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On cellular-compact and related spaces

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ABSTRACT

We solve a problem posed by Tkachuk and Wilson, Question 5.10 in [16], on whether every first countable cellular-compact space is weakly Lindelöf. We actually obtain a stronger result and, as a by-product of it, we present a somewhat different proof of Tkachuk and Wilson theorem on the cardinality of a first countable cellular-compact space, valid for the wider class of Urysohn spaces. Moreover, our result holds for a class of spaces in between cellular-compact and cellular-Lindelöf. We conclude with some comments on the cardinality of a weakly linearly Lindelöf space.

According to [5], a space X is cellular-Lindelöf if for any disjoint collection of non-empty open sets $\mathcal{U}$ there is a Lindelöf subspace L such that $L \cap U \neq \emptyset$ for each $U \in \mathcal{U}$. This notion has been further investigated in [6], [11], [12], [14] and [15]. Among other things, in [6] it was shown that if $2^{<\mathfrak{c}} = \mathfrak{c}$, then the cardinality of a normal first countable cellular-Lindelöf space does not exceed $\mathfrak{c}$.

Recently, Tkachuk and Wilson have considered the narrower class of cellular-compact spaces [16]. A space X is cellular-compact provided that for any disjoint family $\mathcal{U}$ of non-empty open sets there is a compact subspace K such that $K \cap U \neq \emptyset$ for each $U \in \mathcal{U}$. In [16] [Theorem 4.13] the authors proved that the

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cardinality of a regular first countable cellular-compact space does not exceed the continuum and asked [Question 5.1] if this result could be true for every Hausdorff space.

In this short note we give a partial answer, by showing that this happens for the class of Urysohn spaces. The key point of our proof is to show that any first countable Hausdorff cellular-compact space is weakly Lindelöf with respect to closed sets. This result gives a stronger positive answer to Question 5.10 in [16] and Question 5.18 in [14].

Then, the cardinality bound valid for Urysohn spaces can be easily deduced from a theorem of Alas [1].

Recall that a space X is weakly Lindelöf with respect to closed sets provided that for any closed set F and any collection of open sets $\mathcal{U}$ such that $F \subseteq \bigcup \mathcal{U}$ there is a countable subcollection $\mathcal{V} \subseteq \mathcal{U}$ satisfying $F \subseteq \overline{\bigcup \mathcal{U}}$.

For notations and undefined notions we refer to [8]. A space X is Lindelöf (or has the Lindelöf property) if every open cover of X has a countable subcover. A space is Urysohn if distinct points can be separated by disjoint closed neighbourhoods. For a cardinal κ and a space X a set $\{x_\alpha : \alpha < \kappa\} \subseteq X$ is a free sequence if $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \alpha \leq \beta < \kappa\}} = \emptyset$ for each $\alpha < \kappa$. A set $D \subseteq X$ is strongly discrete if it has a disjoint open expansion, i.e. there is a disjoint family of open sets $\{U_d : d \in D\}$ such that $d \in U_d$ for every $d \in D$.

Lemma 1. *If X is a first countable Hausdorff space, then every point has a disjoint local π-base.*

Proof. Let $x \in X$. If x is isolated, then there is nothing to prove. So, assume it is not isolated and fix a local base $\{U_n : n < \omega\}$ at x. We may assume that $U_{n+1} \subseteq U_n$ for each n. Since X is a Hausdorff space, there is some n_0 such that $V_0 = U_0 \setminus \overline{U_{n_0}} \neq \emptyset$. Next, we may choose $n_1 > n_0$ such that $V_1 = U_{n_0} \setminus \overline{U_1} \neq \emptyset$, $n_2 > n_1$ such that $V_2 = U_{n_1} \setminus \overline{U_{n_2}} \neq \emptyset$ and so on. It is clear that the collection $\{V_n : n < \omega\}$ is a disjoint local π-base at x. □

We will say that a space X is strongly cellular-Lindelöf provided that for any disjoint collection $\mathcal{U}$ of non-empty open sets there is a closed Lindelöf subspace Y such that $Y \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

It is evident that every strongly cellular-Lindelöf space is cellular-Lindelöf and every cellular-compact Hausdorff space is strongly cellular-Lindelöf. The usual $\Psi(\mathcal{A})$ space over an uncountable almost disjoint family $\mathcal{A}$ on ω is a cellular-Lindelöf space which is not strongly cellular-Lindelöf. On the other hand, any countable discrete space is strongly cellular-Lindelöf but not cellular-compact.

Lemma 2. *Let X be a Hausdorff first countable strongly cellular-Lindelöf space. If D is a strongly discrete subset of X, then $\overline{D}$ has the Lindelöf property.*

Proof. Let $\{U_d : d \in D\}$ be a disjoint collection of open sets satisfying $d \in U_d$ for each $d \in D$. By Lemma 1, for every $d \in D$ we may fix a disjoint local π-base $\mathcal{E}_d$ at d such that $\bigcup \mathcal{E}_d \subseteq U_d$. The set $\mathcal{E} = \bigcup\{\mathcal{E}_d : d \in D\}$ is a disjoint collection of non-empty open sets and so there exists a closed Lindelöf subspace Y which intersects each member of $\mathcal{E}$. As every member of $\mathcal{E}_d$ meets Y, we have $d \in \overline{Y} = Y$. Therefore, $D \subseteq Y$ and we deduce that $\overline{D} \subseteq Y$ has the Lindelöf property. □

We are now ready to give the announced positive answer to Question 5.10 in [16] and Question 5.18 in [14]. Our proof is inspired by the argument used in Theorem 4.13 of [16].

Theorem 3. *A strongly cellular-Lindelöf first countable Hausdorff space is weakly Lindelöf with respect to closed subsets.*

Proof. Let X be a strongly cellular-Lindelöf first countable Hausdorff space, F a closed subset of X and $\mathcal{U}$ a collection of open subsets of X satisfying $F \subseteq \bigcup \mathcal{U}$. Let $x_0 \in F$ and take any $W_0 = U_0 \in \mathcal{U}$ such that

$x_0 \in W_0$. We proceed by induction to define for each $\alpha < \omega_1$ points $x_\alpha \in F$, open sets $W_\alpha \subseteq U_\alpha \in \mathcal{U}$ with $x_\alpha \in W_\alpha$ and countable families $\mathcal{V}_\alpha \subseteq \mathcal{U}$ in such a way that the following conditions are satisfied.

a) $\overline{\{x_\beta : \beta < \alpha\}} \subseteq \bigcup \mathcal{V}_\alpha$;

b) $W_\alpha \cap (\bigcup(\{W_\beta : \beta < \alpha\} \cup \bigcup\{\mathcal{V}_\beta : \beta < \alpha\})) = \emptyset$.

Fix $\alpha < \omega_1$ and assume to have already defined $\{x_\beta : \beta < \alpha\}$, $\{W_\beta \subseteq U_\beta : \beta < \alpha\}$ and $\{\mathcal{V}_\beta : \beta < \alpha\}$.

If $F \subseteq \overline{\bigcup(\{W_\beta : \beta < \alpha\} \cup \bigcup\{\mathcal{V}_\beta : \beta < \alpha\})}$ we stop because $\mathcal{V} = \{U_\beta : \beta < \alpha\} \cup \bigcup\{\mathcal{V}_\beta : \beta < \alpha\}$ is a countable subfamily of $\mathcal{U}$ satisfying $F \subseteq \overline{\bigcup \mathcal{V}}$. If not, we may pick a point $x_\alpha \in F$, an open set W_α and an element $U_\alpha \in \mathcal{U}$ in such a way that $x_\alpha \in W_\alpha \subseteq U_\alpha$ and $W_\alpha \cap (\bigcup(\{W_\beta : \beta < \alpha\} \cup \bigcup\{\mathcal{V}_\beta : \beta < \alpha\})) = \emptyset$. Finally, as the set $\{x_\beta : \beta < \alpha\}$ is strongly discrete, by Lemma 2 the set $\overline{\{x_\beta : \beta < \alpha\}}$ has the Lindelöf property. Since $\overline{\{x_\beta : \beta < \alpha\}} \subseteq F \subseteq \bigcup \mathcal{U}$, there exists a countable family $\mathcal{V}_\alpha \subseteq \mathcal{U}$ such that $\overline{\{x_\beta : \beta < \alpha\}} \subseteq \bigcup \mathcal{V}_\alpha$.

Now, at the end of the induction, the resulting set $D = \{x_\alpha : \alpha \in \omega_1\}$ turns out to be a free sequence because for each α we have $\overline{\{x_\beta : \beta < \alpha\}} \subseteq \bigcup \mathcal{V}_\alpha$ and $(\bigcup \mathcal{V}_\alpha) \cap \{x_\beta : \alpha \leq \beta < \omega_1\} = \emptyset$.

The set D is also strongly discrete and so by Lemma 2 its closure should have the Lindelöf property. But, a first countable Lindelöf space cannot contain uncountable free sequences and we reach a contradiction. This shows that the induction cannot be carried out for all $\alpha < \omega_1$. As explained before, when the induction stops we get a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ satisfying $F \subseteq \overline{\bigcup \mathcal{V}}$. □

The Ψ-space over any MAD family of ω is a first countable pseudocompact space which is weakly Lindelöf with respect to closed sets, but it is not strongly cellular-Lindelöf.

Tkachuk constructed in [15] a nice cellular-Lindelöf space which is not weakly Lindelöf. In a Hausdorff P-space every Lindelöf subspace is closed. So, any Hausdorff cellular-Lindelöf P-space is strongly cellular-Lindelöf. Tkachuk's example is a P-space, so we actually have a strongly cellular-Lindelöf space which is not weakly Lindelöf.

The previous space is clearly not first countable. On the other hand, in [6] it is shown that under CH any normal first countable cellular-Lindelöf space is weakly Lindelöf. It is not clear whether there exists a regular (or Hausdorff) first countable cellular-Lindelöf space which is not weakly Lindelöf.

Alas in [1] proved that a first countable Urysohn space which is weakly Lindelöf with respect to closed sets has cardinality at most the continuum. So, we immediately get:

Corollary 4. *If X is a first countable Urysohn strongly cellular-Lindelöf (in particular cellular-compact) space, then $|X| \leq \mathfrak{c}$.*

It is still an open problem whether Alas's result is true for Hausdorff spaces. In [2][Corollary 22] Arhangel'skiĭ showed that this happens by using the stronger notion of strict quasi Lindelöfness. Such result makes sense only for first countable spaces, but a more general one has been recently established in [7]. A space X is strictly quasi Lindelöf provided that for any closed set F and any collection of open sets $\mathcal{U} = \bigcup\{\mathcal{U}_n : n < \omega\}$ such that $F \subseteq \bigcup \mathcal{U}$ there are countable subcollections $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each $n < \omega$ satisfying $F \subseteq \bigcup\{\overline{\bigcup \mathcal{V}_n} : n < \omega\}$. Unfortunately, here we did not manage to prove that a first countable Hausdorff cellular-compact space is strictly quasi Lindelöf.

Another partial answer to the question asked by Tkachuk and Wilson can be obtained by strengthening the hypothesis of first countability.

Theorem 5. *If X is a cellular-compact space with a point-countable base, then $|X| \leq \mathfrak{c}$.*

Proof. Since X has a point-countable base, every compact subset is metrizable and hence separable. This in turn implies that every collection of pairwise disjoint open sets is countable. Now, the result follows from the Hajnal-Juhász inequality $|X| \leq 2^{c(X)\chi(X)}$. □

As a Hausdorff cellular-Lindelöf P-space is strongly cellular-Lindelöf, with minor modifications we may prove:

Theorem 6. *A cellular-Lindelöf Urysohn P-space of character at most ω_1 has cardinality not exceeding 2^{ω_1}.*

According to [10], a space is weakly linearly Lindelöf if every family of non-empty open sets has a complete accumulation point. A collection of sets $\mathcal{U}$ in the space X has an accumulation point p if every neighbourhood of p meets $|\mathcal{U}|$-many elements of $\mathcal{U}$.

A space X is almost linearly Lindelöf [15] if every open cover has a subcollection of countable cofinality whose union is dense in X

Corollary 3.3 in [15] shows that every almost linearly Lindelöf space is weakly linearly Lindelöf and Corollary 3.19 in [15] exhibits under CH (but the argument actually works by assuming $\mathfrak{c} < \aleph_\omega$) a cellular-Lindelöf space which is not almost linearly Lindelöf.

Lemma 7. *Let X be a Hausdorff sequential space. If $A \subseteq X$, then $|\overline{A}| \leq |A|^\omega$.*

Lemma 8. *([15] [Theorem 3.11]) Let X be a weakly linearly Lindelöf space. If $\mathcal{W}$ is an open cover of X of regular uncountable cardinality, then there exists a subfamily $\mathcal{W}' \subseteq \mathcal{W}$ such that $\bigcup \mathcal{W}'$ is dense in X and $|\mathcal{W}'| < |\mathcal{W}|$.*

Theorem 9. *Let X be a normal sequential space satisfying $\chi(X) \leq \mathfrak{c}$. If either a) [$2^{<\mathfrak{c}} = \mathfrak{c}$] X is weakly linearly Lindelöf or b) [$\mathfrak{c} < \aleph_\omega$] X is almost linearly Lindelöf, then $|X| \leq \mathfrak{c}$.*

Proof. For each $p \in X$ let $\mathcal{U}_p$ be a local base at p such that $|\mathcal{U}_p| \leq \mathfrak{c}$. We will construct by transfinite recursion an increasing sequence $\{H_\alpha : \alpha < \mathfrak{c}\}$ of closed subsets of X satisfying:
1_α) $|H_\alpha| \leq \mathfrak{c}$;
2_α) if $X \setminus \overline{\bigcup \mathcal{V}} \neq \emptyset$ for some $\mathcal{V} \subseteq \bigcup\{\mathcal{U}_p : p \in H_\alpha\}$ with $|\mathcal{V}| < \mathfrak{c}$ (case a) or $|\mathcal{V}| \leq \omega$ (case b)), then $H_{\alpha+1} \setminus \overline{\bigcup \mathcal{V}} \neq \emptyset$.

Put $H_0 = \{x_0\}$ for some $x_0 \in X$ and let $\phi : \mathcal{P}(X) \to X$ be any choice function such that $\phi(\emptyset) = x_0$. Assume to have already defined the subsequence $\{H_\beta : \beta < \alpha\}$. If α is a limit ordinal, then put $H_\alpha = \overline{\bigcup\{H_\beta : \beta < \alpha\}}$ (Lemma 7 ensures 1_α). If $\alpha = \gamma + 1$, then let H_α be the closure of the set $H_\gamma \cup \{\phi(X \setminus \overline{\bigcup \mathcal{V}}) : \mathcal{V} \subseteq \bigcup\{\mathcal{U}_p : p \in H_\gamma\}$ with $|\mathcal{V}| < \mathfrak{c}$ (case a)) or $|\mathcal{V}| \leq \omega$ (case b)) $\}$. A counting argument (which takes into account $2^{<\mathfrak{c}} = \mathfrak{c}$ in case a)) and again Lemma 7 show that H_α satisfies 1_α.

Then, put $H = \bigcup\{H_\alpha : \alpha < \mathfrak{c}\}$. It is clear that $|H| \leq \mathfrak{c}$. So, the proof will be completed by showing that $X = H$. Suppose the contrary and pick a non-empty open set O such that $\overline{O} \subseteq X \setminus H$. For each $p \in H$ take an element $U_p \in \mathcal{U}_p$ satisfying $U_p \cap O = \emptyset$. As the space is normal, we may also pick an open set W such that $H \subseteq W$ and $\overline{W} \subseteq \bigcup\{U_p : p \in H\}$. The collection $\mathcal{W} = \{U_p : p \in H\} \cup \{X \setminus \overline{W}\}$ is an open cover of X of cardinality not exceeding $\mathfrak{c}$.
Case a). If $|\mathcal{W}| = \mathfrak{c}$, then by Lemma 8 there exists a subfamily $\mathcal{W}' \subseteq \mathcal{W}$ such that $\bigcup \mathcal{W}'$ is dense in X and $|\mathcal{W}'| < \mathfrak{c}$. If $|\mathcal{W}| < \mathfrak{c}$, then let $\mathcal{W}' = \mathcal{W}$.
Case b). By the definition of almost linear Lindelöfness, there exists a subfamily $\mathcal{W}' \subseteq \mathcal{W}$ of countable cofinality such that $\bigcup \mathcal{W}'$ is dense in X. Since $|\mathcal{W}'| \leq |\mathcal{W}| \leq \mathfrak{c} \leq \aleph_\omega$, we must have $|\mathcal{W}'| \leq \omega$.

In both cases, put $\mathcal{V} = \mathcal{W}' \setminus \{X \setminus \overline{W}\}$ and notice that $H \subseteq \overline{\bigcup \mathcal{V}}$.

Moreover, in both cases we have $|\mathcal{V}| < cf(\mathfrak{c})$. Thus, there is an ordinal $\alpha < \mathfrak{c}$ such that $\mathcal{V} \subseteq \{U_p : p \in H_\alpha\}$. Since $H_{\alpha+1} \subseteq H \subseteq \overline{\bigcup \mathcal{V}}$, we reach a contradiction with condition 2_α. This completes the proof. □

As mentioned in [15], it is still an open problem to prove Theorem 9 in ZFC. As in Theorem 5, we observe that a positive solution can be easily obtained by strengthening the first countability assumption.

Theorem 10. *If X is a normal weakly linearly Lindelöf space with a point-countable base, then $|X| \leq \mathfrak{c}$.*

Proof. By Proposition 3.10 in [15], the space X has extent $e(X) \leq \mathfrak{c}$. Now, it suffices to use the inequality $|X| \leq we(X)^{psw(X)}$, established by Hodel in [9] (see also [3] for a short direct proof). Since in our case $we(X) \leq e(X) \leq \mathfrak{c}$ and $psw(X) = \omega$, we are done. □

Recall that the weak extent $we(X)$ of a space X is the smallest cardinal κ such that for any open cover $\mathcal{U}$ there is a set $A \subseteq X$ satisfying $|A| \leq \kappa$ and $X = \bigcup\{U : U \in \mathcal{U}, U \cap A \neq \emptyset\}$.

We conclude this paper with some comments on a recent work of Xuan and Song [13].

Denote by $\tau(X)$ the topology of the space X. A g-function for X is a map $g : \omega \times X \to \tau(X)$ such that $x \in g(n,x)$ and $g(n+1,x) \subseteq g(n,x)$ for each $x \in X$ and $n < \omega$. A g-function $g : \omega \times X \to \tau(X)$ is said to be symmetric if for any $n < \omega$ and $x, y \in X$ $x \in g(n,y)$ if and only if $y \in g(n,x)$. Furthermore, $g^2(n,x) = \bigcup\{g(n,y) : y \in g(n,x)\}$.

In [13] the authors gave several results on weakly linearly Lindelöf spaces. We wish to make a comment on one of them.

Proposition 11. *([13] [Theorem 3.14]) Suppose X is a Baire space with a symmetric function g such that:*

(1) $\bigcap\{g^2(n,x) : n < \omega\} = \{x\}$ *for each* $x \in X$*;*
(2) *for each $n < \omega$ there is a set $F_n \subseteq X$ such that $|F_n| \leq \omega_1$ and $X = \bigcup\{g(n,x) : x \in F_n\}$.*

If every family of non-empty open subsets of X of cardinality ω_1 has a complete accumulation point, then $|X| \leq \mathfrak{c}$.

We observe that in the above result neither the Baire property nor that fragment of weak linear Lindelöfness are needed.

Theorem 12. *Suppose that X is a space with a symmetric g-function $g : \omega \times X \to \tau(X)$ satisfying:*

(1) $\bigcap\{g^2(n,x) : n < \omega\} = \{x\}$ *for each* $x \in X$*;*
(2) *for each $n < \omega$ there is a set $A_n \subseteq X$ such that $X = \bigcup\{g(n,a) : a \in A_n\}$ and $|A_n| \leq \mathfrak{c}$.*

Then $|X| \leq \mathfrak{c}$.

Proof. Let $A = \bigcup\{A_n : n < \omega\}$. Note that $|A| \leq \mathfrak{c}$. Fix a well-ordering $\prec$ on X. We may define a map $f : X \to {}^{\omega}A$ in such a way that for $x \in X$ and $n < \omega$ we have that $f(x)(n) = a$, where a is the $\prec$-first element in A_n satisfying $x \in g(n,a)$. To complete the proof we will show that this mapping is injective.

So fix $x \neq y$. Then we may find $n < \omega$ such that $y \notin g^2(n,x)$. Since we are assuming that g is symmetric, the latter formula is equivalent to $g(n,x) \cap g(n,y) = \emptyset$. Now let $p = f(x)(n)$. Then $x \in g(n,p)$, and then also $p \in g(n,x)$. But, this means that $p \notin g(n,y)$ and therefore $y \notin g(n,p)$. This implies that $p \neq f(y)(n)$. This shows that f is injective and we are done. □

Recall that a space X has a G_δ-diagonal of rank 2 provided that there is a sequence of open covers $\{\mathcal{U}_n : n < \omega\}$ satisfying $\bigcap St^2(x, \mathcal{U}_n) = \{x\}$ for each $x \in X$. From Proposition 11 the authors of [13] derived the following:

Corollary 13. *([13] [Corollary 3.15]) If X is a weakly linearly Lindelöf Baire space with a G_δ-diagonal of rank 2 such that $we(X) \leq \omega_1$, then $|X| \leq \mathfrak{c}$.*

As before, this result has a better formulation, already established in [4].

Corollary 14. *([4] [Proposition 4.3]) If X is a space with a G_δ-diagonal of rank 2 and $we(X) \le \mathfrak{c}$, then $|X| \le \mathfrak{c}$.*

Proposition 4.5 in [4] establishes that the cardinality of a weakly Lindelöf Baire space with a G_δ-diagonal of rank 2 does not exceed $\mathfrak{c}$, while it is still an open problem if a similar result continues to hold without the Baire assumption. In this direction, Questions 5.1 and 5.2 in [13] appear very interesting. Indeed, these questions ask if the cardinality of a weakly linearly Lindelöf (Baire) space with a G_δ-diagonal of rank 2 is bounded by $\mathfrak{c}$. This happens for sure within the class of normal spaces.

Theorem 15. *If X is a weakly linearly Lindelöf normal space with a G_δ-diagonal of rank 2, then $|X| \le \mathfrak{c}$.*

Proof. To obtain a contradiction, assume that $|X| > \mathfrak{c}$ and let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers witnessing a G_δ-diagonal of rank 2. For each $n < \omega$ let $F_n = \{\{x, y\} : x, y \in X, y \notin St^2(x, \mathcal{U}_n)\}$. We have $[X]^2 = \bigcup\{F_n : n < \omega\}$ and so by Erdos-Rado theorem there is some $n_0 \in \omega$ and an uncountable set $H \subseteq X$ such that $[H]^2 \subseteq F_{n_0}$. The collection $\{St(x, \mathcal{U}_{n_0}) : x \in H\}$ consists of pairwise disjoint sets and H is closed in X. By the normality of X there is an open set V satisfying $H \subseteq V$ and $\overline{V} \subseteq \bigcup\{St(x, \mathcal{U}_{n_0}) : x \in H\}$. But then, the uncountable family of open sets $\{V \cap St(x, \mathcal{U}_{n_0}) : x \in H\}$ has no complete accumulation point, contradicting the weak linear Lindelöfness of X. □

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