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A performance measure analysis for traffic networks with random data and general monotone cost functions

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ABSTRACT
We consider a congested traffic network where users behave according to the Wardrop equilibrium principle, but the data are uncertain and only known through their probability distributions. Within this framework, we propose a stochastic equilibrium model to analyse the network performance, which allows for nonlinear cost functions. The effectiveness of our approach is shown through numerical experiments on medium-size networks.

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1. Introduction

A systematic investigation of efficiency and vulnerability of transportation networks only started in the last 15 years and mainly from the topological point of view. In this regards, an interesting approach is the one considered in [1,2] where the authors proposed an efficiency measure for networks which has become very popular among physicists and social scientists. Their measure, combined with other topological measures, has been recently applied to assess the current and future performance of Shanghai urban train transit network [3]. However, a more detailed investigation of network vulnerability and efficiency must take into consideration congestion effects, i.e. requires models which include the analysis of flow distributions. This task has been carried out in the influential papers [4,5] where the authors consider transportation networks where flows are regulated by a central authority. On the other hand, the vulnerability analysis put forward in [6,7] deals with the case where no central authority controls the traffic flows and the users behave according to Wardrop equilibrium principle. A stochastic approach to analyse the efficiency of congested networks with linear costs and uncertain traffic demands has been proposed in [8], where the authors combined the concepts introduced in [6,7] with the theory of stochastic variational inequalities developed in the last decade [9–12].

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In this paper, we extend and improve the model considered in [8] to study the performance and the vulnerability of congested networks where the relevant data of the problem, regarding both costs and demands, are supposed to be uncertain and known through their probability distributions. Our contribution is twofold: from the theoretical point of view, we investigate the network performance and vulnerability by allowing for the possibility that cost functions be nonlinear; the network vulnerability is analysed by means of a measure of the importance of its arcs. From the numerical point of view, we perform experiments on medium-size networks with several origin-destination pairs instead of considering a simple test problem with only one origin-destination pair as in [8].

The paper is structured as follows. In Section 2, we describe the traffic equilibrium problem from the user viewpoint and define the measures of network performance and importance of arcs that will be used throughout the paper. In Section 3, we shortly recall the stochastic variational inequality theory and its application to the traffic equilibrium problem with uncertain data. In Section 4, we carry out the stochastic analysis of network performance and vulnerability in the general case of nonlinear cost functions and prove an approximation theorem which is useful for the numerical computation of the mean values of the considered measures. In Section 5, we first describe in detail the implementation of the numerical approximation procedure for random traffic equilibria. Then, we apply our methodology to three medium-size test networks and show the impact of different probability densities of the random variables on the mean values of the approximated solutions. We also illustrate a discretization strategy that allows us to save CPU time and investigate the scalability of our approach. The main results and possible further developments are shortly discussed in the concluding section. We also provide an appendix in order to explain the numerical approximation sketched in Section 3 and make the paper, to a certain extent, self-consistent.

2. Efficiency measures for traffic networks under the user equilibrium regime

For a comprehensive treatment of all the mathematical aspects of the traffic equilibrium problem, we refer the interested reader to the excellent book of Patriksson [13]. Here, we focus on the basic definitions and on the variational inequality formulation of a network equilibrium flow. In what follows, we denote with $a^\top b$ the scalar product between two vectors and with $A^\top$ the transpose of a given matrix A. A traffic network consists of a triple $G = (N, A, W)$, where $N = \{N_1, \dots, N_p\}$ is the set of nodes, $A = \{A_1, \dots, A_n\}$ represents the set of direct arcs (also called links) connecting pairs of nodes and $W = \{w_1, \dots, w_m\} \subseteq N \times N$ is the set of the origin-destination (OD) pairs. The flow on the arc A_i is denoted by f_i, and we group all the arc flows in a vector $f = (f_1, \dots, f_n)$. For the sake of simplicity, we consider arcs with infinite capacities. A path (or route) is defined as a set of consecutive arcs and we assume that each OD pair w_j is connected by r_j paths whose set is denoted by $P_j, j = 1, \ldots, m$. All the paths in the network are grouped in a vector $(R_1, \ldots, R_k)$. We can describe the arc structure of the paths by using the arc-path incidence matrix $\Delta = (\delta_{ir})$, $i = 1, \ldots, n$ and $r = 1, \ldots, k$, with entries

$$\delta_{ir} = \begin{cases} 1 & \text{if } A_i \in R_r, \\ 0 & \text{if } A_i \notin R_r. \end{cases} \tag{1}$$

To each path R_r it is associated a flow F_r. The path flows are grouped into a vector $(F_1, \ldots, F_k)$ which is called the network path-flow (or simply, the network flow if it is clear that we refer to paths). The flow f_i on the arc A_i is equal to the sum of the path flows on the paths which contain A_i, so that $f = \Delta F$. We now introduce the unit cost of going through A_i as a real function $c_i(f) \geq 0$ of the flows on the network, so that $c(f) = (c_1(f), \ldots, c_n(f))$ denotes the arc cost vector on the network. The meaning of the cost is usually that of travel time and, in the simplest case, the generic component c_i only depends on f_i. Analogously, one can define a cost on the paths as $C(F) = (C_1(F), \ldots, C_k(F))$. Usually, $C_r(F)$ is just the sum of the costs on the arcs which build that path:

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f),$$

or in compact form,

$$C(F) = \Delta^\top c(\Delta F). \tag{2}$$

For each pair w_j, there is a given traffic demand $D_{w_j} = D_j \geq 0$, so that $D = (D_1, \ldots, D_m)$ is the demand vector of the network. Feasible path flows are nonnegative and satisfy the demands, i.e. belong to the set

$$K = \{F \in \mathbb{R}^k : \ F \geq 0, \ \Phi F = D\}, \tag{3}$$

where Φ is the pair-path incidence matrix whose entries, for $j = 1, \ldots, m$ and $r = 1, \ldots, k$ are

$$\varphi_{jr} = \begin{cases} 1 & \text{if the path } R_r \text{ connects the pair } w_j, \\ 0 & \text{elsewhere.} \end{cases} \tag{4}$$

The notion of a user traffic equilibrium is given by the following definition.

Definition 2.1: A network flow $H \in \mathbb{R}^k$ is a user equilibrium if for each OD pair w_j, and for each pair of paths R_r, R_s which connect w_j

$$C_r(H) > C_s(H) \Longrightarrow H_r = 0,$$

that is, if travelling along the path R_r takes more time than travelling R_s, the flow along R_r vanishes.

Remark 2.1: Among the various paths which connect a given OD pair w_j, some will carry a positive flow and others zero flow. It follows from the previous definition that, for a given OD pair, the travel cost is the same for all nonzero flow paths, otherwise users would choose a path with a lower cost. Hence, as an equivalent definition of Wardrop equilibrium we can write that for each OD pair w_j one has

$$C_r(H) \begin{cases} = \lambda_j & \text{if } H_r > 0, \\ \geq \lambda_j & \text{if } H_r = 0. \end{cases} \tag{5}$$

Hence, with the notation λ_j we denote the equilibrium cost shared by all the used paths connecting w_j. The (heaviest) notation λ_w will also be used when we want to stress that we are considering a property depending on the OD pair w only. The variational inequality formulation of the user equilibrium is given by the following theorem (see, e.g. [13]).

Theorem 2.2: *A network flow vector $H \in K$ is a user equilibrium iff it satisfies the variational inequality*

$$C(H)^\top (F - H) \geq 0 \quad \forall F \in K. \tag{6}$$

Sometimes it is useful to decompose the scalar product in (6) according to the various OD pairs:

$$\sum_{w \in W} \sum_{r \in P_w} C_r(H)(F_r - H_r) \geq 0 \quad \forall F \in K.$$

The network efficiency measure put forward in [6] is as follows. For a given network topology G and a given traffic demand D, the performance (or efficiency) of G is measured by

$$\mathcal{E}^G = \frac{1}{m} \sum_{w \in W} \frac{D_w}{\lambda_w}, \tag{7}$$

where m is the total number of OD pairs in the network and λ_w is the equilibrium cost for the OD pair w, see (5). Hence, each term in the sum (7) is the ratio between the traffic demand of a single OD pair and the corresponding equilibrium cost; the overall performance of the network is defined as the average of these quantities. Now, let g be a component of the network (i.e. a node or a link). The importance of g is measured through the relative variation of efficiency after g is removed from the network:

$$\mathcal{I}^g = \frac{\mathcal{E}^G - \mathcal{E}^{G-g}}{\mathcal{E}^G}. \tag{8}$$

Note that (8) can be negative if the efficiency of the network increases after removing the component g. This counterintuitive situation can actually occur due to the so-called Braess' paradox [14,15] which is analysed in detail in [8] in the case where the traffic demand is random. We generalize the above definitions of performance and importance by using the theory of stochastic variational inequalities, which we briefly recall in the following section.

3. Methodology

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $A, B : \mathbb{R}^k \to \mathbb{R}^k$ two given mappings, and $b, c \in \mathbb{R}^k$ two given vectors. Moreover, let R and S be two real-valued random variables defined on Ω, D a random vector in $\mathbb{R}^m$, and $G \in \mathbb{R}^{m\times k}$ a given matrix. For $\omega \in \Omega$, we define a random set $M(\omega) := \{x \in \mathbb{R}^k : \ Gx \le D(\omega)\}$. Consider the following stochastic variational inequality: for almost every $\omega \in \Omega$, find $\hat{x} := \hat{x}(\omega) \in M(\omega)$ such that

$$(S(\omega)A(\hat{x}) + B(\hat{x}))^\top (z - \hat{x}) \ge (R(\omega)c + b)^\top (z - \hat{x}) \quad \forall\, z \in M(\omega). \tag{9}$$

To facilitate the foregoing discussion, we set $T(\omega, x) := S(\omega)A(x) + B(x)$. We assume that A, B and S are such that the map $T : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ is a Carathéodory function. We also assume that $T(\omega, \cdot)$ is monotone for every $\omega \in \Omega$, i.e.

$$(T(\omega, x) - T(\omega, y))^\top (x - y) \ge 0 \quad \forall\, x, y \in \mathbb{R}^k,\ \forall\, \omega \in \Omega,$$

and if equality only holds for $x = y$ we say that T is strictly monotone. Since we are only interested in solutions with finite first- and second-order moments, our approach is to consider an integral variational inequality instead of the parametric variational inequality (9).

Thus, for a fixed $p \ge 2$, consider the Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to $\mathbb{R}^k$ such that the expectation (p-moment) is given by $E^P(\|V\|^p) = \int_\Omega \|V(\omega)\|^p \,\mathrm{d}P(\omega) < \infty$. For subsequent developments, we need the following growth condition:

$$\|T(\omega, z)\| \le \alpha(\omega) + \beta(\omega)\|z\|^{p-1} \quad \forall\, z \in \mathbb{R}^k, \tag{10}$$

where $\alpha \in L^q(\Omega, P)$ and $\beta \in L^\infty(\Omega, P)$. Due to the above growth condition, the Nemytskii operator $\hat{T}$ associated to T, acts from $L^p(\Omega, P, \mathbb{R}^k)$ to $L^q(\Omega, P, \mathbb{R}^k)$, where $p^{-1} + q^{-1} = 1$, and is defined by $\hat{T}(V)(\omega) := T(\omega, V(\omega))$, for any $\omega \in \Omega$. Assuming $D \in L^p_m(\Omega) := L^p(\Omega, P, \mathbb{R}^m)$, we introduce the following nonempty, closed and convex subset of $L^p_k(\Omega)$:

$$M^P := \left\{ V \in L^p_k(\Omega) : \ GV(\omega) \le D(\omega),\ P\text{–a.s.} \right\}.$$

Let $S(\omega) \in L^\infty$, $0 < \underline{s} < S(\omega) < \bar{s}$, and $R(\omega) \in L^q$. Equipped with these notations, we consider the following L^p formulation of (9). Find $\hat{U} \in M^P$ such that for every $V \in M^P$, we have

$$\int_\Omega (S(\omega)\, A[\hat{U}(\omega)] + B[\hat{U}(\omega)])^\top (V(\omega) - \hat{U}(\omega))\, dP(\omega)$$
$$\geq \int_\Omega (b + R(\omega)\, c)^\top (V(\omega) - \hat{U}(\omega))\, \mathrm{d}P(\omega). \tag{11}$$

To get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, D) and work with the special probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where $d{:}=2{+}m$ and $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^d$. For simplicity, we assume that R, S, and D are independent random vectors. We set

$$r = R(\omega), \quad s = S(\omega), \quad t = D(\omega), \quad y = (r, s, t).$$

For each $y \in \mathbb{R}^d$, we define the set $M(y) := \{x \in \mathbb{R}^k : \ Gx \leq t\}$. Consider the space $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ and introduce the closed and convex set

$$M_{\mathbb{P}} := \left\{ v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : \ Gv(r, s, t) \leq t, \ \mathbb{P}\text{−a.s.} \right\}.$$

Without any loss of generality, we assume that $R \in L^q(\Omega, P)$ and $D \in L^p(\Omega, P, \mathbb{R}^m)$ are nonnegative. Moreover, we assume that the support (i.e. the set of possible outcomes) of $S \in L^\infty(\Omega, P)$ is the interval $[\underline{s}, \bar{s}[\subset (0, \infty)$. With these ingredients, we consider the variational inequality problem of finding $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$ we have

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (s\, A[\hat{u}(y)] + B[\hat{u}(y)])^\top (v(y) - \hat{u}(y))\, \mathrm{d}\mathbb{P}(y)$$
$$\geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (b + rc)^\top (v(y) - \hat{u}(y))\, \mathrm{d}\mathbb{P}(y). \tag{12}$$

Details on the numerical approximation of the solution $\hat{u}$ can be found in the appendix, but to allow the reader to understand the subsequent developments without stopping on technicalities, we recall here the main steps:

- the set $M_{\mathbb{P}}$ can be approximated by a sequence $\{M^n_{\mathbb{P}}\}$ of finite-dimensional sets;
- r and s can be approximated by the sequences $\{\rho_n\}$ and $\{\sigma_n\}$ of step functions, with $\rho_n \to \rho$ in L^p and $\sigma_n \to \sigma$ in L^∞, respectively, where $\rho(r, s, t) = r$ and $\sigma(r, s, t) = s$;
- when the solution of (12) is unique, we can compute a sequence of step functions $\{\hat{u}_n\}$ which converges strongly to $\hat{u}$, under suitable hypotheses (see Theorem A.1), by solving for each $n \in \mathbb{N}$ the following discretized variational

inequality: find $\hat{u}_n := \hat{u}_n(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (\sigma_n(y)\, A[\hat{u}_n(y)] + B[\hat{u}_n(y)])^\top (v_n(y) - \hat{u}_n(y))\, \mathrm{d}\mathbb{P}(y)$$
$$\geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (b + \rho_n(y)\, c)^\top (v_n(y) - \hat{u}_n(y))\, \mathrm{d}\mathbb{P}(y). \tag{13}$$

In the absence of strict monotonicity, the solution of (11) and (12) can be not unique and the previous approximation procedure must be coupled with a regularization scheme as follows. We choose a sequence $\{\varepsilon_n\}$ of regularization parameters and choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) \to L^q(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ (see (A3)). We assume that $\varepsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\varepsilon_n \downarrow 0$ as $n \to \infty$.

We can then consider the following regularized stochastic variational inequality: for any $n \in \mathbb{N}$, find $w_n = w_n^{\varepsilon_n}(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \big(\sigma_n(y) A[w_n(y)] + B[w_n(y)] + \varepsilon_n J(w_n(y))\big)^\top (v_n(y) - w_n(y))\, \mathrm{d}\mathbb{P}(y)$$
$$\geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (b + \rho_n(y)\, c)^\top (v_n(y) - w_n(y))\, \mathrm{d}\mathbb{P}(y). \tag{14}$$

As usual, the solution w_n will be referred to as the regularized solution. Weak and strong convergence of w_n to the minimal-norm solution of (12) can be proved under suitable hypotheses (see, e.g. Theorems A.2 and A.4).

In traffic network equilibrium problems, the demand and the cost are often modelled as random variables. In our model, we assume that the main source of uncertainty comes from the demand, but to allow for possible different applications, we consider in this section the general case of random demand and cost. The uncertainties or random fluctuations in the traffic demand, and in the cost functions lead us to consider the stochastic variational inequality model of a traffic equilibrium problem. Thus, let Ω be a sample space and P be a probability measure on Ω, and consider the following feasible set which takes into consideration random fluctuations of the demand:

$$K(\omega) = \{F \in \mathbb{R}^k : \ F \geq 0, \ \Phi F = D(\omega)\}, \quad \omega \in \Omega.$$

Moreover, let $C : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ be the random cost function. We can thus introduce ω as a random parameter in (6) and consider the problem of finding a vector $H(\omega) \in K(\omega)$ such that, $P -$ a.s:

$$C(\omega, H(\omega))^\top (F - H(\omega)) \geq 0 \quad \forall\, F \in K(\omega). \tag{15}$$

Definition 3.1: A random vector $H \in K(\omega)$ is a *random Wardrop equilibrium* if for P-almost every $\omega \in \Omega$, for each OD pair w_j and for each pair of paths R_r, R_s which connect w_j, we get

$$C_r(\omega, (H(\omega)) > C_s(\omega, (H(\omega))) \Longrightarrow H_r(\omega) = 0.$$

Let $D \in L^p(\Omega, P, \mathbb{R}^m)$ and consider then the set

$$K_P = \{F \in L^p(\Omega, P, \mathbb{R}^k) : F_r(\omega) \geq 0,\ P\text{−a.s.},\ \forall\, r = 1, \ldots, k,$$
$$\Phi F(\omega) = D(\omega),\ P\text{−a.s.}\},$$

which is convex, closed and bounded, hence weakly compact. Furthermore, assume that the cost function C satisfies the growth condition:

$$\|C(\omega, z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1} \quad \forall\, z \in \mathbb{R}^k,\ P\text{−a.s.},$$

for some $\alpha \in L^q(\Omega, P), \beta \in L^\infty(\Omega, P)$, and $p^{-1} + q^{-1} = 1$. The Carathéodory function C gives rise to a Nemytskii map $\hat{C} : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k)$ defined through the usual position $\hat{C}(F)(\omega) = C(\omega, F((\omega))$, and, with abuse of a notation, instead of $\hat{C}$, the same symbol C is often used for both the Carathéodory function and the Nemytskii map that it induces. We thus consider the following integral variational inequality: find $H \in K_P$ such that

$$\int_\Omega C(\omega, H(\omega))^\top (F - H(\omega))\, \mathrm{d}P(\omega) \geq 0 \quad \forall\, F \in K_P. \tag{16}$$

A solution of (16) satisfies the random Wardrop conditions in the sense shown by the following lemma (see [8] for the proof).

Lemma 3.2: *If $H \in K_P$ is a solution of (16), then H is a random Wardrop equilibrium.*

As a consequence of the previous lemma, we get that there exists a vector function $\lambda \in L^p(\Omega, P, \mathbb{R}^m)$ such that

$$C_r(\omega, H(\omega)) = \lambda_j(\omega) = \lambda_{w_j} \tag{17}$$

for all paths R_r which connect w_j, with $H_r(\omega) > 0$, P-almost surely. We assume that the operator is the sum of a purely deterministic term and a random term where randomness act as a modulation:

$$C(\omega, H(\omega)) = S(\omega)A[H(\omega)] + B[H(\omega)] - b - R(\omega)c,$$

where $S \in L^\infty(\Omega, P), R \in L^q(\Omega),\ A, B : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k), b, c \in \mathbb{R}^k$. The integral variational inequality now reads: find $H \in K_P$ such that, for all

$F \in K_P$, we have

$$\int_\Omega (S(\omega)(A[H(\omega)])^\top + (B[H(\omega)])^\top)(F - H(\omega))\,\mathrm{d}P(\omega)$$
$$\geq \int_\Omega (b^\top + R(\omega)c^\top)(F - H(\omega))\,\mathrm{d}P(\omega). \tag{18}$$

4. Definitions and approximate computation of the mean values of the efficiency indices of the network

Let us now assume that the traffic demand between the origins and destinations be a random function $D : \Omega \to \mathbb{R}^m$, and $\hat{C} : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k)$ be the cost operator. As usual, we denote by P the probability measure on Ω, while E_P is the expectation (or mean value) with respect to the probability P. We consider the following definitions:

(1) The *average cost at equilibrium* is defined as

$$E_P[\lambda^G(\omega)] = \int_\Omega \lambda^G(\omega)\,\mathrm{d}P(\omega), \tag{19}$$

where $\lambda^G(\omega) = (\lambda_1^G(\omega), \dots, \lambda_m^G(\omega))$ is defined as in (17).

(2) The *average performance* of the network is defined as

$$E_P[\mathcal{E}^G(\omega)] = \frac{1}{m}\sum_{w\in W}\int_\Omega \frac{D_w(\omega)}{\lambda_w^G(\omega)}\,\mathrm{d}P(\omega). \tag{20}$$

(3) We define the *average importance* of an arc l in the network (see (8)) as

$$E_P[\mathcal{I}^l(\omega)] = \int_\Omega \frac{\mathcal{E}^G(\omega) - \mathcal{E}^{G-l}(\omega)}{\mathcal{E}^G(\omega)}\,\mathrm{d}P(\omega). \tag{21}$$

Remark 4.1: Let us note that the integral in (19) is different from zero under the natural assumption that in each path R_r there is a link where the cost is bounded from below by a positive number (uniformly in $\omega \in \Omega$). This hypothesis is fulfilled in real networks because the cost is positive for positive flows, but also the cost at zero flow (called the free flow time) is positive, because it represents the travel time without congestion. We also assume that $0 < \alpha \leq D_j(\omega) \leq \beta$ holds P-a.s.. The integrals in (20) and (21) are thus finite. We shall make these two blanket assumptions throughout this section.

As explained in Section 3, the random variable $t = D(\omega)$ and the two random variables $r = R(\omega)$, $s = S(\omega)$ generate a probability $\mathbb{P}$ in the image space $\mathbb{R}^{2+m}$ of (r, s, t) from the probability P on the abstract sample space Ω. Hence, we can express the earlier defined quantities in terms of the image space variables, thus obtaining functions which can be approximated through a discretization procedure. The integration now runs over the image space variables, but to keep notation simple we just write $\int$ instead of $\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+}$. The transformed expressions read as follows:

$$E_{\mathbb{P}}[\lambda^G(r,s,t)] = \int \lambda^G(r,s,t)\, \mathrm{d}\mathbb{P}(r,s,t), \tag{22}$$

$$E_{\mathbb{P}}[\mathcal{E}^G(r,s,t)] = \frac{1}{m} \sum_{w \in W} \int \frac{t_w}{\lambda^G_w(r,s,t)}\, \mathrm{d}\mathbb{P}(r,s,t), \tag{23}$$

$$E_{\mathbb{P}}[(\mathcal{I}^l(r,s,t)] = \int \frac{\mathcal{E}^G(r,s,t) - \mathcal{E}^{G-l}(r,s,t)}{\mathcal{E}^G(r,s,t)} \mathrm{d}\mathbb{P}(r,s,t). \tag{24}$$

Let us recall that the solution $H(r,s,t)$ of the stochastic variational inequality which describes the network equilibrium can be approximated by a sequence $\{H^n\}$ of step functions such that $H^n \to H$ in L^p. In the theorem that follows, we give converging approximations for the mean values defined previously.

Theorem 4.1: *Let $\lambda^{G,n}_w(r,s,t) = C^G_i[r,s,t,H^n(r,s,t)]$, where $H_i(r,s,t) > 0$, $\mathbb{P}$-a.s. for all paths R_i which connect w, and, for $t = (t_1, \dots, t_m)$, let $\{T_n\}$ be any sequence of L^p functions such that $T_n \to t$ in L^p. Moreover, assume that there exists $a > 0$ such that $C^G_i(r,s,t,F) > a$, for each i, and $\mathbb{P}$-a.s. and that there exist $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha \le t_w = D_w(\omega) \le \beta$, for every OD pair $w \in W$. We then have*

(1) *The sequence*

$$\{E_{\mathbb{P}}[\lambda^{G,n}(r,s,t)]\}_n = \left\{ \int \lambda^{G,n}(r,s,t)\, \mathrm{d}\mathbb{P}(r,s,t) \right\}_n$$

converges to $E_{\mathbb{P}}[\lambda^G(r,s,t)]$.

(2) *The sequence*

$$\{E_{\mathbb{P}}[\mathcal{E}^{G,n}(r,s,t)]\}_n = \left\{ \frac{1}{m} \sum_{w \in W} \int \frac{T_{w,n}}{\lambda^{G,n}_w(r,s,t)}\, \mathrm{d}\mathbb{P}(r,s,t) \right\}_n$$

converges to $E_{\mathbb{P}}[\mathcal{E}^G(r,s,t)]$.

(3) *The sequence*

$$\{E_{\mathbb{P}}[\mathcal{I}^{l,n}(r,s,t)]\}_n = \left\{ \int \frac{\mathcal{E}^{G,n}(r,s,t) - \mathcal{E}^{G-l,n}(r,s,t)}{\mathcal{E}^{G,n}(r,s,t)}\, \mathrm{d}\mathbb{P}(r,s,t) \right\}_n$$

converges to $E_{\mathbb{P}}[\mathcal{I}^l(r,s,t)]$.

Proof: (1) Since $H^n \to H$ strongly in L^p, it follows that $A[H^n] \to A[H]$ and $B[H^n] \to B[H]$, strongly in $L^q = L^{\frac{p}{p-1}}$ because of the continuity of the Nemytskii operators A and B. Moreover, $\rho_n \to \rho$ strongly in L^q and $\sigma_n \to \sigma$ strongly in L^∞. As a consequence,

$$\sigma_n A[H^n] + B[H^n] - b - \rho_n c \to \sigma A[H] + B[H] - b - \rho c$$

strongly in L^q, and also strongly in L^1 because $\mathbb{P}$ is a probability measure. Hence, for each $i = 1, \dots, k$, we get $C_i^n[\rho_n, \sigma_n, H^n] \to C_i[r, s, H]$ strongly in L^1 and, by the definitions of λ and λ^n, the thesis is proved.

(2) We prove convergence of each summand. We have

$$\begin{aligned}
\int \left| \frac{T_{w,n}}{\lambda_w^{G,n}} - \frac{t_w}{\lambda_w^G} \right| \mathrm{d}\mathbb{P}(r,s,t) &\leq \frac{1}{a^2} \int |T_{w,n}\lambda_w^G - t_w \lambda_w^{G,n}| \, \mathrm{d}\mathbb{P}(r,s,t) \\
&\leq \frac{1}{a^2} \int |T_{w,n}| \, |\lambda^G - \lambda_w^{G,n}| \, \mathrm{d}\mathbb{P}(r,s,t) \\
&\quad + \frac{1}{a^2} \int |\lambda_w^{G,n}| \, |T_{w,n} - t_w| \, \mathrm{d}\mathbb{P}(r,s,t) \\
&\leq c_1 \|\lambda_w^G - \lambda_w^{G,n}\|_{L^q} + c_2 \|T_{w,n} - t_w\|_{L^p} \longrightarrow 0
\end{aligned}$$

as $n \to \infty$.

(3) Finally,

$$\begin{aligned}
|E_{\mathbb{P}}[\mathcal{I}^l(r,s,t)] - E_{\mathbb{P}}[\mathcal{I}^{l,n}(r,s,t)]| &\leq \int \left| \frac{\mathcal{E}^{G-l,n}}{\mathcal{E}^{G,n}} - \frac{\mathcal{E}^{G-l}}{\mathcal{E}^G} \right| \mathrm{d}\mathbb{P}(r,s,t) \\
&\leq \int \left| \frac{\mathcal{E}^{G-l}(\mathcal{E}^{G,n} - \mathcal{E}^G)}{\mathcal{E}^G \mathcal{E}^{G,n}} \right| \mathrm{d}\mathbb{P}(r,s,t) \\
&\quad + \int \left| \frac{\mathcal{E}^G(\mathcal{E}^{G-l,n} - \mathcal{E}^{G-l})}{\mathcal{E}^G \mathcal{E}^{G,n}} \right| \mathrm{d}\mathbb{P}(r,s,t) \\
&\leq k_1 \|\mathcal{E}^{G,n} - \mathcal{E}^G\|_{L^1} + k_2 \|\mathcal{E}^{G-l,n} - \mathcal{E}^{G-l}\|_{L^1}
\end{aligned}$$

and the last expression vanishes when $n \to \infty$ because of the convergence proved in the previous point. ■

5. Numerical experiments

We now report some numerical results for the network efficiency indices defined in Section 4. In what follows, the traffic demand is randomly distributed as $D = d + \delta e$, where d and e are deterministic vectors in $\mathbb{R}^m$ and δ is a random variable with support in the interval $[a, b]$, distributed according to a probability measure

P. Moreover, the link cost functions are supposed to be exactly known and of the BPR form [16]:

$$c_i(f_i) = t_i^0 \left[1 + 0.15 \left(\frac{f_i}{u_i} \right)^\beta \right], \tag{25}$$

where t_i^0 and u_i represent the free flow travel time and the capacity of link i, respectively, and $\beta > 0$ is a network parameter. Hence, the average cost at equilibrium (19), the average performance of the network (20) and the average importance of arcs (21) depend only on the random vector $t = D(\omega)$.

The numerical computation of random Wardrop equilibria has been implemented in Matlab 2020a and tested on an Intel Core i7 system at 2.5 GHz with 16 GB of RAM running under macOS 10.15.

The rest of this section is organized as follows. Section 5.1 describes in detail the implementation of the approximation procedure reported in the appendix. Section 5.2 shows the convergence of the approximated mean values of the three efficiency measures on three test networks. Section 5.3 shows that a non-uniform discretization may improve the convergence rate of the approximated mean values. Finally, Section 5.4 reports the scalability of the approximation procedure for medium-large test networks.

5.1. Implementation of the numerical approximation procedure

We describe how the approximation procedure for the solution of stochastic variational inequalities, reported in the appendix, is applied to the case of our traffic problems with random demand.

First, transform the link cost function c (see (25)) to the path cost function C according to (2). Then, consider a partition of $[a, b]$ into N subintervals according to: $a = \delta_0^N < \delta_1^N < \cdots < \delta_N^N = b$, and let $I_j^N = [\delta_{j-1}^N, \delta_j^N[$, $j = 1, \dots, N$. We recall that if the map C is strongly monotone, then the random Wardrop equilibrium that solves the integral variational inequality (16) can be approximated by a sequence of step functions $\{u^N\}$, where each u^N is the solution of the discretized variational inequality (13) that can be split in N finite-dimensional variational inequalities. In order to derive such variational inequalities for each index $j = 1, \dots, N$, we must first specify the corresponding feasible sets. For this, for any $j = 1, \dots, N$, we define the vector

$$\bar{q}_j^N = \frac{1}{P(I_j^N)} \int_{\delta_{j-1}^N}^{\delta_j^N} [d + \delta e] \, \mathrm{d}P,$$

that represents the mean value of the traffic demand in the interval I_j^N, and the set

$$K_j^N = \{v_j \in \mathbb{R}^k : \ v_j \geq 0, \ \Phi v_j = \bar{q}_j^N\}, \tag{26}$$

where the matrix Φ is defined in (4), that represents the set of path flows satisfying the demand $\bar{q}_j^N$. We can now write the finite-dimensional variational inequality for each j as: find $u_j^N \in K_j^N$ such that

$$[C(u_j^N)]^\top (v_j - u_j^N) \geq 0 \quad \forall\, v_j \in K_j^N. \tag{27}$$

The step function u^N approximating the random Wardrop equilibrium is then given by

$$u^N = \sum_{j=1}^{N} u_j^N \mathbf{1}_{I_j^N}, \tag{28}$$

where $\mathbf{1}_A$ is the characteristic function of a set A. Notice that the solution u_j^N of (27) is the deterministic Wardrop equilibrium on the network where $\bar{q}_j^N$ is the traffic demand vector. In the following numerical experiments, we used the algorithm designed in [17] to approximate the deterministic Wardrop equilibria.

When the path cost operator is monotone but not strictly monotone, we need to apply the regularization procedure described in the appendix. Remember that the exponent p in L^p is fixed by the growth condition (10), that is, in our traffic application, we have $p = \beta + 1$, where β is the degree of the polynomial cost of BPR type (25).

In the case $\beta = 1$ (linear cost functions), we get $p = 2$, thus the duality map J is the identity and it is sufficient to add to the cost term C in (27) the term, where I is the $k \times k$ identity matrix. Hence, the approximating step function u^N is of the form (28), where $u_j^N \in K_j^N$ solves the finite-dimensional variational inequality

$$[C(u_j^N) + \varepsilon_N u_j^N]^\top (v_j - u_j^N) \geq 0 \quad \forall\, v_j \in K_j^N. \tag{29}$$

We note that also the solution of (29) can be interpreted as the deterministic Wardrop equilibrium on the network where the traffic demand is $\bar{q}_j^N$ and the path cost operator is modified according to the regularization term.

In the case $\beta > 1$ (nonlinear cost functions), we get $p > 2$, thus the duality map J is different from the identity map (see (A3)) and the regularized variational inequality (14) cannot be split into N finite-dimensional variational inequalities. For the subsequent development it is useful to notice that for a step function $v^N = \sum_{j=1}^N v_j^N \mathbf{1}_{I_j^N}$, where $v_j^N \in \mathbb{R}^k$, we have

$$\|v^N\|_{L^p} = \left[\sum_{j=1}^{N} \left(\sqrt{(v_{j1}^N)^2 + \cdots + (v_{jk}^N)^2} \right)^p P(I_j^N) \right]^{1/p}, \tag{30}$$

and define $f(v_1^N, \ldots, v_N^N) := \|v^N\|_{L^p}^{2-p}$. It is important to specify how the elements $v_j^N \in \mathbb{R}^k$ are ordered in a vector $(\tilde{v}_\alpha^N) \in \mathbb{R}^{k \times N}$, but in our example of one random variable this can be done in the simple manner suggested by (30), specifically

$$\tilde{v}^N = (\tilde{v}^N_\alpha) = (v^N_{11}, \dots, v^N_{1k}, v^N_{21}, \dots, v^N_{2k}, \dots, v^N_{N1}, \dots, v^N_{Nk}).$$

The feasible set to consider in this case is

$$K^N = \{\tilde{v}^N \in \mathbb{R}^{k \times N} : v^N_j \in K^N_j \text{ for any } j = 1, \dots, N\}, \tag{31}$$

where K^N_j has been defined in (26). With these ingredients, the regularized variational inequality (14) in our application is equivalent to: find $\tilde{u}^N \in K^N$ such that

$$\sum_{j=1}^{N} [C(u^N_j) + \varepsilon_N f(u^N_1, \dots, u^N_N) \|u^N_j\|_2^{p-2} u^N_j]^\top [v^N_j - u^N_j] \geq 0 \quad \forall \tilde{v}^N \in K^N. \tag{32}$$

We note that the above variational inequality cannot be split into N variational inequalities with dimension k because of the term $f(u^N_1, \dots, u^N_N)$ involving all the sub-vectors of $\tilde{u}^N$. Moreover, the solution of (32) can be interpreted as the deterministic Wardrop equilibrium in a network with N connected components, each of which has the same topology as the original network and the traffic demand vector of the jth component is equal to $\bar{q}^N_j$.

5.2. Convergence of approximated mean values of the network efficiency indices

In this section, we compute the approximated mean values of the considered network efficiency measures on three test networks. In Example 5.1 the path cost operator C is strongly monotone, in Example 5.2 the operator C is monotone and linear, while in Example 5.3 the operator C is monotone and nonlinear.

Example 5.1: We consider the grid network with 36 nodes and 60 arcs shown in Figure 1. The arc cost functions are defined as in (25) with $\beta = 4$ for all the links, while $t^0_i = 1$ and $u_i = 25$ for any $i = 1, \dots, 30$, and $t^0_i = 5$ and $u_i = 50$ for any $i = 31, \dots, 60$. We consider five OD pairs: (1,12), (7,18), (13,24), (19,30), (25,36). We assume that the traffic demand is $D_j = 150 + \delta$, for any $j = 1, \dots, 5$, where δ is a random variable which varies in the interval $[-50, 50]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 5. Let us mention that uniform and truncated normal distributions are widely used in stochastic traffic equilibrium models (see e.g. [18,19]).

Notice that each OD pair is connected by six paths and any arc A_i, with $i = 31, \dots, 60$, belongs to a unique path, thus the arc-path incidence matrix Δ has full column rank and the path cost operator is strongly monotone (see [20, Lemma 1]). The approximation procedure considers a uniform partition of the interval $[-50, 50]$ into N subintervals and solves problems as (27) for each N. Moreover, the regularization procedure is not needed for this instance.

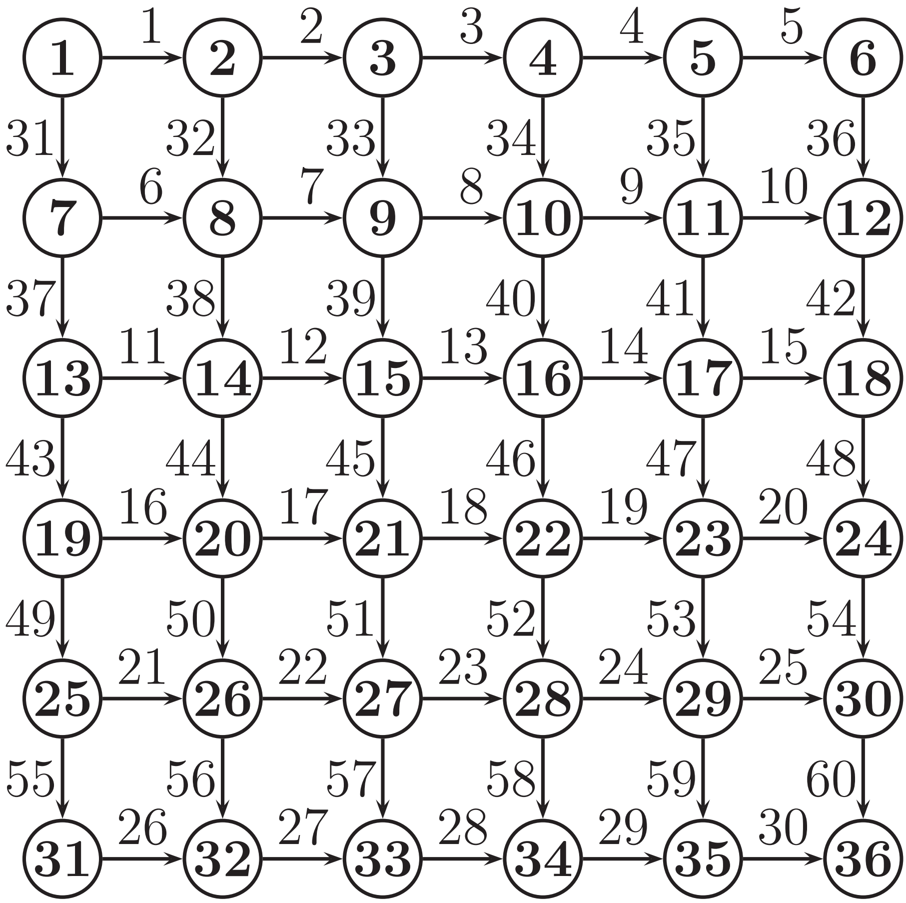

Figure 1. Grid network of Example 5.1.

Table 1 shows the convergence of the mean values of the approximate performance and cost at equilibrium for different values of N when the random variable δ varies in the interval $[-50, 50]$ with uniform distribution. Table 2 shows the convergence of the mean values of the approximate performance and cost at equilibrium when δ varies in the interval $[-50, 50]$ with truncated normal distribution with mean 0 and standard deviation 5. We note that the mean values of the cost at equilibrium decrease by about 16% from uniform to truncated normal distribution, but also the mean value of the network performance decreases by about 19%.

Table 3 reports the ranking and the approximated average importance of the ten most important arcs for two different probability distributions of the random variable δ (uniform and truncated normal). The approximated values of the average importance have been computed by uniformly partitioning the interval $[-50, 50]$ into 100 subintervals. In contrast to performance and cost at

Table 1. The convergence of the approximated mean values of performance and cost at equilibrium in Example 5.1 for $\delta \sim \mathcal{U}(-50, 50)$.

		Avg cost at equilibrium				
N	Avg performance	(1,12)	(7,18)	(13,24)	(19,30)	(25,36)
10	0.3775	590.4129	599.9754	602.6772	599.8602	590.3997
20	0.3782	591.2331	600.8086	603.5153	600.6935	591.2210
50	0.3784	591.4631	601.0429	603.7496	600.9275	591.4499
100	0.3784	591.4958	601.0758	603.7833	600.9606	591.4832
200	0.3785	591.5039	601.0840	603.7916	600.9689	591.4915
300	0.3785	591.5055	601.0858	603.7931	600.9706	591.4928

Table 2. The convergence of the approximated mean values of performance and cost at equilibrium in Example 5.1 for $\delta \sim \mathcal{N}(0, 5)$ on $[-50, 50]$.

		Avg cost at equilibrium				
N	Avg performance	(1,12)	(7,18)	(13,24)	(19,30)	(25,36)
10	0.3076	487.2105	495.0727	497.2941	494.9780	487.1997
20	0.3080	487.7426	495.6136	497.8375	495.5188	487.7318
50	0.3081	487.9447	495.8190	498.0438	495.7241	487.9338
100	0.3081	487.9758	495.8506	498.0756	495.7557	487.9650
200	0.3081	487.9833	495.8580	498.0834	495.7637	487.9733
300	0.3081	487.9849	495.8597	498.0850	495.7652	487.9746

Table 3. Average importance for the 10 most important arcs in Example 5.1 for $\delta \sim \mathcal{U}(-50, 50)$ (on the left) and for $\delta \sim \mathcal{N}(0, 5)$ on $[-50, 50]$ (on the right).

	$\delta \sim \mathcal{U}(-50, 50)$		$\delta \sim \mathcal{N}(0, 5)$ on $[-50, 50]$	
Rank	Arc	Avg importance	Arc	Avg importance
1	1	0.520024	1	0.522308
2	60	0.520013	60	0.522296
3	59	0.449418	59	0.451680
4	3	0.449417	3	0.451678
5	58	0.379124	58	0.381267
6	5	0.379122	5	0.381265
7	14	0.329059	14	0.330633
8	51	0.329057	51	0.330631
9	49	0.326574	49	0.328540
10	16	0.326572	16	0.328539

equilibrium, the average importance of the arcs and the corresponding ranking do not seem to depend significantly on the probability distribution of δ.

Example 5.2: We consider the Sioux Falls network shown in Figure 2 consisting of 24 nodes and 76 links. The link cost functions are of the form (25) with $\beta = 1$ for all the links, while the parameters t_i^0 and u_i are given in [21] (see Sioux Falls 2). We assume that the traffic demand for the 528 OD pairs is $D_j = d_j + \delta$ if $d_j \geq 1100$, and $D_j = d_j$ otherwise, where the deterministic demand d is given in [21] and δ is a random variable which varies in the interval $[-1000, 1000]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 100.

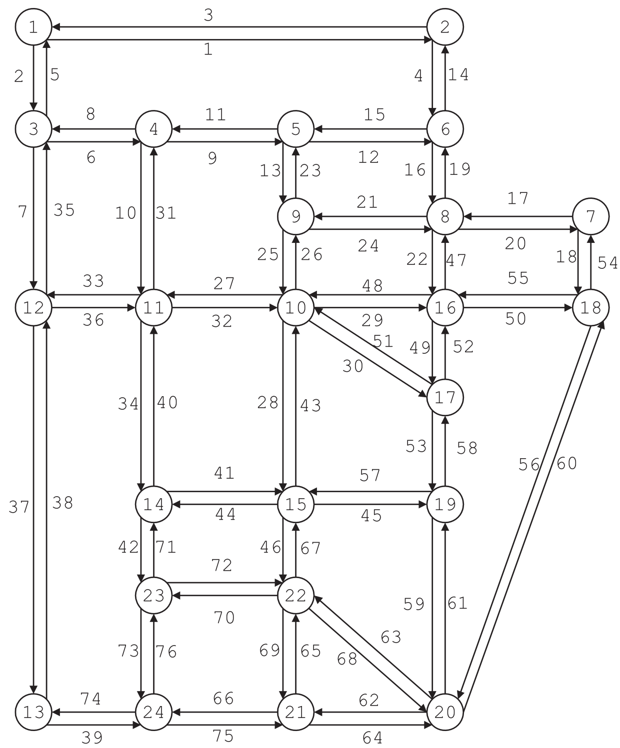

Figure 2. Sioux Falls network of Example 5.2.

Notice that in this case the arc-path incidence matrix Δ has not full column rank and the path cost operator is monotone but not strongly monotone. Hence, the discretization procedure, coupled with the regularization scheme with $p = 2$, solves problems as (29). The interval $[-1000, 1000]$ has been uniformly partitioned into N subintervals in the approximation procedure and the regularization parameter ε_N has been chosen equal to $1/N^2$.

Table 4 shows the convergence of the mean values of the approximate performance and cost at equilibrium of five selected OD pairs: (4,11), (10,13), (14,15), (16,22) and (20,17), for different values of N when the random variable δ varies in the interval $[-1000, 1000]$ with uniform distribution. Table 5 shows the convergence of the mean values of the approximate performance and cost at equilibrium when δ varies in $[-1000, 1000]$ with truncated normal distribution with mean 0 and standard deviation 100. We note that the probability distribution of δ has a weaker impact both on average performance and average cost at equilibrium than in Example 5.1.

Table 6 reports the ranking and the approximated average importance of the ten most important arcs for two different probability distributions of the random variable δ (uniform and truncated normal). The approximated values of the average importance have been computed by partitioning the interval $[-1000, 1000]$

Table 4. The convergence of the approximated mean values of performance and cost at equilibrium in Example 5.2 for $\delta \sim \mathcal{U}(-1000, 1000)$.

			Avg cost at equilibrium				
N	ε_N	Avg performance	(4,11)	(10,13)	(14,15)	(16,22)	(20,17)
10	1.0e−02	4.0790	118.6775	342.3687	148.4124	222.4286	182.1626
20	2.5e−03	4.4868	106.8025	332.8790	137.1493	212.8895	174.4598
50	4.0e−04	4.6307	103.4114	330.0372	133.9382	210.1524	172.3478
100	1.0e−04	4.6527	102.9465	329.6548	133.5142	209.7731	172.0823
200	2.5e−05	4.6582	102.8327	329.5449	133.4239	209.6765	171.9907

Table 5. The convergence of the approximated mean values of performance and cost at equilibrium in Example 5.2 for $\delta \sim \mathcal{N}(0, 100)$ on $[-1000, 1000]$.

			Avg cost at equilibrium				
N	ε_N	Avg performance	(4,11)	(10,13)	(14,15)	(16,22)	(20,17)
10	1.00e−02	4.1151	116.8386	343.6712	149.7758	223.2194	181.5447
20	2.50e−03	4.5248	104.6739	331.5677	137.3640	213.3099	174.0132
50	4.00e−04	4.6696	101.3736	328.4031	134.2994	210.5215	171.8286
100	1.00e−04	4.6917	100.9129	327.9967	133.8407	210.1110	171.5318
200	2.50e−05	4.6973	100.8069	327.9054	133.7464	210.0099	171.4756

Table 6. Average importance for the 10 most important arcs in Example 5.2 for $\delta \sim \mathcal{U}(-1000, 1000)$ (on the left) and for $\delta \sim \mathcal{N}(0, 100)$ on $[-1000, 1000]$ (on the right).

	$\delta \sim \mathcal{U}(-1000, 1000)$		$\delta \sim \mathcal{N}(0, 100)$ on $[-1000, 1000]$	
Rank	Arc	Avg importance	Arc	Avg importance
1	37	0.06381	37	0.06421
2	26	0.06231	26	0.06383
3	25	0.06171	25	0.06281
4	43	0.06105	43	0.06133
5	56	0.06103	38	0.06102
6	60	0.06088	28	0.06066
7	28	0.06077	56	0.06057
8	38	0.06069	60	0.06037
9	50	0.05250	50	0.05242
10	55	0.05232	55	0.05224

into 50 subintervals. We note that the average importance of each arc slightly depends on the probability distribution of δ, while the ranking of arcs is more sensitive to it.

Example 5.3: We consider again the grid network shown in Figure 1, where the arc cost functions are defined as in (25) with $\beta = 4$ for all the links, while $t_i^0 = 1$ and $u_i = 50$ for any $i = 1, \ldots, 30$, and $t_i^0 = 5$ and $u_i = 100$ for any $i = 31, \ldots, 60$. We consider three OD pairs: (1,18), (13,30) and (19,36). We assume that the traffic demand is $D_j = 150 + \delta$, for any $j = 1, \ldots, 3$, where δ is a random variable which varies in the interval $[-100, 100]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 10.

Since each OD pair is connected by 21 paths, the arc-path incidence matrix Δ has not full column rank. Hence, the path cost operator is monotone but not

Table 7. The convergence of the approximated mean values of performance and cost at equilibrium in Example 5.3 for $\delta \sim \mathcal{U}(-100, 100)$.

			Avg cost at equilibrium		
N	ε_N	Avg performance	(1,18)	(13,30)	(19,36)
40	6.25e−04	6.0606	22.8241	26.6419	26.5954
50	4.00e−04	6.0599	22.8399	26.6358	26.5968
60	2.78e−04	6.0597	22.8478	26.6342	26.5980
70	2.04e−04	6.0596	22.8516	26.6341	26.5993
80	1.56e−04	6.0595	22.8544	26.6337	26.5998
90	1.23e−04	6.0594	22.8563	26.6335	26.6003
100	1.00e−04	6.0594	22.8575	26.6334	26.6006

Table 8. The convergence of the approximated mean values of performance and cost at equilibrium in Example 5.3 for $\delta \sim \mathcal{N}(0, 10)$ on $[-100, 100]$.

			Avg cost at equilibrium		
N	ε_N	Avg performance	(1,18)	(13,30)	(19,36)
40	6.25e−04	7.3299	19.1499	21.2067	21.1678
50	4.00e−04	7.3293	19.1672	21.2007	21.1693
60	2.78e−04	7.3290	19.1744	21.1986	21.1712
70	2.04e−04	7.3288	19.1779	21.1974	21.1723
80	1.56e−04	7.3287	19.1802	21.1968	21.1733
90	1.23e−04	7.3287	19.1819	21.1964	21.1740
100	1.00e−04	7.3286	19.1831	21.1961	21.1746

strongly monotone and the discretization procedure, along with the regularization scheme with $p = 5$, solves problems as (32). The interval $[-100, 100]$ has been uniformly partitioned into N subintervals and the regularization parameter $\varepsilon_N = 1/N^2$.

Tables 7 and 8 show the convergence of the mean values of the approximate performance and cost at equilibrium for different values of N when δ varies in the interval $[-100, 100]$ with uniform distribution and with truncated normal distribution with mean 0 and standard deviation 10, respectively.

5.3. The impact of a non-uniform discretization

In this section, we show the effect of uniform and non-uniform discretizations on the convergence rate of the network efficiency indices. We consider the experimental setting described in Example 5.1, i.e. the grid network shown in Figure 1 with five OD pairs whose traffic demands are $D_j = 150 + \delta$ for any $i = 1, \dots, 5$, where δ is a random variable which varies in the interval $[-50, 50]$ with truncated normal distribution with mean 0 and standard deviation 5. Since the probability density function of δ is concentrated around the mean value, it is reasonable to choose a non-uniform discretization of the interval $[-50, 50]$ in N subintervals for the approximation procedure of the network efficiency indices. To this end, we consider three non-uniform discretizations of $[-50, 50]$ defined as follows:

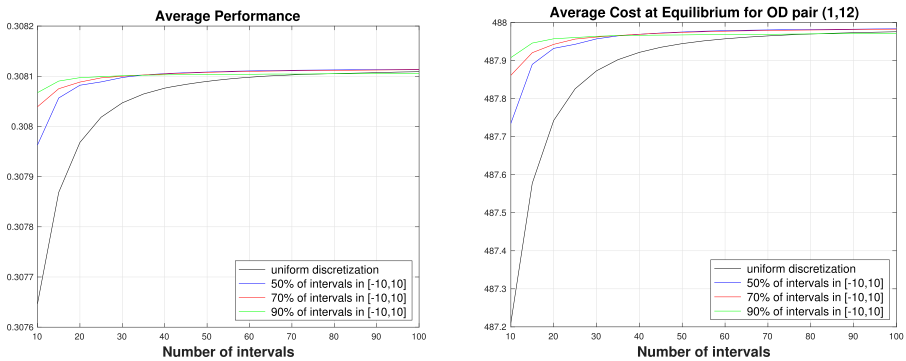

Figure 3. The impact of the discretization of the interval $[-50, 50]$ on the convergence of the average network performance (on the left) and of the average cost at equilibrium for OD pair (1,12) (on the right).

(1) 50% of the N subintervals are uniformly distributed in the interval $[-10, 10]$, 25% are uniformly distributed in $[-50, -10]$ and 25% are uniformly distributed in $[10, 50]$;
(2) 70% of the N subintervals are uniformly distributed in the interval $[-10, 10]$, 15% are uniformly distributed in $[-50, -10]$ and 15% are uniformly distributed in $[10, 50]$;
(3) 90% of the N subintervals are uniformly distributed in the interval $[-10, 10]$, 5% are uniformly distributed in $[-50, -10]$ and 5% are uniformly distributed in $[10, 50]$.

We run the approximation procedure with the uniform discretization and the three non-uniform discretizations mentioned above to compare the convergence rates of the efficiency indices. Figure 3 shows the approximated mean values of the network performance (on the left) and the cost at equilibrium for OD pair (1,12) (on the right) for each of the four considered discretizations.

The results show that non-uniform discretizations speed up the convergence of both indices. In particular, Table 9 reports the relative errors of the approximated average performance and cost at equilibrium for (1,12) found by non-uniform discretizations by using a small number of subintervals (from 10 to 30) with respect to the values found by the uniform discretization with 100 subintervals. The results show that non-uniform discretizations (especially the last one) need very few subintervals to get values close to those of the uniform discretization with 100 subintervals.

5.4. Scalability of the proposed approach

In this section, we show how the proposed approximation method scales for medium-large size networks. We generated a set of grid networks of dimension $6 \times Q$ (see Figure 4), where Q varies from 6 (as in the network considered in

Table 9. Relative error of the approximated average performance and cost at equilibrium for (1,12) found by non-uniform discretizations of the interval [−50, 50] with respect to the values found by the uniform discretization with 100 intervals.

	Avg performance			Avg cost at equilibrium for (1,12)		
N	50%	70%	90%	50%	70%	90%
10	4.736e−04	2.265e−04	1.352e−04	4.936e−04	2.347e−04	1.392e−04
15	1.700e−04	1.101e−04	6.052e−05	1.757e−04	1.128e−04	6.094e−05
20	8.739e−05	6.700e−05	3.841e−05	8.951e−05	6.787e−05	3.779e−05
25	6.580e−05	3.983e−05	3.198e−05	6.752e−05	3.943e−05	3.106e−05
30	3.773e−05	2.905e−05	2.579e−05	3.837e−05	2.842e−05	2.458e−05

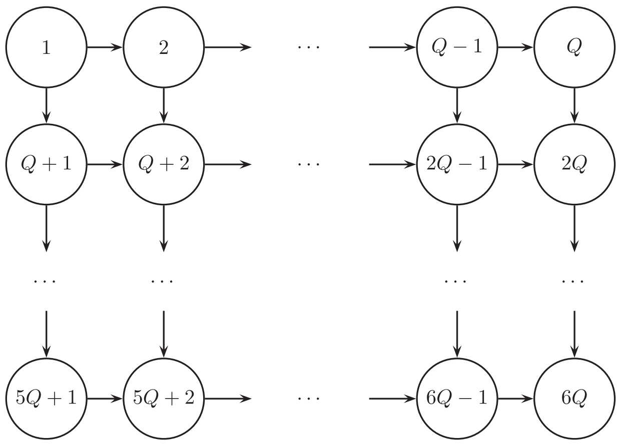

Figure 4. Grid networks considered in the scalability analysis.

Example 5.1) to 100. The arc cost functions are of the BPR form (25) with $\beta = 4$ for all the links, while $t_i^0 = 1$ and $u_i = 25$ for any horizontal arc, and $t_i^0 = 5$ and $u_i = 50$ for any vertical arc. For any generated network we consider five OD pairs: $(1, 2Q)$, $(Q+1, 3Q)$, $(2Q+1, 4Q)$, $(3Q+1, 5Q)$, $(4Q+1, 6Q)$. We assume that the traffic demand is $D_j = 150 + \delta$, for any $j = 1, \dots, 5$, where δ is a random variable which varies in the interval $[-50, 50]$ with uniform distribution.

Table 10 shows the CPU times needed for solving the generated instances. In particular, columns 1–3 report the number of nodes, arcs and paths of each instance, respectively; columns 4–6 report the CPU times (in seconds) of the numerical approximation procedure, where the interval $[-50, 50]$ is uniformly divided into 50, 100 or 200 subintervals, respectively. The results show that the approximation method finds the network efficiency indices with good accuracy within satisfactory times and the CPU times increase linearly with respect to the number of nodes of the network. Finally, we remark that the approximation procedure has been implemented using a sequential algorithm (i.e. the deterministic variational inequalities are solved one at a time), hence the running times could be significantly improved by implementing suitable parallel computing techniques.

Table 10. Scalability of the approximation procedure.

Instances			CPU times (seconds)		
Nodes	Arcs	Paths	$N = 50$	$N = 100$	$N = 200$
36	60	30	2.22	4.38	7.81
60	104	50	4.03	7.71	15.16
120	214	100	9.15	18.66	37.28
180	324	150	15.61	32.04	65.78
240	434	200	22.58	45.58	91.66
300	544	250	31.92	64.56	124.27
360	654	300	37.85	77.87	154.35
420	764	350	53.53	109.69	220.57
480	874	400	57.11	114.61	242.87
540	984	450	61.28	123.25	253.20
600	1094	500	83.92	170.63	351.91

6. Conclusions and further research perspectives

In this paper, we developed the approach proposed in [8] to analyse the performance and the vulnerability of traffic networks operating under the user equilibrium regime and random perturbations of data. We extended the previous analysis, which was restricted to linear cost functions, to the case of nonlinear monotone functions, and also included a regularization procedure when strict monotonicity does not hold in the path-flow variables. From the application point of view we considered here medium-size networks instead of the elementary, though paradigmatic, Braess network investigated previously. We also performed numerical experiments to show the impact of non-uniform discretization and got the encouraging evidence that a carefully chosen discretization procedure allows to reduce the number of intervals used without significantly deteriorating the accuracy of the approximation. Moreover, by using the class of grid networks we investigated the scalability of our approach and found that the CPU time increases linearly with respect to the number of nodes.

Further numerical work could be done to include parallel computing techniques in our procedure, thus increasing the number of independent random variables in the model. Another interesting research avenue is the performance and vulnerability analysis of stochastic traffic networks regulated by a central authority, as in the case of train or metro networks.

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No potential conflict of interest was reported by the author(s).

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Appendix. Numerical approximation procedure

In this appendix, we provide some technical details about our approximation procedure of (12). We start with a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. We then introduce a sequence $\{\pi_n\}$ of partitions of the support

$$\Upsilon := [0, \infty[\times [\underline{s}, \bar{s}[\times \mathbb{R}^m_+$$

of the probability measure $\mathbb{P}$ induced by the random elements R, S, and D. For this, we set

$$\pi_n = (\pi_n^R, \pi_n^S, \pi_n^D),$$

where

$$\begin{aligned}
&\pi_n^R := (r_n^0, \dots, r_n^{N_n^R}), \quad \pi_n^S := (s_n^0, \dots, s_n^{N_n^S}), \quad \pi_n^{D_i} := (t_{n,i}^0, \dots, t_{n,i}^{N_n^{D_i}}),\\
&0 = r_n^0 < r_n^1 < \cdots r_n^{N_n^R} = n,\\
&\underline{s} = s_n^0 < s_n^1 < \cdots s_n^{N_n^S} = \bar{s},\\
&0 = t_{n,i}^0 < t_{n,i}^1 < \cdots t_{n,i}^{N_n^{D_i}} = n \quad (i = 1, \dots, m),\\
&|\pi_n^R| := \max\{r_n^j - r_n^{j-1} : j = 1, \dots, N_n^R\} \to 0 \quad (n \to \infty),\\
&|\pi_n^S| := \max\{s_n^k - s_n^{k-1} : k = 1, \dots, N_n^S\} \to 0 \quad (n \to \infty),\\
&|\pi_n^{D_i}| := \max\{t_{n,i}^{h_i} - t_{n,i}^{h_i-1} : h_i = 1, \dots, N_n^{D_i}\} \to 0 \quad (i = 1, \dots, m; n \to \infty).
\end{aligned}$$

These partitions give rise to an exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ, where each Υ_n is given by the finite disjoint union of the intervals:

$$I_{jkh}^n := [r_n^{j-1}, r_n^j[\times [s_n^{k-1}, s_n^k[\times I_h^n,$$

where we use the multi-index $h = (h_1, \dots, h_m)$ and

$$I_h^n := \prod_{i=1}^m [t_{n,i}^{h_i-1}, t_{n,i}^{h_i}[.$$

For each $n \in \mathbb{N}$, we consider the space of the $\mathbb{R}^k$-valued step functions on Υ_n, extended by 0 outside of Υ_n:

$$X_n^k := \left\{ v_n : \; v_n(r, s, t) = \sum_j \sum_k \sum_h v_{jkh}^n \mathbf{1}_{I_{jkh}^n}(r, s, t), \; v_{jkh}^n \in \mathbb{R}^k \right\},$$

where $\mathbf{1}_I$ denotes the $\{0, 1\}$-valued characteristic function of a subset I. To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$, we employ the mean value truncation operator μ_0^n

associated to the partition π_n given by

$$\mu_0^n w := \sum_{j=1}^{N_n^R} \sum_{k=1}^{N_n^S} \sum_h (\mu_{jkh}^n w)\, \mathbf{1}_{I_{jkh}^n}, \tag{A1}$$

where

$$\mu_{jkh}^n w := \begin{cases} \dfrac{1}{\mathbb{P}(I_{jkh})} \displaystyle\int_{I_{jkh}^n} w(y)\, \mathrm{d}\mathbb{P}(y), & \text{if } \mathbb{P}(I_{jkh}^n) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Analogously, for a L^p vector function $v = (v_1, \dots, v_k)$, we define

$$\mu_0^n v := (\mu_0^n v_1, \dots, \mu_0^n v_k),$$

for which one can prove that $\mu_0^n v$ converges to v in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$.

To construct approximations for the set

$$M_{\mathbb{P}} = \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : \ Gv(r, s, t) \le t, \ \mathbb{P} - \text{a.s.}\},$$

we introduce the orthogonal projector $q : (r, s, t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and define, for each elementary cell I_{jkh}^n, the quantities

$$\overline{q}_{jkh}^n = (\mu_{jkh}^n q) \in \mathbb{R}^m \quad \text{and} \quad (\mu_0^n q) = \sum_{jkh} \overline{q}_{jkh}^n \mathbf{1}_{I_{jkh}^n} \in X_n^m.$$

This leads to the following sequence of polyhedra:

$$M_{\mathbb{P}}^n := \{v \in X_n^k : \ Gv_{jkh}^n \le \overline{q}_{jkh}^n, \ \forall j, k, h\}.$$

Since our objective is to approximate the random variables R and S, we introduce

$$\rho_n = \sum_{j=1}^{N_n^R} r_n^{j-1} 1_{[r_n^{j-1}, r_n^j[} \in X_n \quad \text{and} \quad \sigma_n = \sum_{k=1}^{N_n^S} s_n^{k-1}\, 1_{[s_n^{k-1}, s_n^k[} \in X_n.$$

Note that

$$\sigma_n(r, s, t) \to \sigma(r, s, t) = s \text{ in } L^\infty(\mathbb{R}^d, \mathbb{P}), \quad \rho_n(r, s, t) \to \rho(r, s, t) = r \text{ in } L^p(\mathbb{R}^d, \mathbb{P}).$$

Combining the above ingredients, for any $n \in \mathbb{N}$ we consider the variational inequality (13) of Section 3, which we rewrite below for the reader convenience: find $\hat{u}_n := \hat{u}_n(y) \in M_{\mathbb{P}}^n$ such that, for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} (\sigma_n(y) A[\hat{u}_n(y)] + B[\hat{u}_n(y)])^\top (v_n(y) - \hat{u}_n(y))\, \mathrm{d}\mathbb{P}(y)$$
$$\ge \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} (b + \rho_n(y)\, c)^\top (v_n(y) - \hat{u}_n(y))\, \mathrm{d}\mathbb{P}(y).$$

We also assume that the probability measures P_R, P_S and P_{D_i} have the probability densities φ_R, φ_S and φ_{D_i}, with $i = 1, \dots, m$, respectively. Therefore, for $i = 1, \dots, m$, we have

$$\mathrm{d}P_R(r) = \varphi_R(r)\, \mathrm{d}r, \quad \mathrm{d}P_S(s) = \varphi_S(s)\, \mathrm{d}s, \quad \mathrm{d}P_{D_i}(t_i) = \varphi_{D_i}(t_i)\, \mathrm{d}t_i.$$

For actual implementation it is important to notice that (13) can be split in a finite number of finite dimensional variational inequalities: for every $n \in \mathbb{N}$, and for every j, k, h, find $\hat{u}_{jkh}^n \in$

M^n_{jkh} such that

$$[\tilde{T}^n_k(\hat{u}^n_{jkh})]^\top[v^n_{jkh}-\hat{u}^n_{jkh}] \geq [\tilde{c}^n_j]^\top[v^n_{jkh}-\hat{u}^n_{jkh}], \quad \text{for every } v^n_{jkh} \in M^n_{jkh}, \tag{A2}$$

where

$$\begin{aligned} M^n_{jkh} &:= \{v^n_{jkh} \in \mathbb{R}^k : Gv^n_{jkh} \leq \bar{q}^n_{jkh}\}, \\ \tilde{T}^n_k &:= s_n^{k-1}A + B \\ \tilde{c}^n_j &:= b + r_n^{j-1}c. \end{aligned}$$

We can then reconstruct the step-function solution as follows:

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}^n_{jkh} 1_{I^n_{jkh}} \in X^k_n.$$

The following convergence result was proved in [11].

Theorem A.1: *Assume that the growth condition (10) holds and $T(\omega,\cdot)$ is strongly monotone, uniformly with respect to $\omega \in \Omega$, that is there exists $\tau > 0$ such that*

$$(T(\omega,x)-T(\omega,y))^\top(x-y) \geq \tau\|x-y\|^2 \quad \forall\, x,y, \text{ a.e. } \omega \in \Omega.$$

Then the sequence $\{\hat{u}_n\}$, where $\hat{u}_n$ is the unique solution of (13), converges strongly in $L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$ to the unique solution $\hat{u}$ of (12).

In the absence of strict monotonicity, the solution of (11) and (12) is not unique. In this case (which often occurs in our application) the previous approximation procedure must be coupled with a *regularization* scheme as follows. We choose a sequence $\{\varepsilon_n\}$ of regularization parameters and choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) \to L^q(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$. We assume that $\varepsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\varepsilon_n \downarrow 0$ as $n \to \infty$. We can then consider, for any $n \in \mathbb{N}$, the regularized stochastic variational inequality (14) of Section 3: find $w_n = w_n^{\varepsilon_n}(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\begin{aligned} &\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \big(\sigma_n(y)A[w_n(y)] + B[w_n(y)] + \varepsilon_n J(w_n(y))\big)^\top (v_n(y)-w_n(y))\,\mathrm{d}\mathbb{P}(y) \\ &\quad \geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (b+\rho_n(y)c)^\top(v_n(y)-w_n(y))\,\mathrm{d}\mathbb{P}(y). \end{aligned}$$

As usual, the solution w_n will be referred to as the regularized solution. Weak and strong convergence of $\{w_n\}$ to the minimal-norm solution of (12) can be proved under suitable hypotheses (see below). We also recall (see e.g. [22]) that in L^p we have

$$J(u)(y) = \|u\|^{2-p}_{L^p}\|u(y)\|_2^{p-2}\,u(y), \tag{A3}$$

thus, in the case $p=2$ the duality map is the Riesz isometry $I : L^2(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) \to L^2(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$.

The following results (see [12]) highlight some of the features of the regularized solutions.

Theorem A.2: *The following statements hold.*

(1) *For every $n \in \mathbb{N}$, the regularized stochastic variational inequality (14) has the unique solution w_n.*
(2) *Any weak limit of the sequence of regularized solutions $\{w_n\}$ is a solution of (12).*

(3) *The sequence of regularized solutions $\{w_n\}$ is bounded provided that the following coercivity condition holds: there exists a bounded sequence $\{\delta_n\}$, with $\delta_n \in M^n_{\mathbb{P}}$, such that*

$$\frac{\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} [\sigma_n(y)A(u_n(y)) + B(u_n(y))]^\top (u_n(y) - \delta_n(y))\,\mathrm{d}\mathbb{P}(y)}{\|u_n\|} \to \infty$$

as $\|u_n\| \to \infty$.

To obtain strong convergence, we need to use the concept of fast Mosco convergence [23], as given by the following definition.

Definition A.3: Let X be a normed space, let $\{K_n\}$ be a sequence of closed and convex subsets of X and let $K \subset X$ be closed and convex. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. The sequence $\{K_n\}$ is said to converge to K in the fast Mosco sense, related to ε_n, if

(1) For each $x \in K$, $\exists\{x_n\} \in K_n$ such that $\varepsilon_n^{-1}\|x_n - x\| \to 0$;
(2) For $\{x_m\} \subset X$, if $\{x_m\}$ weakly converges to $x \in K$, then $\exists\{z_m\} \in K$ such that $\varepsilon_m^{-1}(z_m - x_m)$ weakly converges to 0.

Theorem A.4: *Assume that $M^n_{\mathbb{P}}$ converges to $M_{\mathbb{P}}$ in the fast Mosco sense related to ε_n. Moreover, assume that $\varepsilon_n^{-1}\|\sigma_n - \sigma\| \to 0$ and $\varepsilon_n^{-1}\|\rho_n - \rho\| \to 0$ as $n \to \infty$. Then the sequence of regularized solutions $\{w_n\}$ converges strongly to the minimal-norm solution of the stochastic variational inequality* (12)*, provided that $\{w_n\}$ is bounded.*

In the case $p > 2$, a thorough analysis of the implementation of (14) has been carried out in the forthcoming paper [24].