limsup_{\|u\|_X \to +\infty} \frac{\psi(u)}{I(u)} < \frac{1}{\lambda} < \sup_{u \in X \setminus \{0\}} \frac{\psi(u)}{I(u)} \,.$$

This readily implies that $\theta^* < \lambda < \tilde{\theta}$ and the conclusion is directly provided by Theorem 1. $\square$

Corollary 1. *Let the assumptions of Theorem 3 be satisfied and let* $\lambda \in \left]\frac{\lambda_1}{2\sigma}, \frac{\lambda_1}{2\max\{\rho,0\}}\right[$ *satisfy*

$$\sup_{\xi\in\mathbf{R}}(\lambda g(\xi) - F(\xi)f(\xi))\xi \leq 0\,. \tag{16}$$

Then, for every convex set $S \subseteq L^\infty(\Omega)$ *dense in* $L^2(\Omega)$*, there exists* $\alpha \in S$ *such that the problem*

$$\begin{cases} -\Delta u = \alpha(x)f(u) + \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ *of the functional*

$$u \to \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \alpha(x)F(u(x))dx - \lambda\int_\Omega G(u(x))dx\,.$$

Proof. It suffices to observe that, in view of (16), 0 is the only solution of (8) and then to apply Theorem 3. □

Finally, notice the following remarkable corollary of Corollary 1:

Corollary 2. *Let* $q > 1$*, with* $q < \frac{n+2}{n-2}$ *if* $n \geq 3$*. Let* $h : \mathbf{R} \to \mathbf{R}$ *be a non-negative continuous function, with* $\inf_{[0,1]} h > 0$*, satisfying conditions* (a) *and* (b) *of Theorem 3 for* $f = h$*.*

Then, for every $\lambda > \lambda_1$ *and for every convex set* $S \subseteq L^\infty(\Omega)$ *dense in* $L^2(\Omega)$*, there exists* $\alpha \in S$ *such that the problem*

$$\begin{cases} -\Delta u = \alpha(x)h(u) + \lambda(u^+ - (u^+)^q) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ *of the functional*

$$u \to \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \alpha(x)H(u(x))dx - \lambda\int_\Omega \left(\frac{1}{2}|u^+(x)|^2 - \frac{1}{q+1}|u^+(x)|^{q+1}\right)dx\,.$$

Proof. Fix $\lambda > \lambda_1$. Notice that, since $\inf_{[0,1]} h > 0$, the number

$$\gamma := \inf_{\xi\in]0,1]} \frac{H(\xi)h(\xi)}{\xi}$$

is positive. Now, we are going to apply Corollary 1 taking

$$f(\xi) = \sqrt{\frac{\lambda}{\gamma}}h(\xi)$$

and

$$g(\xi) = \xi^+ - (\xi^+)^q\,.$$

Of course (with the notations of Theorem 3), $\rho = 0$ and $\sigma = \frac{1}{2}$. Since f in non-negative, Ff is so in $[0,+\infty[$ and non-positive in $]-\infty,0]$. Therefore, (16) is satisfied for all $\xi \in \mathbf{R}\setminus[0,1]$ since g has the opposite sign of Ff in that set. Now, let $\xi \in]0,1]$. We have

$$\frac{F(\xi)f(\xi)}{\xi} = \frac{\lambda}{\gamma}\frac{H(\xi)h(\xi)}{\xi} \geq \lambda \geq \lambda(1-\xi^{q-1})$$

which gives (16). Now, let $S \subseteq L^\infty(\Omega)$ be any convex set dense in $L^2(\Omega)$. Then, the set $\sqrt{\frac{\gamma}{\lambda}}S$ is convex and dense in $L^2(\Omega)$ and the conclusion follows applying Corollary 1 with this set. □

Remark 1. *We are not aware of known results close enough to Theorems 1 and 3 in order to do a proper comparison. We refer to the monographs [9,10] for an account on multiplicity results for nonlinear PDEs.*

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References

1. Ricceri, B. Multiplicity of global minima for parametrized functions. *Rend. Lincei Mat. Appl.* **2010**, *21*, 47–57. [CrossRef]
2. Ricceri, B. A strict minimax inequality criterion and some of its consequences. *Positivity* **2012**, *16*, 455–470. [CrossRef]
3. Ricceri, B. On a minimax theorem: An improvement, a new proof and an overview of its applications. *Minimax Theory Appl.* **2017**, *2*, 99–152.
4. Ricceri, B. A range property related to non-expansive operators. *Mathematika* **2014**, *60*, 232–236. [CrossRef]
5. Ricceri, B. Singular points of non-monotone potential operators. *J. Nonlinear Convex Anal.* **2015**, *16*, 1123–1129.
6. Ricceri, B. Miscellaneous applications of certain minimax theorems II. *Acta Math. Vietnam.* **2020**, in press.
7. Ricceri, B. An alternative theorem for gradient systems. *arXiv* **2020**, arXiv:2002.01413. Available online: https://arxiv.org/abs/2002.01413 (accessed on 4 February 2020).
8. Zeidler, E. *Nonlinear Functional Analysis and Its Applications*; Springer: New York, NY, USA, 1985; Volume III.
9. Kristály, A.; Rădulescu, V.D.; Varga, C. *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*; Cambridge University Press: Cambridge, UK, 2010.
10. Molica Bisci, G.M.; Radulescu, V.D.; Servadei, R. *Variational Methods for Nonlocal Fractional Problems*; Cambridge University Press: Cambridge, UK, 2016.