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Regularity Criteria for the 3D Magneto-Hydrodynamics Equations in Anisotropic Lorentz Spaces

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Abstract: In this paper, we investigate the regularity of weak solutions to the 3D incompressible MHD equations. We provide a regularity criterion for weak solutions involving any two groups functions $(\partial_1 u_1, \partial_1 b_1)$, $(\partial_2 u_2, \partial_2 b_2)$ and $(\partial_3 u_3, \partial_3 b_3)$ in anisotropic Lorentz space.

Keywords: MHD equations; weak solution; regularity criteria; anisotropic Lorentz space

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1. Introduction

In this paper, we are concerned with regularity criteria for the weak solutions to the incompressible magneto-hydrodynamic (MHD) equations in $\mathbb{R}^3$ [1,2]:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - \Delta b = (b\cdot\nabla)u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), b(x,0) = b_0(x), \end{cases} \tag{1}$$

where $u = (u_1, u_2, u_3)$ is the fluid velocity field, $b = (b_1, b_2, b_3)$ is the magnetic field, p is a scalar pressure, and u_0, b_0 is the prescribed initial data satisfying the compatibility condition $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$ in the distributional sense. Physically, Equation (1) govern the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water.

Besides its physical applications, the MHD equations (1) have also mathematically significant. Duvaut and Lions [1] developed a global weak solution to (1) for initial data withfinite energy, that is,

$$u, b \in L^\infty\left(0, T; L^2(\mathbb{R}^3)\right) \cap L^2\left(0, T; H^1\left(\mathbb{R}^3\right)\right) \quad \text{for any} \quad T > 0.$$

It is well known that the issue of regularity for weak solutions to the 3D incompressible Navier- Stokes equations has been one of the most challenging open problem in mathematical fluid mechanics [3], as well as that for the 3D incompressible magneto-hydrodynamics (MHD) equations (see Sermange and Temam [2]). Many sufficient conditions (see e.g., [4–14] and the references therein) were derived to guarantee the regularity of the weak solution. He and Xin [15] first extended the classical Prodi-Serrin conditions of Navier-Stokes equations to the MHD equations, they obtained regularity criteria involving only on velocity u, i.e.,

$$u \in L^q\left(0, T; L^p\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \le 1 \quad \text{and} \quad 3 < p \le \infty \tag{2}$$

or

$$\nabla u \in L^q\Big(0,T;L^p\Big(\mathbb{R}^3\Big)\Big) \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty. \tag{3}$$

Later, He and Wang [16] showed that a weak solution (u,b) is regular, provided only $\nabla\omega^+ = (u+b)$ or $\nabla\omega^- = (u-b)$ belongs to Beirao da Veiga's class, that is,

$$\nabla\omega^+ \quad \text{or} \quad \nabla\omega^- \in L^q\Big(0,T;L^{p,\infty}\Big(\mathbb{R}^3\Big)\Big) \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}=2 \quad \text{and} \quad 3\le p\le\infty. \tag{4}$$

Ni et al. [17] showed that one of the following conditions hold

$$\begin{cases} \nabla_h u \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \\ \partial_3 b \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty. \end{cases} \tag{5}$$

$$\begin{cases} u_3 \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 1 \quad \text{and} \quad 3<p\le\infty, \\ \partial_3 u \in L^p(0,T;L^q(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \\ b_3 \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 1 \quad \text{and} \quad 3<p\le\infty, \\ \partial_3 b \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \end{cases} \tag{6}$$

$$\begin{cases} \nabla_h u \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \\ \nabla_h b \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \end{cases} \tag{7}$$

then the weak solution (u,b) is regular on $(0,T]$, where $\nabla_h = (\partial_1,\partial_2)$. Recently, Jia [18] showed that condition (7) can be replaced by

$$\begin{cases} \nabla_h \tilde{u} \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \\ \nabla_h \tilde{b} \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}\le 2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \end{cases} \tag{8}$$

where $\tilde{f} = (f_1, f_2)$. Regularity condition (8) was further improved by Xu et al. [19], more precisely, they proved that if any two quantities of

$$\begin{cases} A_i^{q,p}(T) := \partial_i u_i \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}=2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \\ B_i^{q,p}(T) := \partial_i b_i \in L^q(0,T;L^p(\mathbb{R}^3)) & \text{with} \quad \frac{2}{q}+\frac{3}{p}=2 \quad \text{and} \quad \frac{3}{2}<p\le\infty, \end{cases} \tag{9}$$

where $i = 1,2,3$, then the solution is smooth on interval (0,T]. For readers interested in this topic for partial components, please refer to [20–26] for recent progresses.

Motivated by papers cited above, the aim of this article is to study the regularity of weak solutions for the 3D MHD equations (1) in term of the two partial derivative of the velocity components and magnetic components on framework of the anisotropic Lorentz space. Before stating our main Theorem, we shall first recall the definitions of some function spaces [27].

Lorentz Spaces

Given a measurable function $f:\mathbb{R}^n\to\mathbb{R}$ define the distribution function of f by

$$d_f(\alpha) = \mu(\{x : |f(x)| > \alpha\}),$$

where $\mu(A)$ (or $|A|$) denotes the Lebesgue measure of a set A. We now define its decreasing rearrangement $f^* : [0,\infty) \to [0,\infty]$ as

$$f^*(t) = \inf\Big\{\alpha : d_f(\alpha) \le t\Big\},$$

with the convention that $\inf \emptyset = \infty$. The point of this definition is that f and f^* have the same distribution function,

$$d_{f^*}(\alpha) = d_f(\alpha),$$

but f^* is a positive non-increasing scalar function.

Definition 1. *Let $(p,q) \in [1,\infty]^2$, the Lorentz space $L^{p,q}(\mathbb{R}^3)$ consists of all measurable functions f for which the quantity*

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & q < \infty, \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & q = \infty, \end{cases}$$

is finite.

In order to give the following definition involving anisotropic Lorentz space, we denote $f = f(x_1, x_2, x_3)$ be a measurable function defined on $\mathbb{R}^3$, $f^*(t) = f^{*_1,*_2,*_3}(t_1, t_2, t_3)$. Here $f^{*_1,*_2,*_3}(t_1, t_2, t_3)$ is the multivariate decreasing rearrangement of $f(x_1, x_2, x_3)$ obtained by applying decreasing rearrangement $f^{*_1}(t_1, x_2, x_3)$ of $f(x_1, x_2, x_3)$ relating to the first variable x_1, under fixed the second, the third variables x_2, x_3, and then applying decreasing rearrangement $f^{*_1,*_2}(t_1, t_2, x_3)$ of $f^{*_1}(t_1, x_2, x_3)$ with respect to the second variable x_2 under fixed the first variable t_1 of $f^{*_1}(t_1, x_2, x_3)$ and variable x_3, finally for variable x_3, by the same trick, we obtain the multivariate decreasing rearrangement $f^{*_1,*_2,*_3}(t_1, t_2, t_3)$.

Recently, many works have been done for mixed-norm spaces. Stefanov-Torres [28] obtained the boundedness of Calderón-Zygmund operators on mixed-norm Lebesgue spaces. Georgiadis et al. [29] obtained various properties of anisotropic Triebel-Lizorkin spaces with mixed norms. In [30], Chen-Sun introduced the iterated weak and weak mixed-norm spaces and given some applications to geometric inequalities.

Definition 2. *Let multi indexes $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3)$ be such that if $0 < p_i < \infty$, then $0 < q_i \le \infty$, and if $p_i = \infty$, then $q_i = \infty$ for every $i = 1,2,3$ [31]. An anisotropic Lorentz space $L^{p_1,q_1}(\mathbb{R}_{x_1}; L^{p_2,q_2}(\mathbb{R}_{x_2}; L^{p_3,q_3}(\mathbb{R}_{x_3})))$ is the set of functions for which the following norm is finite:*

$$\Big\|\Big\|\|f\|_{L^{p_1,q_1}_{x_1}}\Big\|_{L^{p_2,q_2}_{x_2}}\Big\|_{L^{p_3,q_3}_{x_3}} := \left(\int_0^\infty \left(\int_0^\infty \left(\int_0^\infty [t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} t_3^{\frac{1}{p_3}} f^{*_1,*_2,*_3}(t_1,t_2,t_3)]^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2}\right)^{\frac{q_3}{q_2}} \frac{dt_3}{t_3}\right)^{\frac{1}{q_3}}.$$

Now, our main result reads:

Theorem 1. *Suppose that $(u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in distributional sense. Let (u, b) be the Leray-Hopf weak solution of (1) on $(0, T]$. If any two quantities*

$$\begin{cases} A_i(T) := \int_0^T \Big\|\Big\|\|\partial_i u_i(t)\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}} dt, \\ B_i(T) := \int_0^T \Big\|\Big\|\|\partial_i b_i(t)\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}} dt, \end{cases} \tag{10}$$

are finite, where $i = 1,2,3$ with $2 < p, q, r \le \infty$ and $1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) \ge 0$, then the weak solution (u, b) is actually smooth on interval $(0, T]$.

Remark 1. *While $L^p(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3)$, clearly $L^{p,\infty}$ is a larger space than L^p. Therefore, from this point of view, condition (10) can be regarded as an extension of (7)–(9). In addition, our regularity criteria only depends on any two groups functions of $(\partial_1 u_1, \partial_1 b_1)$, $(\partial_2 u_2, \partial_2 b_2)$ and $(\partial_3 u_3, \partial_3 b_3)$. Hence, (10) can be as a significant improvement of condition (7) and (8). In addition, when $b = 0$, it is note that Theorem 1 is also new to the incompressible Navier-Stokes equations.*

Remark 2. *According to embedding relation $L^p(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3)$, we can obtain the following regularity criteria on framework of anisotropic Lebesgue space,*

$$\begin{cases} A_i(T) := \int_0^T \left\| \left\| \|\partial_i u_i(t)\|_{L^p_{x_1}} \right\|_{L^q_{x_2}} \right\|_{L^r_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}} dt < \infty, \\ B_i(T) := \int_0^T \left\| \left\| \|\partial_i b_i(t)\|_{L^p_{x_1}} \right\|_{L^q_{x_2}} \right\|_{L^r_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}} dt < \infty, \end{cases} \tag{11}$$

where we should point out that for Equation (1), the regularity criterion (11) still new.

Remark 3. *Notice that when fix $p = q = r$ in condition (11), the conditions (9) naturally turn out as stated in [19]. Furthermore, if let $p = q = r$ in condition (10), it is not difficult to find that our result improves the condition (4) significantly. Hence, regularity criteria (10) or (11) is much better. In other words, Theorem 1 can be regarded as a generalization of [16,18,19,23].*

Before ending this section, we state the following lemmas, which will be used in the proof of our main result.

Lemma 1. *(Young's Inequality for Lorentz Spaces [32,33]) Let $1 < p < \infty, 1 \le q \le \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1, \frac{1}{q'} + \frac{1}{q'} = 1$. Suppose as well that $1 < p_1 < p'$ and $q' \le q \le \infty$. If $\frac{1}{p_2} + 1 = \frac{1}{p} + \frac{1}{p_1}$ and $\frac{1}{q_2} = \frac{1}{q} + \frac{1}{q_1}$, then the convolution operator,*

$$* : L^{p,q}(\mathbb{R}^n) \times L^{p_1,q_1}(\mathbb{R}^n) \mapsto L^{p_2,q_2}(\mathbb{R}^n)$$

is a bounded bilinear operator.

Lemma 2. *(Hölder's inequality in Lorentz spaces [33]) If $1 \le p_1, p_2, q_1, q_2 \le \infty$, then for any $f \in L^{p_1,q_1}(\mathbb{R}^n), g \in L^{p_2,q_2}(\mathbb{R}^n)$,*

$$\|fg\|_{L^{p,q}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^n)},$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

For any $s \ge 0$, even if s not an integer, we can define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$:

$$\dot{H}^s(\mathbb{R}^n) = \{f \in \mathcal{S}' : \hat{f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{f(\xi)}|^2 d\xi < \infty\}$$

with the natural norm

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{f(\xi)}|^2 d\xi\right)^{\frac{1}{2}},$$

where $\mathcal{S}'$ denotes the space of the tempered distributions on $\mathbb{R}^n$.

Lemma 3. *For $2 < p < \infty$, there exists a constant $C = C(p)$ such that $f \in \dot{H}^{\frac{1}{p}}(\mathbb{R})$, then $f \in L^{\frac{2p}{p-2},2}(\mathbb{R})$ and*

$$\|f\|_{L^{\frac{2p}{p-2},2}} \le C\|f\|_{\dot{H}^{\frac{1}{p}}}. \tag{12}$$

Proof. We first make the pointwise definition, $\gamma(\xi) = |\xi|^{\frac{1}{p}} \hat{f}(\xi)$; since $f \in \dot{H}^{\frac{1}{p}}(\mathbb{R})$, $\gamma \in L^2(\mathbb{R})$. If we set $g = \mathcal{F}^{-1}\gamma$, then $g \in L^2(\mathbb{R})$ and $\|g\|_{L^2} = \|\gamma\|_{L^2} = \|f\|_{\dot{H}^{\frac{1}{p}}}$. Now,

$$\hat{f}(\xi) = \frac{|\xi|^{\frac{1}{p}} \hat{f}(\xi)}{|\xi|^{\frac{1}{p}}} = \hat{g}(\xi)|\xi|^{-\frac{1}{p}}.$$

Combining the fact that if $P_\alpha(x) = |x|^{-\alpha}$, then $\widehat{P_\alpha(\xi)} = C_\alpha P_{1-\alpha}(\xi)$. Thus we obtain $f = g * C^{-1}_{1-\frac{1}{p}} P_{1-\frac{1}{p}}$. The function $P_{1-\frac{1}{p}} = |x|^{-\frac{p-1}{p}} \in L^{\frac{p}{p-1},\infty}(\mathbb{R})$ but not in $L^{\frac{p}{p-1}}(\mathbb{R})$. Applying Lemma 1, we find that

$$\begin{aligned} \|f\|_{L^{\frac{2p}{p-2},2}} &= \left\| g * C^{-1}_{1-\frac{1}{p}} P_{1-\frac{1}{p}} \right\|_{L^{\frac{2p}{p-2},2}} \\ &\le C\|g\|_{L^2} \left\| |x|^{-\frac{p-1}{p}} \right\|_{L^{\frac{p}{p-1},\infty}} \le C\|f\|_{\dot{H}^{\frac{1}{p}}}. \end{aligned} \tag{13}$$

□

Lemma 4. *There exists a positive constant C such that*

$$\left\| \left\| \|f\|_{L^{\frac{2p}{p-2},2}_{x_1}} \right\|_{L^{\frac{2q}{q-2},2}_{x_2}} \right\|_{L^{\frac{2r}{r-2},2}_{x_3}} \le C\|\partial_1 f\|^{\frac{1}{p}}_{L^2} \|\partial_2 f\|^{\frac{1}{q}}_{L^2} \|\partial_3 f\|^{\frac{1}{r}}_{L^2} \|f\|^{1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}_{L^2}, \tag{14}$$

for every $f \in C_0^\infty(\mathbb{R}^3)$ *where* $2 < p, q, r \le \infty, 1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) \ge 0$.

Proof. Let Λ_1^p be the Fourier multiplier defined as

$$\mathcal{F}_1\left(\Lambda_1^p f\right)(\xi_1, x_2, x_3) = |\xi_1|^p \mathcal{F}_1 f(\xi_1, x_2, x_3)$$

with

$$\mathcal{F}_1 f(\xi_1, x_2, x_3) = \int_{\mathbb{R}} e^{-i\xi_1 x_1} f(x_1, x_2, x_3) \mathrm{d}x_1,$$

Λ_2^p and Λ_3^p can be defined analogously. Then by Lemma 3, Minkowski's inequality and Hölder's inequality to obtain

$$\begin{aligned} \left\| \left\| \|f\|_{L^{\frac{2p}{p-2},2}_{x_1}} \right\|_{L^{\frac{2q}{q-2},2}_{x_2}} \right\|_{L^{\frac{2r}{r-2},2}_{x_3}} &\le C \left\| \left\| \left\| \Lambda_1^{\frac{1}{p}} f \right\|_{L^2_{x_1}} \right\|_{L^{\frac{2q}{q-2},2}_{x_2}} \right\|_{L^{\frac{2r}{r-2},2}_{x_3}} \le \left\| \left\| \left\| \Lambda_1^{\frac{1}{p}} f \right\|_{L^{\frac{2q}{q-2},2}_{x_2}} \right\|_{L^2_{x_1}} \right\|_{L^{\frac{2r}{r-2},2}_{x_3}} \\ &\le C \left\| \left\| \Lambda_2^{\frac{1}{q}} \Lambda_1^{\frac{1}{p}} f \right\|_{L^2_{x_1,x_2}} \right\|_{L^{\frac{2r}{r-2},2}_{x_3}} \le C \left\| \left\| \Lambda_2^{\frac{1}{q}} \Lambda_1^{\frac{1}{p}} f \right\|_{L^{\frac{2r}{r-2},2}_{x_3}} \right\|_{L^2_{x_1,x_2}} \\ &\le C \left\| \Lambda_3^{\frac{1}{r}} \Lambda_2^{\frac{1}{q}} \Lambda_1^{\frac{1}{p}} f \right\|_{L^2}. \end{aligned} \tag{15}$$

Combining the Fourier-Plancherel formula and the Hölder's inequality, we have

$$
\begin{aligned}
&C\left\|\Lambda_3^{\frac{1}{r}}\Lambda_2^{\frac{1}{q}}\Lambda_1^{\frac{1}{p}}f\right\|_{L^2}\le C\left(\int_{\mathbb{R}^3}|\xi_1|^{\frac{2}{p}}|\xi_2|^{\frac{2}{q}}|\xi_3|^{\frac{2}{r}}|\mathcal{F}f(\xi_1,\xi_2,\xi_3)|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)^{\frac{1}{2}}\\
&=C\left(\int_{\mathbb{R}^3}|\xi_1|^{\frac{2}{p}}|\mathcal{F}f(\xi)|^{\frac{2}{p}}|\xi_2|^{\frac{2}{q}}|\mathcal{F}f(\xi)|^{\frac{2}{q}}|\xi_3|^{\frac{2}{r}}|\mathcal{F}f(\xi)|^{\frac{2}{r}}|\mathcal{F}f(\xi)|^{2-\left(\frac{2}{p}+\frac{2}{q}+\frac{2}{r}\right)}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)^{\frac{1}{2}}\\
&\le C\|\mathcal{F}f\|_{L^2}^{1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}}\left(\int_{\mathbb{R}^3}|\xi_1|^2|\mathcal{F}f|^2\mathrm{d}\xi\right)^{\frac{1}{2p}}\left(\int_{\mathbb{R}^3}|\xi_2|^2|\mathcal{F}f|^2\mathrm{d}\xi\right)^{\frac{1}{2q}}\left(\int_{\mathbb{R}^3}|\xi_3|^2|\mathcal{F}f|^2\mathrm{d}\xi\right)^{\frac{1}{2r}}\\
&\le C\|\partial_1 f\|_{L^2}^{\frac{1}{p}}\|\partial_2 f\|_{L^2}^{\frac{1}{q}}\|\partial_3 f\|_{L^2}^{\frac{1}{r}}\|f\|_{L^2}^{1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}.
\end{aligned}
\tag{16}
$$

□

Remark 4. *In fact, since* $L^{\frac{2p}{p-2},2}\hookrightarrow L^{\frac{2p}{p-2},\frac{2p}{p-2}}$ *for* $2<p<\infty$*, we have similar result for estimate* (14) *in anisotropic Lebesgue space (for more details refer to [34]). However, we should point out that Lemma 4 holds in Lorentz space mainly depends on the Sobolev's embedding in Lemma 3.*

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. The proof is based on the establishment of a priori estimates under condition (10).

Firstly, we note that, by the energy inequality, for weak solution (u,b), we have

$$
\|u\|_{L^2}^2+\|b\|_{L^2}^2+2\int_0^T\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2dt\le\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2. \tag{17}
$$

Next, let us convert (1) into a symmetric form. Writing

$$
\omega^{\pm}=u\pm b,
$$

we find by adding and subtracting $(1)_1$ with $(1)_2$,

$$
\begin{cases}
\partial_t\omega^+ +(\omega^-\cdot\nabla)\omega^+ -\Delta\omega^+ +\nabla p=0,\\
\partial_t\omega^- +(\omega^+\cdot\nabla)\omega^- -\Delta\omega^- +\nabla p=0,\\
\nabla\cdot\omega^+=\nabla\cdot\omega^-=0,\\
\omega^+(0)=\omega_0^+\equiv u_0+b_0,\quad \omega^-(0)=\omega_0^-\equiv u_0-b_0.
\end{cases}
\tag{18}
$$

Taking the inner product of the i-th equation of $(18)_1$ with $|\omega_i^+|^2\omega_i$ and $(18)_2$ with $|\omega_i^-|^2\omega_i$ (for i = 1, 2, 3) and integrating by parts in $\mathbb{R}^3$ to get

$$
\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega_i^+\|_{L^4}^4+\|\omega_i^-\|_{L^q}^q\right)+\frac{1}{2}\left(\left\|\nabla|\omega_i^+|^2\right\|_{L^2}^2+\left\|\nabla|\omega_i^-|^2\right\|_{L^2}^2\right)\\
&+\||\omega_i^+|\cdot|\nabla\omega_i^+|\|_{L^2}^2+\||\omega_i^-|\cdot|\nabla\omega_i^-|\|_{L^2}^2\\
&=-\int_{\mathbb{R}^3}\partial_i p|\omega_i^+|^2\omega_i^+\mathrm{d}x-\int_{\mathbb{R}^3}\partial_i p|\omega_i^-|^2\omega_i^-\mathrm{d}x\equiv I+J,
\end{aligned}
\tag{19}
$$

we consider the (u,b) satisfying condition (10) with any two quantities of $A_i(T)$ and $B_i(T)$ for $(i=1,2,3)$:

$$
\begin{cases}
A_i(T):=\int_0^T\left\|\left\|\|\partial_i u_i(t)\|_{L_{x_1}^{p,\infty}}\right\|_{L_{x_2}^{q,\infty}}\right\|_{L_{x_3}^{r,\infty}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}dt<\infty,\\
B_i(T):=\int_0^T\left\|\left\|\|\partial_i b_i(t)\|_{L_{x_1}^{p,\infty}}\right\|_{L_{x_2}^{q,\infty}}\right\|_{L_{x_3}^{r,\infty}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}dt<\infty.
\end{cases}
$$

In order to estimate the term I and J of (19), let us first establish an estimate between the p and the ω. Taking the divergence operator $\nabla\cdot$ on both sides of the first equations of (18), it follows that

$$-\Delta p = \mathrm{div}\big(w^- \cdot \nabla w^+\big) = \mathrm{div}\,\mathrm{div}\big(w^- \otimes w^+\big).$$

Similarly, taking ∇div operator on both sides of the first equation of (18) to obtain

$$-\Delta(\nabla p) = \nabla\,\mathrm{div}\big(w^- \cdot \nabla w^+\big) = \nabla\,\mathrm{div}\big(w^+ \cdot \nabla w^-\big).$$

By using the boundedness of Riesz transformations in L^p $(1 < p < \infty)$ space, so we have

$$\begin{cases} \|p\|_{L^p} \le C\|w^+\|_{L^{2p}}\|w^-\|_{L^{2p}}, \\ \|\nabla p\|_{L^p} \le C\|w^+ \cdot \nabla w^-\|_{L^p}, \\ \|\nabla p\|_{L^p} \le C\|w^- \cdot \nabla w^+\|_{L^p}. \end{cases} \tag{20}$$

Using the Hölder's inequality, Young's inequality, Lemma 4 and (20), we can deduce that

$$\begin{aligned} I =& -\int_{\mathbb{R}^3} \partial_i p|\omega_i^+|^2\omega_i^+ dx \le C\left|\int_{\mathbb{R}^3} p|\omega_i^+|^2\partial_i\omega_i^+ \mathrm{d}x\right| \\ \le& \left\|\left\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}} \left\|\left\|\|p\|_{L^{\frac{2p}{p-2},2}_{x_1}}\right\|_{L^{\frac{2q}{q-2},2}_{x_2}}\right\|_{L^{\frac{2r}{r-2},2}_{x_3}} \left\||\omega_i^+|^2\right\|_{L^2} \\ \le& C\left\|\left\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}} \|\partial_1 p\|_{L^2}^{\frac{1}{p}}\|\partial_2 p\|_{L^2}^{\frac{1}{q}}\|\partial_3 p\|_{L^2}^{\frac{1}{r}}\|p\|_{L^2}^{1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}\||\omega_i^+|^2\|_{L^2} \\ \le& C\left\|\left\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}} \|\nabla p\|_{L^2}^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}}\|p\|_{L^2}^{1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}\||\omega_i^+|^2\|_{L^2} \\ \le& C\left\|\left\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}} \|\nabla p\|_{L^2}^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}}\|\omega^+\|_{L^4}^{3-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}\|\omega^-\|_{L^4}^{1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)} \\ \le& \epsilon\Big(\|w^+ \cdot \nabla w^-\|_{L^2}^2 + \|w^- \cdot \nabla w^+\|_{L^2}^2\Big) + C\left\|\left\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}} \Big(\|\omega^+\|_{L^4}^4 + \|\omega^-\|_{L^4}^4\Big). \end{aligned} \tag{21}$$

Similarly, for J, we have

$$\begin{aligned} J =& -\int_{\mathbb{R}^3} \partial_i p|\omega_i^-|^2\omega_i^+ dx \le C\left|\int_{\mathbb{R}^3} p|\omega_i^-|^2\partial_i\omega_i^- \mathrm{d}x\right| \\ \le& \epsilon\Big(\|w^+ \cdot \nabla w^-\|_{L^2}^2 + \|w^- \cdot \nabla w^+\|_{L^2}^2\Big) + C\left\|\left\|\|\partial_i\omega_i^-\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}} \Big(\|\omega^+\|_{L^4}^4 + \|\omega^-\|_{L^4}^4\Big). \end{aligned} \tag{22}$$

Inserting (21) and (22) into (19) and summing up with respect to the index i from 1 to 3, we get

$$
\begin{aligned}
&\frac{1}{4}\Big(\|\omega^+\|_{L^4}^4+\|\omega^-\|_{L^4}^4\Big)+\frac{1}{2}\int_0^t\Big(\Big\|\nabla|\omega^+|^2\Big\|_{L^2}^2+\Big\|\nabla|\omega^-|^2\Big\|_{L^2}^2\Big)ds\\
&+\int_0^t\Big(\||\omega^+|\cdot|\nabla\omega^+|\|_{L^2}^2+\||\omega^-|\cdot|\nabla\omega^-|\|_{L^2}^2\Big)ds\\
\leq& C\int_0^t\sum_{i=1}^3\left(\Big\|\Big\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}+\Big\|\Big\|\|\partial_i\omega_i^-\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}\right)\\
&\cdot\Big(\|\omega^+\|_{L^4}^4+\|\omega^-\|_{L^4}^4\Big)ds+\epsilon\int_0^t\Big(\|w^+\cdot\nabla w^-\|_{L^2}^2+\|w^-\cdot\nabla w^+\|_{L^2}^2\Big)ds+C\Big(\|\omega_0^+\|_{L^4}^4+\|\omega_0^-\|_{L^4}^4\Big),
\end{aligned}
\tag{23}
$$

where we have used that for any $p\geq 1$ and some constant $C_{\gamma,p}>0$,

$$
C_{\gamma,p}^{-1}\|u\|_{L^p}^{\gamma}\leq\sum_{i=1}^3\|u_i\|_{L^p}^{\gamma}\leqslant C_{\gamma,p}\|u\|_{L^p}^{\gamma}.
$$

Due to the fact

$$
\Big|\nabla|w^+|^2\Big|\leq 2|w^+||\nabla w^+|
$$

and the inequality

$$
\|u(t)\|_{L^4}\leq\frac{1}{2}\big(\|w^+(t)\|_{L^4}+\|w^-(t)\|_{L^4}\big),
$$

$$
\|b(t)\|_{L^4}\leq\frac{1}{2}\big(\|w^+(t)\|_{L^4}+\|w^-(t)\|_{L^4}\big).
$$

We rewrite inequality (23) as follows

$$
\begin{aligned}
&\frac{1}{4}\Big(\|u(t)\|_{L^4}^4+\|b(t)\|_{L^4}^4\Big)+\frac{1}{4}\int_0^t\Big(\Big\|\nabla|u|^2\Big\|_{L^2}^2+\Big\|\nabla|b|^2\Big\|_{L^2}^2\Big)ds\\
&+\int_0^t\Big(\||u|\cdot|\nabla u|\|_{L^2}^2+\||u|\cdot|\nabla b|\|_{L^2}^2+\||b|\cdot|\nabla u|\|_{L^2}^2+\||b|\cdot|\nabla b|\|_{L^2}^2\Big)ds\\
\leq& C\int_0^t\sum_{i=1}^3\left(\Big\|\Big\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}+\Big\|\Big\|\|\partial_i\omega_i^-\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}\right)\\
&\cdot\Big(\|u\|_{L^4}^4+\|b\|_{L^4}^4\Big)ds+\epsilon\int_0^t\Big(\|u\cdot\nabla u\|_{L^2}^2+\|b\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla b\|_{L^2}^2+\|b\cdot\nabla b\|_{L^2}^2\Big)ds\\
&+C\Big(\|\omega_0^+\|_{L^4}^4+\|\omega_0^-\|_{L^4}^4\Big),
\end{aligned}
\tag{24}
$$

and hence we get

$$
\begin{aligned}
&\frac{1}{4}\Big(\|u(t)\|_{L^4}^4+\|b(t)\|_{L^4}^4\Big)+\frac{1}{4}\int_0^t\Big(\Big\|\nabla|u|^2\Big\|_{L^2}^2+\Big\|\nabla|b|^2\Big\|_{L^2}^2\Big)ds\\
&+\frac{1}{4}\int_0^t\Big(\||u|\cdot|\nabla u|\|_{L^2}^2+\||u|\cdot|\nabla b|\|_{L^2}^2+\||b|\cdot|\nabla u|\|_{L^2}^2+\||b|\cdot|\nabla b|\|_{L^2}^2\Big)ds\\
\leq& C\int_0^t\sum_{i=1}^3\left(\Big\|\Big\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}+\Big\|\Big\|\|\partial_i\omega_i^-\|_{L^{p,\infty}_{x_1}}\Big\|_{L^{q,\infty}_{x_2}}\Big\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}\right)\\
&\cdot\Big(\|u\|_{L^4}^4+\|b\|_{L^4}^4\Big)ds+C\Big(\|u_0\|_{L^4}^4+\|b_0\|_{L^4}^4\Big).
\end{aligned}
\tag{25}
$$

Applying the Gronwall's inequality to obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|u(t)\|_{L^4}^4+\|b(t)\|_{L^4}^4\right)\\
\le& C\exp C\int_0^T\sum_{i=1}^3\left(\left\|\left\|\|\partial_i\omega_i^+\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}+\left\|\left\|\|\partial_i\omega_i^-\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}\right)dt\\
\le& C\exp C\int_0^T\sum_{i=1}^3\left(\left\|\left\|\|\partial_i u_i\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}+\left\|\left\|\|\partial_i b_i\|_{L^{p,\infty}_{x_1}}\right\|_{L^{q,\infty}_{x_2}}\right\|_{L^{r,\infty}_{x_3}}^{\frac{2}{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)}}\right)dt\\
<&\infty.
\end{aligned}
\tag{26}
$$

Since

$$u,b\in L^\infty\left(0,T;L^4\left(\mathbb{R}^3\right)\right)\subset L^8\left(0,T;L^4\left(\mathbb{R}^3\right)\right),$$

combining the classical Serrin-type regularity criterion (2), as in [15], then we complete the proof of Theorem 1.

3. Conclusions

This paper studies the MHD equations, and obtains the a regularity criterion only involving the partial components of the ∇u and ∇b. In addition, the anisotropic Lorentz space used in this article is broader than the general Lebesgue and Lorentz spaces. It seems that a slightly modified the technique in Theorem 1 can be applied to other incompressible fluid equations such as micropolar equations and the magneto-micropolar equations.

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