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Research Article

Fractional Integral Inequalities via Atangana-Baleanu Operators for Convex and Concave Functions

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Recently, many fractional integral operators were introduced by different mathematicians. One of these fractional operators, Atangana-Baleanu fractional integral operator, was defined by Atangana and Baleanu (Atangana and Baleanu, 2016). In this study, firstly, a new identity by using Atangana-Baleanu fractional integral operators is proved. Then, new fractional integral inequalities have been obtained for convex and concave functions with the help of this identity and some certain integral inequalities.

1. Introduction

Mathematics is a tool that serves pure and applied sciences with its deep-rooted history as old as human history and sheds light on how to express and then solve problems. Mathematics uses various concepts and their relations with each other while performing this task. By defining spaces and algebraic structures built on spaces, mathematics creates structures that contribute to human life and nature. The concept of function is one of the basic structures of mathematics, and many researchers have focused on new function classes and made efforts to classify the space of functions. One of the types of functions defined as a product of this intense effort is the convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis. This interesting class of functions is defined as follows.

Definition 1. The mapping $f : [\theta_1, \theta_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1}$$

is valid for all $x, y \in [\theta_1, \theta_2]$ and $\lambda \in [0, 1]$.

Many inequalities have been obtained by using this unique function type and varieties in inequality theory, which is one of the most used areas of convex functions. We will continue by introducing the Hermite-Hadamard inequality that generate limits on the mean value of a convex function and the famous Bullen inequality as follows.

Assume that $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$, where $\theta_1 < b$. The following statement:

$$f\left(\frac{\theta_1+\theta_2}{2}\right) \le \frac{1}{\kappa_2-\theta_1}\int_{\theta_1}^{\theta_2} f(x)dx \le \frac{f(\theta_1)+f(\theta_2)}{2}, \quad (2)$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

Bullen's integral inequality can be presented as

$$\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} f(x)dx \le \frac{1}{2}\left[f\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{f(\theta_1)+f(\theta_2)}{2}\right], \quad (3)$$

where $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a convex mapping on the interval I of $\mathbb{R}$ where $\kappa_1, \theta_2 \in I$ with $\theta_1 < \theta_2$.

To provide detail information on convexity, let us consider some earlier studies that have been performed by many researchers. In [1], Jensen introduced the concept of convex function to the literature for the first time and drew attention to the fact that it seems to be the basis of the concept of incremental function. In [2], Beckenbach has mentioned about the concept of convexity and emphasized several features of this useful function class. In [3], the authors have focused the relations between convexity and Hermite-Hadamard's inequality. This study has led many researchers to the link between convexity and integral inequalities, which has guided studies in this field. Based on these studies, many papers have been produced for different kinds of convex functions. In [4], Akdemir et al. have proved several new integral inequalities for geometric-arithmetic convex functions via a new integral identity. Several new Hadamard's type integral inequalities have been established with applications to special means by Kavurmaci et al. in [5]. Therefore, a similar argument has been carried out by Zhang et al. but now for $s-$geometrically convex functions in [6]. On all of these, Xi et al. have extended the challenge to $m-$ and $(\alpha, m)-$convex functions by providing Hadamard type inequalities in [7].

Although fractional analysis has been known since ancient times, it has recently become a more popular subject in mathematical analysis and applied mathematics. The adventure that started with the question of whether the solution will exist if the order is fractional in a differential equation has developed with many derivative and integral operators. By defining the derivative and integral operators in fractional order, the researchers who aimed to propose more effective solutions to the solution of physical phenomena have turned to new operators with general and strong kernels over time. This orientation has provided mathematics and applied sciences several operators with kernel structures that differ in terms of locality and singularity, as well as generalized operators with memory effect properties. The struggle that started with the question of how the order in the differential equation being a fraction would have consequences has now evolved into the problem of how to explain physical phenomena and find the most effective fractional operators that will provide effective solutions to real-world problems. Let us introduce some fractional derivative and integral operators that have broken ground in fractional analysis and have proven their effectiveness in different fields by using by many researchers.

We will remember the Caputo-Fabrizio derivative operators. Also, we would like to note that the functions belong to Hilbert spaces denoted by $H^1(0, \theta_2)$.

Definition 2. [8]. Let $f \in H^1(0, \theta_2), \theta_2 > \theta_1, \alpha \in [0, 1]$, then the definition of the new Caputo fractional derivative is

$$^{CF}D^{\alpha} f(\tau_1) = \frac{M(\alpha)}{1-\alpha}\int_{\kappa_1}^{\tau_1} f'(s)\exp\left[-\frac{\alpha}{(1-\alpha)}(\tau_1 - s)\right]ds, \quad (4)$$

where $M(\alpha)$ is normalization function.

Depending on this interesting fractional derivative operator, the authors have defined the Caputo-Fabrizio fractional integral operator as follows.

Definition 3. [9] Let $f \in H^1(0, \theta_2), \theta_2 > \theta_1, \alpha \in [0, 1]$, then the definition of the left and right side of Caputo-Fabrizio fractional integral is

$$\left({}^{CF}_{:::\theta_1}I^{\alpha}\right)(\tau_1) = \frac{1-\alpha}{B(\alpha)} f(\tau_1) + \frac{\alpha}{B(\alpha)}\int_{\theta_1}^{\tau_1} f(y)dy, \quad (5)$$

and

$$\left({}^{CF}I^{\alpha}_{\theta_2}\right)(\tau_1) = \frac{1-\alpha}{B(\alpha)} f(\tau_1) + \frac{\alpha}{B(\alpha)}\int_{\tau_1}^{\kappa_2} f(y)dy, \quad (6)$$

where $B(\alpha)$ is the normalization function.

The Caputo-Fabrizio fractional derivative, which is used in dynamical systems, physical phenomena, disease models, and many other fields, is a highly functional operator by definition, but has a deficiency in terms of not meeting the initial conditions in the special case $\alpha = 1$. The improvement to eliminate this deficiency has been provided by the new derivative operator developed by Atangana-Baleanu, which has versions in the sense of Caputo and Riemann. In the sequel of this paper, we will denote the normalization function with $B(\alpha)$ with the same properties with the $M(\alpha)$ which is defined in Caputo-Fabrizio definition.

Definition 4. [10] Let $f \in H^1(\theta_1, \theta_2)$, $\theta_2 > \kappa_1$, $\alpha \in [0, 1]$, then the definition of the new fractional derivative is given:

$$^{ABC}_{::::::\theta_1}D^{\alpha}_{\tau_1}[f(\tau_1)] = \frac{B(\alpha)}{1-\alpha}\int_a^{\tau_1} f'(x)E_{\alpha}\left[-\alpha\frac{(\tau_1 - x)^{\alpha}}{(1-\alpha)}\right]dx. \quad (7)$$

Definition 5. [10] Let $f \in H^1(\theta_1, \theta_2)$, $\theta_2 > \kappa_1$, $\alpha \in [0, 1]$, then the definition of the new fractional derivative is given:

$$^{ABR}_{::::::\theta_1}D^{\alpha}_{\tau_1}[f(\tau_1)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\tau_1}\int_{\theta_1}^{\tau_1} f(x)E_{\alpha}\left[-\alpha\frac{(\tau_1 - x)^{\alpha}}{(1-\alpha)}\right]dx. \quad (8)$$

Equations (7) and (8) have a nonlocal kernel. Also, in Equation (8), when the function is constant, we get zero.

The associated fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 6. [10] The fractional integral associate to the new fractional derivative with nonlocal kernel of a function $f \in H^1(\kappa_1, \theta_2)$ as defined:

$$ {}^{AB}_{::::\theta_1}I^{\alpha}\{f(\tau_1)\} = \frac{1-\alpha}{B(\alpha)}f(\tau_1) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{\tau_1} f(y)(\tau_1-y)^{\alpha-1}dy, \quad (9) $$

where $\theta_2 > \theta_1$ and $\alpha \in [0,1]$.

In [11], Abdeljawad and Baleanu introduced the right hand side of integral operator as follows: the right fractional new integral with ML kernel of order $\alpha \in [0,1]$ is defined by

$$ \left({}^{AB}I^{\alpha}_{\theta_2}\right)\{f(\tau_1)\} = \frac{1-\alpha}{B(\alpha)}f(\tau_1) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \cdot \int_{\tau_1}^{\theta_2} f(y)(y-\tau_1)^{\alpha-1}dy. \quad (10) $$

In [9], Abdeljawad and Baleanu has presented some new results based on fractional-order derivatives and their discrete versions. Conformable integral operators have been defined by Abdeljawad in [12]. This useful operator has been used to prove some new integral inequalities in [13]. Another important fractional operator—Riemann-Liouville fractional integral operators—have been used to provide some new Simpson type integral inequalities in [14]. Ekinci and Ozdemir have proved several generalizations by using Riemann-Liouville fractional integral operators in [15], and the authors have established some similar results with this operator in [16]. In [17], Akdemir et al. have presented some new variants of celebrated Chebyshev inequality via generalized fractional integral operators. The argument has been proceed with the study of Rashid et al. (see [18]) that involves new investigations related to generalized k-fractional integral operators. In [19], Rashid et al. have presented some motivated findings that extend the argument to the Hilbert spaces. For more information related to different kinds of fractional operators, we recommend to consider [20]. The applications of fractional operators have been demonstrated by several researchers; we suggest to see the papers [21–23].

The main motivation of this paper is to prove an integral identity that includes the Atangana-Baleanu integral operator and to provide some new Bullen type integral inequalities for differentiable convex and concave functions with the help of this integral identity. Some special cases are also considered.

2. Main Results

We will start with a new integral identity that will be used as proofs of our main findings.

Lemma 7. *Let $f : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ be differentiable function on (θ_1, θ_2) with $\kappa_1 < \theta_2$. Then, we have the following identity for Atangana-Baleanu fractional integral operators:*

$$ \begin{aligned} &\frac{2(\theta_2-\theta_1)^{\alpha} + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1) + f(\theta_2) + 2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] \\ &\quad - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1) + {}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right. \\ &\quad \left. + {}^{AB}_{::((\theta_1+\theta_2)/2)}I^{\alpha}f(\theta_2) + {}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right] \\ &= \int_0^1 ((1-\tau_1)^{\alpha} - \tau_1^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)d\tau_1 \\ &\quad + \int_0^1 (\tau_1^{\alpha} - (1-\tau_1)^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\kappa_1\right)d\tau_1, \end{aligned} \quad (11) $$

where $\alpha, \tau_1 \in [0,1]$, $\Gamma(.)$ is the gamma function, and $B(\alpha)$ is the normalization function.

Proof. By adding I_1 and I_2, we have

$$ \begin{aligned} I_1 + I_2 &= \int_0^1 ((1-\tau_1)^{\alpha} - \tau_1^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\kappa_2\right)d\tau_1 \\ &\quad + \int_0^1 (\tau_1^{\alpha} - (1-\tau_1)^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_2 + \frac{1-\tau_1}{2}\theta_1\right)d\tau_1. \end{aligned} \quad (12) $$

By using integration, we have

$$ \begin{aligned} I_1 &= \int_0^1 ((1-\tau_1)^{\alpha} - \tau_1^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\kappa_2\right)d\tau_1 \\ &= \left.\frac{((1-\tau_1)^{\alpha} - \tau_1^{\alpha})f(((1+\tau_1)/2)\theta_1 + ((1-\tau_1)/2)\theta_2)d\tau_1}{(\theta_1-\theta_2)/2}\right|_1^0 \\ &\quad - \frac{2\alpha}{\kappa_2-\theta_1}\int_0^1 ((1-\tau_1)^{\alpha-1} + \tau_1^{\alpha-1})f\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\kappa_2\right)d\tau_1 \\ &= -\frac{2}{\theta_1-\theta_2}f(\theta_1) - \frac{2}{\kappa_1-\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right) \\ &\quad - \frac{2\alpha}{\kappa_2-\theta_1}\int_0^1 (1-\tau_1)^{\alpha-1}f\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)d\tau_1 \\ &\quad - \frac{2\alpha}{\theta_2-\theta_1}\int_0^1 \tau_1^{\alpha-1}f\left(\frac{1+\tau_1}{2}\theta_1 + \frac{1-\tau_1}{2}\theta_2\right)d\tau_1 \\ &= \frac{2}{\theta_2-\theta_1}\left(f(\theta_1) + f\left(\frac{\kappa_1+\theta_2}{2}\right)\right) \\ &\quad - \frac{2^{\alpha+1}\alpha}{(\kappa_2-\theta_1)^{\alpha+1}}\int_{\theta_1}^{(\theta_1+\kappa_2)/2}(x-\theta_1)^{\alpha-1}f(x)dx \\ &\quad - \frac{2^{\alpha+1}\alpha}{(\theta_2-\theta_1)^{\alpha+1}}\int_{\theta_1}^{(\theta_1+\theta_2)/2}\left(\frac{\kappa_1+\theta_2}{2} - x\right)^{\alpha-1}f(x)dx. \end{aligned} \quad (13) $$

Multiplying both sides of (13) identity by $(\kappa_2-\theta_1)^{\alpha+1}/(2^{\alpha+1}B(\alpha)\Gamma(\alpha))$, we have

$$\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}I_1=\frac{(\theta_2-\theta_1)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(f(\theta_1)+f\left(\frac{\theta_1+\kappa_2}{2}\right)\right)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\kappa_1}^{(\theta_1+\theta_2)/2}(x-\theta_1)^{\alpha-1}f(x)dx-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{(\theta_1+\theta_2)/2}\left(\frac{\theta_1+\theta_2}{2}-x\right)^{\alpha-1}\cdot f(x)dx. \tag{14}$$

Similarly, by using integration, we get

$$\begin{aligned}I_2&=\int_0^1(\tau_1^{\alpha}-(1-\tau_1)^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\kappa_1\right)d\tau_1\\&=\frac{(\tau_1^{\alpha}-(1-\tau_1)^{\alpha})f(((1+\tau_1)/2)\theta_2+((1-\tau_1)/2)\theta_1)d\tau_1}{(\theta_2-\theta_1)/2}\Bigg|_1^0\\&\quad-\frac{2\alpha}{\kappa_2-\theta_1}\int_0^1(\tau_1^{\alpha-1}+(1-\tau_1)^{\alpha-1})f\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\kappa_1\right)d\tau_1\\&=\frac{2}{\theta_2-\theta_1}\left(f(\theta_2)+f\left(\frac{\kappa_1+\theta_2}{2}\right)\right)\\&\quad-\frac{2^{\alpha+1}\alpha}{(\kappa_2-\theta_1)^{\alpha+1}}\int_{(\theta_1+\theta_2)/2}^{\theta_2}\left(x-\frac{\theta_1+\theta_2}{2}\right)^{\alpha-1}f(x)dx\\&\quad-\frac{2^{\alpha+1}\alpha}{(\theta_2-\theta_1)^{\alpha+1}}\int_{(\theta_1+\theta_2)/2}^{\theta_2}(\theta_2-x)^{\alpha-1}f(x)dx.\end{aligned} \tag{15}$$

Multiplying both sides of (13) identity by $(\theta_2-\kappa_1)^{\alpha+1}/(2^{\alpha+1}B(\alpha)\Gamma(\alpha))$, we get

$$\begin{aligned}&\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}I_2\\&\quad=\frac{(\theta_2-\theta_1)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(f(\theta_2)+f\left(\frac{\theta_1+\kappa_2}{2}\right)\right)\\&\quad-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\theta_1+\theta_2)/2}^{\theta_2}\left(x-\frac{\theta_1+\kappa_2}{2}\right)^{\alpha-1}f(x)dx\\&\quad-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\theta_1+\kappa_2)/2}^{\theta_2}(\theta_2-x)^{\alpha-1}f(x)dx.\end{aligned} \tag{16}$$

By adding identities (14) and (16), we obtain

$$\begin{aligned}&\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}[I_1+I_2]\\&\quad=\frac{(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha}\Gamma(\alpha)}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(\theta_1)+f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\\&\quad-\frac{1-\alpha}{B(\alpha)}f(\theta_1)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{(\theta_1+\theta_2)/2}(x-\theta_1)^{\alpha-1}f(x)dx\\&\quad-\frac{1-\alpha}{B(\alpha)}f\left(\frac{\theta_1+\theta_2}{2}\right)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\theta_1}^{(\theta_1+\theta_2)/2}\\&\quad\cdot\left(\frac{\theta_1+\theta_2}{2}-x\right)^{\alpha-1}f(x)dx+\frac{(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha}\Gamma(\alpha)}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\\&\quad\cdot\left[f(\theta_2)+f\left(\frac{\theta_1+\theta_2}{2}\right)\right]-\frac{1-\alpha}{B(\alpha)}f(\theta_2)\\&\quad-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\kappa_1+\theta_2)/2}^{\theta_2}(\theta_2-x)^{\alpha-1}f(x)dx\\&\quad-\frac{1-\alpha}{B(\alpha)}f\left(\frac{\theta_1+\theta_2}{2}\right)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{(\kappa_1+\theta_2)/2}^{\theta_2}\\&\quad\cdot\left(x-\frac{\theta_1+\theta_2}{2}\right)^{\alpha-1}f(x)dx.\end{aligned} \tag{17}$$

Using the definition of Atangana-Baleanu fractional integral operators, we get

$$\begin{aligned}&\frac{(\theta_2-\theta_1)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\int_0^1((1-\tau_1)^{\alpha}-\tau_1^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)d\tau_1\right.\\&\quad\left.+\int_0^1(\tau_1^{\alpha}-(1-\tau_1)^{\alpha})f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)d\tau_1\right]\\&=\frac{(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha}\Gamma(\alpha)}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\\&\quad-{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\kappa_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\theta_1+\theta_2}{2}\right)\\&\quad+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\Bigg].\end{aligned} \tag{18}$$

□□ □

Theorem 8. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\&\quad-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\&\quad\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\&\leq\frac{2\left[|f'(\theta_1)|+|f'(\theta_2)|\right]}{\alpha+1},\end{aligned} \tag{19}$$

where $\alpha\in[0,1]$ and $B(\alpha)$ is the normalization function.

Proof. By using Lemma 7, we can write

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1. \tag{20}$$

By using convexity of $|f'|$, we get

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|+\frac{1-\tau_1}{2}|f'(\theta_2)|\right]d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|+\frac{1-\tau_1}{2}|f'(\theta_2)|\right]d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|+\frac{1-\tau_1}{2}|f'(\kappa_1)|\right]d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|+\frac{1-\tau_1}{2}|f'(\theta_1)|\right]d\tau_1. \tag{21}$$

By computing the above integral, we obtain

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\frac{2\left[|f'(\theta_1)|+|f'(\theta_2)|\right]}{\alpha+1}, \tag{22}$$

and the proof is completed.□□ □

Corollary 9. *In Theorem 8, if we choose $\alpha=1$, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\theta_1+\kappa_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\theta_2-\kappa_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|$$
$$\leq\frac{|f'(\theta_1)|+|f'(\kappa_2)|}{2}. \tag{23}$$

Theorem 10. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\frac{2}{(\alpha p+1)^{1/p}}\left[\left(\frac{3|f'(\theta_1)|^q+|f'(\kappa_2)|^q}{4}\right)^{1/q}\right.$$
$$\left.+\left(\frac{3|f'(\theta_2)|^q+|f'(\kappa_1)|^q}{4}\right)^{1/q}\right], \tag{24}$$

where $p^{-1}+q^{-1}=1$, $\alpha\in[0,1]$, $q>1$, and $B(\alpha)$ is the normalization function.

Proof. By using the identity that is given in Lemma 7, we have

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1. \tag{25}$$

By applying Hölder inequality, we have

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\kappa_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}$$
$$+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}$$
$$+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}$$
$$+\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}. \tag{26}$$

By using convexity of $|f'|^q$, we obtain

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_1)|^q\right.\right.$$
$$\left.\left.+\frac{1-\tau_1}{2}|f'(\kappa_2)|^q\right]dv\right)^{1/q}+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}$$
$$\cdot\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_1)|^q+\frac{1-\tau_1}{2}|f'(\kappa_2)|^q\right]d\tau_1\right)^{1/q}$$
$$+\left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_2)|^q\right.\right.$$
$$\left.\left.+\frac{1-\tau_1}{2}|f'(\kappa_1)|^q\right]d\tau_1\right)^{1/q}+\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}$$
$$\cdot\left(\int_0^1\left[\frac{1+\tau_1}{2}|f'(\kappa_2)|^q+\frac{1-\tau_1}{2}|f'(\kappa_1)|^q\right]d\tau_1\right)^{1/q}. \tag{27}$$

By calculating the integrals that is in the above inequalities, we get desired result.□□ □

Corollary 11. *In Theorem 10, if we choose $\alpha=1$, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\kappa_1+\theta_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\kappa_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|$$
$$\le\frac{1}{(p+1)^{1/p}}\left[\left(\frac{3|f'(\theta_1)|^q+|f'(\kappa_2)|^q}{4}\right)^{1/q}\right.$$
$$\left.+\left(\frac{3|f'(\theta_2)|^q+|f'(\theta_1)|^q}{4}\right)^{1/q}\right]. \tag{28}$$

Theorem 12. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\left(\frac{1}{\alpha+1}\right)^{1-(1/q)}\left[\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\kappa_1)|^q\right.\right.$$
$$\left.+\frac{1}{2(\alpha+2)}|f'(\kappa_2)|^q\right)^{1/q}+\left(\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\theta_1)|^q\right.$$
$$\left.+\frac{1}{2(\alpha+1)(\alpha+2)}|f'(\theta_2)|^q\right)^{1/q}$$
$$+\left(\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\theta_2)|^q+\frac{1}{2(\alpha+1)(\alpha+2)}|f'(\theta_1)|^q\right)^{1/q}$$
$$\left.+\left(\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}|f'(\theta_2)|^q+\frac{1}{2(\alpha+2)}|f'(\theta_1)|^q\right)^{1/q}\right], \tag{29}$$

where $\alpha\in[0,1]$, $q\ge1$, and $B(\alpha)$ is the normalization function.

Proof. By Lemma 7, we get

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\le\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\kappa_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1$$
$$+\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1. \tag{30}$$

By applying power mean inequality, we get

$$\begin{aligned}&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\&\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\&\le\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1\right.\right.\right.\\&\left.\left.\left.+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\\&\cdot\left(\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q}\\&+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1\tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}\\&+\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2\right.\right.\right.\\&\left.\left.\left.+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}.\end{aligned}\tag{31}$$

By using convexity of $|f'|^q$, we obtain

$$\begin{aligned}&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\&\left.\left.+{}^{AB}_{::(\theta_1+\theta_2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\&\le\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|^q\right.\right.\\&\left.\left.+\frac{1-\tau_1}{2}|f'(\theta_2)|^q\right]d\tau_1\right)^{1/q}+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\\&\cdot\left(\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_1)|^q+\frac{1-\tau_1}{2}|f'(\theta_2)|^q\right]d\tau_1\right)^{1/q}\\&+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)^{1-(1/q)}\left(\int_0^1\tau_1^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|^q\right.\right.\\&\left.\left.+\frac{1-\tau_1}{2}|f'(\theta_1)|^q\right]d\tau_1\right)^{1/q}+\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)^{1-(1/q)}\\&\cdot\left(\int_0^1(1-\tau_1)^{\alpha}\left[\frac{1+\tau_1}{2}|f'(\theta_2)|^q+\frac{1-\tau_1}{2}|f'(\theta_1)|^q\right]d\tau_1\right)^{1/q}.\end{aligned}\tag{32}$$

By computing the above integrals, the proof is completed.□□ □

Corollary 13. *In Theorem 12, if we choose $\alpha = 1$, we obtain*

$$\begin{aligned}&\left|\frac{2[f(\theta_1)+f(\theta_2)+2f((\kappa_1+\theta_2)/2)]}{\theta_2-\theta_1}-\frac{8}{(\theta_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|\\&\le\left(\frac{1}{2}\right)^{1-(1/q)}\left[\left(\frac{2|f'(\theta_1)|^q+|f'(\theta_2)|^q}{6}\right)^{1/q}\right.\\&+\left(\frac{5|f'(\theta_1)|^q+|f'(\theta_2)|^q}{12}\right)^{1/q}\\&+\left(\frac{5|f'(\theta_2)|^q+|f'(\theta_1)|^q}{12}\right)^{1/q}\\&\left.+\left(\frac{2|f'(\theta_2)|^q+|f'(\theta_1)|^q}{6}\right)^{1/q}\right].\end{aligned}\tag{33}$$

Theorem 14. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\begin{aligned}&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\&\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\&\le\frac{4}{p(\alpha p+1)}+\frac{2\left[|f'(\kappa_1)|^q+|f'(\theta_2)|^q\right]}{q},\end{aligned}\tag{34}$$

where $p^{-1}+q^{-1}=1$, $\alpha\in[0,1]$, $q>1$, and $B(\alpha)$ is the normalization function.

Proof. By using identity that is given in Lemma 7, we get

$$\begin{aligned}&\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.\\&-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.\\&\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|\\&\le\int_0^1(1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1\end{aligned}$$

$$+\int_0^1 \tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|d\tau_1$$
$$+\int_0^1 \tau_1^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1$$
$$+\int_0^1 (1-\tau_1)^{\alpha}\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|d\tau_1. \tag{35}$$

By using the Young inequality as $xy \leq (1/p)x^p + (1/q)y^q$

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq \frac{1}{p}\int_0^1(1-\tau_1)^{\alpha p}d\tau_1+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1$$
$$+\frac{1}{p}\int_0^1\tau_1^{\alpha p}d\tau_1+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1$$
$$+\frac{1}{p}\int_0^1\tau_1^{\alpha p}d\tau_1+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1$$
$$+\frac{1}{p}\int_0^1(1-\tau_1)^{\alpha p}d\tau_1+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1. \tag{36}$$

By using convexity of $|f'|^q$ and by a simple computation, we have the desired result.□□ □

Corollary 15. *In Theorem 14, if we choose $\alpha = 1$, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\theta_1+\kappa_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\theta_2-\kappa_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right|$$
$$\leq \frac{2}{p^2+p}+\frac{|f'(\theta_1)|^q+|f'(\theta_2)|^q}{q}. \tag{37}$$

Theorem 16. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|$ is a concave for $q>1$, then we have*

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\left(\frac{1}{\alpha+1}\right)\left[\left|f'\left(\frac{\theta_1(\alpha+3)+\theta_2(\alpha+1)}{2(\alpha+2)}\right)\right|\right.$$
$$+\left|f'\left(\frac{\theta_1(2\alpha+3)+\theta_2}{2(\alpha+2)}\right)\right|+\left|f'\left(\frac{\theta_2(2\alpha+3)+\theta_1}{2(\alpha+2)}\right)\right|$$
$$\left.+\left|f'\left(\frac{\kappa_2(\alpha+3)+\theta_1(\alpha+1)}{2(\alpha+2)}\right)\right|\right], \tag{38}$$

where $\alpha\in[0,1]$ and $B(\alpha)$ is the normalization function.

Proof. From Lemma 7 and the Jensen integral inequality, we have

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)$$
$$\cdot\left|f'\left(\frac{\int_0^1(1-\tau_1)^{\alpha}(((1+\tau_1)/2)\theta_1+((1-\tau_1)/2)\theta_2)d\tau_1}{\int_0^1(1-\tau_1)^{\alpha}d\tau_1}\right)\right|$$
$$+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)\left|f'\left(\frac{\int_0^1\tau_1^{\alpha}(((1+\tau_1)/2)\theta_1+((1-\tau_1)/2)\theta_2)d\tau_1}{\int_0^1\tau_1^{\alpha}d\tau_1}\right)\right|$$
$$+\left(\int_0^1\tau_1^{\alpha}d\tau_1\right)\left|f'\left(\frac{\int_0^1\tau_1^{\alpha}(((1+\tau_1)/2)\theta_2+((1-\tau_1)/2)\theta_1)d\tau_1}{\int_0^1\tau_1^{\alpha}d\tau_1}\right)\right|$$
$$+\left(\int_0^1(1-\tau_1)^{\alpha}d\tau_1\right)$$
$$\cdot\left|f'\left(\frac{\int_0^1(1-\tau_1)^{\alpha}(((1+\tau_1)/2)\theta_2+((1-\tau_1)/2)\theta_1)d\tau_1}{\int_0^1(1-\tau_1)^{\alpha}d\tau_1}\right)\right|. \tag{39}$$

By computing the above integrals, we have

$$\left|\frac{2(\theta_2-\theta_1)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right.$$
$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\left(\frac{1}{\alpha+1}\right)\left[\left|f'\left(\frac{\theta_1(\alpha+3)+\theta_2(\alpha+1)}{2(\alpha+2)}\right)\right|\right.$$
$$+\left|f'\left(\frac{\theta_1(2\alpha+3)+\theta_2}{2(\alpha+2)}\right)\right|+\left|f'\left(\frac{\theta_2(2\alpha+3)+\theta_1}{2(\alpha+2)}\right)\right|$$
$$\left.+\left|f'\left(\frac{\kappa_2(\alpha+3)+\theta_1(\alpha+1)}{2(\alpha+2)}\right)\right|\right]. \tag{40}$$

So, the proof is completed.□□ □

Corollary 17. *In Theorem 16, if we choose α = 1, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\kappa_1+\theta_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\kappa_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right| \leq \left(\frac{1}{4}\right)\left[\left|f'\left(\frac{2\theta_1+\theta_2}{3}\right)\right|+\left|f'\left(\frac{5\theta_1+\theta_2}{6}\right)\right| + \left|f'\left(\frac{5\theta_2+\theta_1}{6}\right)\right|+\left|f'\left(\frac{2\theta_2+\theta_1}{3}\right)\right|\right]. \tag{41}$$

Theorem 18. *Let $f:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be differentiable function on (θ_1,θ_2) with $\kappa_1<\theta_2$ and $f'\in L_1[\theta_1,\theta_2]$. If $|f'|^q$ is a concave function, we have*

$$\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right| \leq \frac{2}{(\alpha p+1)^{1/p}}\left[\left|f'\left(\frac{3\theta_1+\theta_2}{4}\right)\right|+\left|f'\left(\frac{3\theta_2+\theta_1}{4}\right)\right|\right], \tag{42}$$

where $p^{-1}+q^{-1}=1$, $\alpha\in 0,1]$, and $q>1$.

Proof. By using the Lemma 7 and Hölder integral inequality, we can write

$$\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{(\theta_1+\theta_2)/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::(\theta1+\theta2)/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right|$$
$$\leq\left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\kappa_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q} + \left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1\right)^{1/q} + \left(\int_0^1\tau_1^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q} + \left(\int_0^1(1-\tau_1)^{\alpha p}d\tau_1\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1\right)^{1/q}. \tag{43}$$

By using concavity of $|f'|^q$ and Jensen integral inequality, we get

$$\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1 = \int_0^1\tau_1^0\left|f'\left(\frac{1+\tau_1}{2}\theta_1+\frac{1-\tau_1}{2}\theta_2\right)\right|^q d\tau_1 \leq \left(\int_0^1\tau_1^0 d\tau_1\right)\left|f'\left(\frac{\int_0^1\tau_1^0(((1+\tau_1)/2)\theta_1+((1-\tau_1)/2)\theta_2)d\tau_1}{\int_0^1\tau_1^0 d\tau_1}\right)\right|^q = \left|f'\left(\frac{3\theta_1+\theta_2}{4}\right)\right|^q. \tag{44}$$

Similarly,

$$\int_0^1\left|f'\left(\frac{1+\tau_1}{2}\theta_2+\frac{1-\tau_1}{2}\theta_1\right)\right|^q d\tau_1 \leq \left|f'\left(\frac{3\theta_2+\theta_1}{4}\right)\right|^q, \tag{45}$$

so we obtain

$$\left|\frac{2(\theta_2-\theta_1)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(\theta_2-\theta_1)^{\alpha+1}}\left[f(\kappa_1)+f(\theta_2)+2f\left(\frac{\theta_1+\theta_2}{2}\right)\right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(\theta_2-\kappa_1)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\theta_1+\theta_2/2}f(\theta_1)+{}^{AB}_{::::\theta_1}I^{\alpha}f\left(\frac{\kappa_1+\theta_2}{2}\right) + {}^{AB}_{::\theta_1+\theta_2/2}I^{\alpha}f(\theta_2)+{}^{AB}I^{\alpha}_{\theta_2}f\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right| \leq \frac{2}{(\alpha p+1)^{1/p}}\left[\left|f'\left(\frac{3\theta_1+\theta_2}{4}\right)\right|+\left|f'\left(\frac{3\theta_2+\theta_1}{4}\right)\right|\right]. \tag{46}$$

□□ □

Corollary 19. *In Theorem 18, if we choose α = 1, we obtain*

$$\left|\frac{f(\theta_1)+f(\theta_2)+2f((\kappa_1+\theta_2)/2)}{\theta_2-\theta_1}-\frac{4}{(\kappa_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}f(x)dx\right| \leq \frac{1}{(p+1)^{1/p}}\left[\left|f'\left(\frac{3\theta_1+\theta_2}{4}\right)\right|+\left|f'\left(\frac{3\theta_2+\theta_1}{4}\right)\right|\right]. \tag{47}$$

3. Conclusion

In this study, an integral identity including Atangana-Baleanu integral operators has been proved. Some integral inequalities are established by using Hölder inequality, power-mean inequality, Young inequality, and convex functions with the help of Lemma 7 which has the potential to produce Bullen type inequalities. Some special cases of the results in this general form have been pointed out. Researchers can establish new equations such as the integral identity in the study and reach similar inequalities of these equality-based inequalities.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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