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An optimization model for the management of green areas

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Abstract

In this paper, we present an optimization model for the management of the green area in order to find the optimal green surface to absorb CO_2 emissions of industrialized cities. We obtain a minimization problem and the related variational inequality. We study the Lagrange theory to better understand the process that regulates the possible increase in green space. Then, we propose a computational procedure, based on the Euler method, to find the optimal solution to the variational inequality associated with our minimization problem and, finally, some numerical examples based on real scenarios.

Keywords: CO_2 emissions; green area; Lagrange theory; variational inequalities

1. Introduction

Global warming, especially in recent decades, has been and continues to be one of the most worrying problems to solve. According to United States Environmental Protection Agency (see United States Environmental Protection Agency), Carbon dioxide (CO_2) is the primary greenhouse gas emitted through human activities. The progressive increase of the world population and all the related activities, such as energy consumption, transports, industrial activities, will cause an increase in CO_2 concentrations, and not only, in the atmosphere. According to IPCC, Intergovernmental Panel on Climate Change, most of the observed increase in globally averaged temperatures since the mid-20th century is very likely due to the observed increase in anthropogenic greenhouse gas concentrations (see Intergovernmental Panel on Climate Change, 2014). CO_2-increasing concentrations cause global warming followed by climate change and a progressive environmental degradation. Moreover, climate change could increase environmental risks for people, assets, economies, and ecosystems, including risks from heat stress, storms and extreme precipitation, air pollution, melting glaciers, sea level rise, extinctions, and reduced water resources.

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The gradual increase in scientific data available on global warming has fueled a growing political debate since the 1970s that led the major Institutions to consider as a priority also the containment of greenhouse gas emissions and the use of alternative and renewable energy sources. The main international agreement for the control of global warming is the Kyoto Protocol, an amendment of the United Nations Framework Convention on Climate Change negotiated in 1997, covering 180 countries and more than 55% of global greenhouse gas emissions (see UNFCCC, 1997). More recently, in 2015, all United Nations countries negotiated the Paris Agreement, which aims at containing the increase in the average global temperature below the 2°C threshold beyond preindustrial levels and to limit this increase to 1.5°C, as this would substantially reduce the risks and effects of climate change. The content of the agreement was negotiated by representatives of 196 states at the 21st Conference of the Parties of the UNFCCC (see UNFCCC, 2015).

Urban pollution is one of the main causes of the increase in greenhouse gas concentrations in the atmosphere. A strategy to mitigate the environmental effects of urban pollution is to increase green areas (see, for instance, Gill et al., 2007; Gómez et al., 2011; Yoon et al., 2019).

In general, the three components of sustainable development are economy, environment, and society. So, sustainable urban growth is related not only to the ecological aspect, but also the social and societal ones. The development of cities has caused many consequences such as environmental degradation, loss of natural habitat, local climate changes, the increase in the level of air, and noise pollution. The presence of green spaces is fundamental to reduce the effects of urbanization and improve the quality of life, by providing people with natural settings for leisure and recreation, and by safeguarding the quality of precious resources such as air and water. Local governments involve local communities in the decision-making process and, finally, the local community benefits most directly from a greening project and will confirm how successful the project is.

In many cities, in order to reduce the emission of CO_2, some strategies have been planned aimed at improving air quality for the health of the population. In urban areas, administrations are enhancing infrastructure and transport, thereby promoting the movement through public transport and the sharing of mobility (car sharing) and reducing the consumption of private cars. Other effective policies are to promote the purchase of zero-emission vehicles or to allow the movement of vehicles with alternate plates. Sometimes the use of mitigation strategies after reaching critical pollution levels may not resolve the pollution emergency. For this reason, it is preferable to proceed toward a preventive approach to emergency by promoting effective measures before reaching critical levels of pollution. Therefore, the authorities should promote new technologies for monitoring air pollution such as air quality monitoring networks. With the monitoring of air quality, if a particular pollutant has exceeded the threshold value established by law, it is necessary to act with appropriate strategies to mitigate the problem of air pollution. That is why increasing public space that can be used to create new parks and open spaces for both recreational and commercial purposes is often a very good solution. Furthermore, the creation of the green barriers along the fast roads allows to reduce the pollution levels that reach the inhabitants both in the street and in their homes. Indeed, several studies have shown that the plant barriers, suitably located, act as real biological filters, removing gaseous components from the air.

Of course, all this process involves costs for acquiring new spaces, for training personnel, for converting the land in green areas, for managing pre-existent types of green areas, and our purpose is to find the optimal amount of total green areas in every city, the optimal urban flow, and the optimal number of personnel to be employed in the managing of green spaces, under the request that the quantity of CO_2 absorbed by the total green area in every city exceeds the total emissions of that city.

This paper is organized as follows. In Section 2, we present the model and we introduce the cost functions associated with acquisition, transformation, management, and maintenance of a green area. We determine optimality conditions for an external institution that has the responsibility to manage green areas and we derive the associated variational inequality. Then, we give conditions of existence and uniqueness of the solutions to the variational inequality. In Section 3, we study the Lagrange theory related to the model in order to better understand the behavior of green areas adjustment process, providing an interpretation of the Lagrange multipliers. In Section 4, we recall the Euler method that we apply in the successive Section 5 to solve numerical examples that describe real scenarios of application of our model. Section 6 is dedicated to the conclusions.

2. The mathematical model

The optimal green area model we are examining consists of n cities, with a typical one denoted by i. Unlike Indrawati et al. (2014), we consider different types of green areas depending on their location and their efficacy in absorbing CO_2. Such hypothesis is justified by the fact that different types of green areas require different maintenance and administration works as well as produce different CO_2 absorption potential (see, for example, Azaria et al., 2018).

Therefore, we consider m different types of green areas, with a typical one denoted by j. Let $x_{ij} \in \mathbb{R}_+$ be the decision variable indicating the surface of green area of type j in city i (expressed in km^2), with $i = 1, \dots, n$ and $j = 1, \dots, m$. For a given city i, $i = 1, \dots, n$, we group the different types of green area into the vector $x_i \in \mathbb{R}^m_+$ and we group all such vectors into the vector $x \in \mathbb{R}^{nm}$. Moreover, we denote by $\underline{x}_{ij} \in \mathbb{R}_+$ the parameter representing the pre-existent surface of green area of type j in city i (expressed in km^2), $j = 1, \dots, m$ and $i = 1, \dots, n$. We suppose that the management of green area in every city is a responsibility of an external institution, for example, a region and a province.

Indonesian law (see Indrawati et al., 2014) requires that green area of Jakarta be at least 30% of the city's total area. Hence, it is plausible to assume that the optimal space of green area is not less than a quantity imposed by law and, if in a country such a constraint does not exist, then the minimum quantity of green area imposed by law corresponds to the pre-existing one. Anyway, we can assume that the optimal space of green area is not less than the maximum between the above two quantities. For a given city i, $i = 1, \dots, m$, l_i denotes the parameter that represents the minimum surface of green area imposed by the law and that m_i denotes the quantity:

$$m_i := \max\left\{ l_i, \sum_{j=1}^{m} \underline{x}_{ij} \right\}, \quad \forall i = 1, \dots, n. \tag{1}$$

Therefore, the following condition has to be satisfied:

$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \geq m_i, \quad \forall i = 1, \dots, n. \tag{2}$$

Moreover, we impose that the total surface of green area in every city i is less than a fixed parameter u_i related to the total area of the city i , $i = 1, \ldots, n$, that is,

$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \leq u_i, \quad \forall i = 1, \ldots, n. \tag{3}$$

We can rewrite inequality constraints (2) and (3) as follows:

$$m_i \leq \sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \leq u_i, \quad \forall i = 1, \ldots, n.$$

The continuous increase in the population and the presence of industrial plants are the main causes of the increase in the concentration of CO_2 in the atmosphere. So, it is often necessary to enlarge the urban green areas in order to compensate for the massive emission of greenhouse gases produced by industrial plants and urban circulation. For a given city i, $i = 1, \ldots, n$, $f_i \in \mathbb{R}_+$ indicates the decision variable indicating the flow of urban circulation in that city. We group these quantities into the n-dimensional vector $f \in \mathbb{R}^n_+$.

Let c^a_{ij} be the transaction cost associated with the expansion of green area of type j in city i and we assume c^a_{ij} as a function of x_{ij}:

$$c^a_{ij} = c^a_{ij}(x_{ij}), \quad \forall i = 1, \ldots, n, \ \forall j = 1, \ldots, m. \tag{4}$$

If the existing flow exceeds the optimal one, the local authorities would be forced to take containment measures, such as, for example, the movement of vehicles with alternate plates, the promotion for the purchase of zero emission vehicles, and the intensification of the public infrastructure.

Hence, we denote by c^c_i the costs related to the additional containment measures that must be incurred in the city i, $i = 1, \ldots, N$, and we suppose that

$$c^c_i := c^c_i(f_i), \quad \forall i = 1, \ldots, n. \tag{5}$$

Let g_i be the quantity of employees to train for the management and maintenance of public green spaces and we group these quantities into the vector $g \in \mathbb{R}^n_+$. We suppose that $g_i \geq \underline{g}_i$, $i = 1, \ldots, n$, where $\underline{g}_i$ represents a parameter that indicates the amount of employees needed to maintain the pre-existing green area.

Let c^T_i be the training cost for such personnel and we assume c^T_i as a function of g_i:

$$c^T_i = c^T_i(g_i), \quad \forall i = 1, \ldots, n. \tag{6}$$

Let c^t_{ij} be the transformation cost that is required to convert the land in green area of type j in city i and we assume c^t_{ij} as a function of x_{ij}:

$$c^t_{ij} = c^t_{ij}(x_{ij}), \quad \forall i = 1, \ldots, n, \ \forall j = 1, \ldots, m. \tag{7}$$

Such costs include planning, design, and construction material costs.

Let γ_{ij}^m be the management cost of green area of type j in city i. Such costs include also personnel remuneration costs. We assume γ_{ij}^m as a function of x_{ij} and g_i:

$$\gamma_{ij}^m = \gamma_{ij}^m(x_{ij}, g_i), \quad \forall i = 1, \ldots, n, \ \forall j = 1, \ldots, m. \tag{8}$$

It is also necessary to consider, for a given city i, $i = 1, \ldots, n$, the management cost of pre-existent types of green areas, namely $\underline{\gamma}_{ij}^m$, for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We suppose that such costs depend on the green area x_{ij} of type j that is set up in city i, $j = 1, \ldots, m$, $i = 1, \ldots, n$ as well as on a constant term that depends on the parameters $\underline{x}_{ij}$ and $\underline{g}_i$. We define the total management costs of green area of type j in city i, denoted by c_{ij}^m, as the sum of these two costs, that is,

$$c_{ij}^m = \underline{\gamma}_{ij}^m(x_{ij}) + \gamma_{ij}^m(x_{ij}, g_i), \quad \forall i = 1, \ldots, n, \ \forall j = 1, \ldots, m. \tag{9}$$

From Equation (8), we can consider c_{ij}^m as a function of x_{ij}:

$$c_{ij}^m = c_{ij}^m(x_{ij}, g_i), \quad \forall i = 1, \ldots, n, \ \forall j = 1, \ldots, m. \tag{10}$$

Let $e_i(f_i)$ be the total amount of CO_2 emissions of city i, $i = 1, \ldots, n$, and we suppose that this quantity is the sum of three different contributes, namely e_{i1}, e_{i2}, and e_{i3} that are the amount of CO_2 emissions due to population in city i, the amount of CO_2 due to industrial activities in city i, and the amount of CO_2 emissions due to vehicles in city i, $i = 1, \ldots, n$, respectively. We suppose that e_{i3} depends on the flow of urban circulation in the city i, that is,

$$e_{i3} = e_{i3}(f_i), \quad \forall\, i = 1, \ldots, n. \tag{11}$$

We assume that the following conservation law is satisfied:

$$e_i(f_i) = \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i), \quad \forall\, i = 1, \ldots, n. \tag{12}$$

Let e_i^a be the quantity of CO_2 absorbed by the overall green area in city i, $i = 1, \ldots, n$. The quantity of CO_2 absorbed by a type j of green area is proportional to this type of green surface, $j = 1, \ldots, m$. Hence, we have

$$e_i^a = \sum_{j=1}^{m} \alpha_{ij} x_{ij}, \quad \forall i = 1, \ldots, n, \tag{13}$$

where $\alpha_{ij} \in \mathbb{R}_+$ is a fixed coefficient that expresses the capacity of the green area of type j to absorb CO_2 in city i, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Thereby, the parameter α_{ij} captures both the city and the type of the green area. Indeed, it is plausible to suppose that the absorption depends on the city as well that inherently captures the population and arrangement of it.

In this model, our aim is to minimize the total costs incurred by the external institution to adapt the green area surface in each city to its real needs. So the optimality conditions are as follows:

$$\min \left\{ \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}^{a}(x_{ij}) + \sum_{i=1}^{n} c_i^T(g_i) + \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}^{t}(x_{ij}) + \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}^{m}(x_{ij}, g_i) + \sum_{i=1}^{n} c_i^c(f_i) \right\} \tag{14}$$

subject to the constraints:

$$x_{ij} \geq \underline{x}_{ij}, \quad \forall i = 1, \ldots, n, \ \forall j = 1, \ldots, m; \tag{15}$$

$$g_i \geq \underline{g}_i, \quad \forall i = 1, \ldots, n; \tag{16}$$

$$f_i \geq \underline{f}_i, \quad \forall i = 1, \ldots, n; \tag{17}$$

$$m_i \leq \sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \leq u_i, \quad \forall i = 1, \ldots, n; \tag{18}$$

$$\sum_{j=1}^{m} \alpha_{ij} x_{ij} \geq \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i) - \sum_{j=1}^{m} \alpha_{ij} \underline{x}_{ij}, \quad \forall i = 1, \ldots, n; \tag{19}$$

$$g_i \leq \sum_{j=1}^{m} \gamma_j x_{ij}, \quad \forall i = 1, \ldots, n. \tag{20}$$

Constraint (15) ensures that the amount of every type of green space is not less than the amount of pre-existing green space of the same type.

Constraints (16) ensures that the amount of employees is no less than the pre-existing staff for the maintenance of green areas.

Constraints (17) ensure that the flow of urban circulation in every city is no less than the positive parameter $\underline{f}_i$ that is strictly connected to the population, the number of registered vehicles, and people's travel behavior in that city as well as on the possibility of using alternative means of transport to private cars.

Constraint (18), as already seen, is given by the combination of constraints (2) and (3).

Constraint (19) ensures that the quantity of CO_2 absorbed by the total green area in every city, which is given by the pre-existent green area and the new one, is not less than the total emissions of city i, $i = 1, \ldots, n$.

Constraint (20) ensures that the number of workers to train for the management and maintenance of public green spaces in every city is not greater than the quantity $\sum_{j=1}^{m} \gamma_j x_{ij}$, where $\gamma_j \in \mathbb{R}_+$ represents a parameter indicating the maximum number of employees that are needed for the maintenance of a unit of green area of type j, $j = 1, \ldots, m$.

Problem (14) can be characterized by the following variational inequality:

Determine $(x^*, g^*, f^*) \in \mathbb{K}$ such that:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{\partial c_{ij}^a(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^t(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial x_{ij}}\right] \times [x_{ij} - x_{ij}^*]$$
$$+ \sum_{i=1}^{n}\left[\frac{\partial c_i^T(g_i^*)}{\partial g_i} + \sum_{j=1}^{m}\frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial g_i}\right] \times [g_i - g_i^*] \tag{21}$$
$$+ \sum_{i=1}^{n}\left[\frac{\partial c_i^c(f_i^*)}{\partial f_i}\right] \times [f_i - f_i^*] \geq 0, \quad \forall (x, g, f) \in \mathbb{K},$$

where:

$$\mathbb{K} = \Bigg\{ (x, g, f) \in \mathbb{R}_+^{nm+2n} :$$
$$x_{ij} \geq \underline{x}_{ij},\ f_i \geq \underline{f}_i,\ g_i \geq \underline{g}_i,\ \forall i = 1, \ldots, n,\ \forall j = 1, \ldots, m,$$
$$m_i \leq \sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \leq u_i,\ \forall\, i = 1, \ldots, n, \tag{22}$$
$$\sum_{j=1}^{m}\alpha_{ij}x_{ij} \geq \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i) - \sum_{j=1}^{m}\alpha_{ij}\underline{x}_{ij}, \quad \forall\, i = 1, \ldots, n$$
$$g_i \leq \sum_{j=1}^{m}\gamma_j x_{ij}, \quad \forall\, i = 1, \ldots, n \Bigg\}.$$

We make the following fundamental assumption:

Hp 2.1. Let all the involved functions be continuously differentiable and strictly convex with respect to all variables.

We now put variational inequality (21) into standard form, that is, determine $X^* \in \mathcal{K} \subset \mathbb{R}^N$ such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{23}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the $nm + 2n$-dimensional Euclidean space, F is a given function from $\mathcal{K}$ to $\mathbb{R}^N$, and $\mathcal{K}$ is a closed and convex set. We define the $(nm + 2n)$-dimensional column vector $X = (x, g, f)$ and the $(nm + 2n)$-dimensional column vector $F(X) = (F_1(X), F_2(x), F_3(X))$, where the (i, j)-th component, F_{ij}^1, of $F^1(X)$ is given by

$$F_{ij}^1(X) \equiv \frac{\partial c_{ij}^a(x_{ij})}{\partial x_{ij}} + \frac{\partial c_{ij}^t(x_{ij})}{\partial x_{ij}} + \frac{\partial c_{ij}^m(x_{ij}, g_i)}{\partial x_{ij}},$$

the ith component, $F_i^2(X)$, of $F^2(X)$ is given by

$$F_i^2(X) \equiv \frac{\partial c_i^T(g_i)}{\partial g_i} + \sum_{j=1}^{m} \frac{\partial c_{ij}^m(x_{ij}, g_i)}{\partial g_i},$$

the ith component, $F_i^3(X)$, of $F^3(X)$ is given by

$$F_i^3(X) \equiv \frac{\partial c_i^c(f_i)}{\partial f_i},$$

and the feasible set $\mathcal{K}$ is defined as $\mathbb{K}$.

Following Kinderlehrer and Stampacchia (1980), the existence of a solution to (23), as the feasible set $\mathcal{K}$ is closed, convex, and bounded, is guaranteed by the next theorem:

Theorem 2.1 (Existence). *Let us assume that Assumption 2.1 is satisfied. Then, there exists at least one solution to variational inequality (23).*

We want now to demonstrate the following theorem:

Theorem 2.2 (Strictly Monotonicity). *The function F defining variational inequality (23) is strictly monotone on $\mathcal{K}$, that is*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle > 0, \quad \forall X^1,\ X^2 \in \mathcal{K},\ X^1 \neq X^2.$$

Proof. Let $X^1,\ X^2 \in \mathcal{K}$ be two elements such that $X^1 \neq X^2$. We have

$$\begin{aligned}
&\langle F(X^1) - F(X^2), X^1 - X^2 \rangle \\
&\quad = \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\frac{\partial c_{ij}^a(x_{ij}^1)}{\partial x_{ij}} - \frac{\partial c_{ij}^a(x_{ij}^2)}{\partial x_{ij}} \right] \times [x_{ij}^1 - x_{ij}^2] \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\frac{\partial c_{ij}^t(x_{ij}^1)}{\partial x_{ij}} - \frac{\partial c_{ij}^t(x_{ij}^2)}{\partial x_{ij}} \right] \times [x_{ij}^1 - x_{ij}^2] \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\frac{\partial c_{ij}^m(x_{ij}^1, g_i^1)}{\partial x_{ij}} - \frac{\partial c_{ij}^m(x_{ij}^2, g_i^2)}{\partial x_{ij}} \right] \times [x_{ij}^1 - x_{ij}^2]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\left[\frac{\partial c_i^T(g_i^1)}{\partial g_i}-\frac{\partial c_i^T(g_i^2)}{\partial x_{ij}}\right]\times[g_i^1-g_i^2]\\
&+\sum_{i=1}^{n}\left[\sum_{j=1}^{m}\left(\frac{\partial c_{ij}^m(x_{ij}^1,g_i^1)}{\partial g_i}-\frac{\partial c_{ij}^m(x_{ij}^2,g_i^2)}{\partial g_i}\right)\right]\times[g_i^1-g_i^2] \qquad (24)\\
&+\sum_{i=1}^{n}\left[\frac{\partial c_i^c(f_i^1)}{\partial f_i}-\frac{\partial c_i^c(f_i^2)}{\partial f_i}\right]\times[f_i^1-f_i^2].
\end{aligned}
$$

In virtue of Assumption 2.1, we know that each of the three terms in (24) is strictly greater than zero if $X^1 \neq X^2$. Hence, we have established that $F(X)$ is strictly monotone. □

We now provide a uniqueness result.

Theorem 2.3 (Uniqueness). *Under the assumptions of Theorem 2.1 and Theorem 2.2, as the function $F(X)$ in (23) is strictly monotone on $\mathcal{K}$, variational inequality (21) admits a unique solution.*

3. Lagrange theory

In this section, we explore the Lagrange theory associated with variational inequality (21) in order to better understand the behavior of green areas adjustment process (see also Daniele, 2001, 2004, 2006; Daniele et al., 2007; Barbagallo et al., 2012; Toyasaki et al., 2014; Daniele and Giuffrè, 2015; Giuffrè et al., 2015; Daniele et al., 2017; Nagurney and Shukla, 2017; Caruso and Daniele, 2018 for an application of the Lagrange theory to various network models). Our aim is to find an alternative formulation of variational inequality (23) governing the minimization problem for the optimal green area model by means of the Lagrange multipliers associated with the constraints defining the feasible set $\mathcal{K}$. To this aim, we set:

$$
\begin{aligned}
&a_i = m_i - \sum_{j=1}^{m} x_{ij} - \sum_{j=1}^{m} \underline{x}_{ij} \le 0, \quad \forall i = 1,\ldots,n;\\
&b_i = \sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} - u_i \le 0, \quad \forall i = 1,\ldots,n;\\
&q_i = \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i) - \sum_{j=1}^{m} \alpha_{ij}x_{ij} - \sum_{j=1}^{m} \alpha_{ij}\underline{x}_{ij} \le 0, \quad \forall i = 1,\ldots,n;\\
&k_i = g_i - \sum_{j=1}^{m} \gamma_j x_{ij} \le 0, \quad \forall i = 1,\ldots,n; \qquad (25)\\
&e_{ij} = -x_{ij} + \underline{x}_{ij} \le 0, \quad \forall i = 1,\ldots,n,\ \forall j = 1,\ldots,m;
\end{aligned}
$$

$$n_i = -g_i + \underline{g}_i \leq 0, \quad \forall i = 1, \dots, n;$$

$$h_i = -f_i + \underline{f}_i \leq 0, \quad \forall i = 1, \dots, n;$$

and

$$\Gamma(X) = (a_i, b_i, q_i, k_i, e_{ij}, n_i, h_i)_{i=1,\dots,n,\ j=1,\dots,m}.$$

So, the constraints set $\mathcal{K}$ can be rewritten as $\mathcal{K} = \{X \in \mathbb{R}_+^{2nm+2n} : \Gamma(X) \leq 0\}$.

Now, let us set

$$V(x, g, f) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{\partial c_{ij}^a(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^t(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial x_{ij}}\right] \times [x_{ij} - x_{ij}^*]$$
$$+ \sum_{i=1}^{n}\left[\frac{\partial c_i^T(g_i^*)}{\partial g_i} + \sum_{j=1}^{m}\frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial g_i}\right] \times [g_i - g_i^*]$$
$$+ \sum_{i=1}^{n}\left[\frac{\partial c_i^c(f_i^*)}{\partial f_i}\right] \times [f_i - f_i^*].$$

We consider the following Lagrange function:

$$\begin{aligned}\mathcal{L}(X, \omega, \varphi, \vartheta, \lambda, \psi, \mu, \varepsilon) = {} & V(x, g, f) + \sum_{i=1}^{n}\omega_i a_i + \sum_{i=1}^{n}\varphi_i b_i + \sum_{i=1}^{n}\vartheta_i q_i + \sum_{i=1}^{n}\lambda_i k_i \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m}\psi_{ij}e_{ij} + \sum_{i=1}^{n}\mu_i n_i + \sum_{i=1}^{n}\varepsilon_i h_i,\end{aligned} \tag{26}$$

$$\forall X \in \mathbb{R}_+^{nm+2n},\ \forall\omega \in \mathbb{R}_+^n,\ \forall\varphi \in \mathbb{R}_+^n,\ \forall\vartheta \in \mathbb{R}_+^n,$$

$$\forall\lambda \in \mathbb{R}_+^n,\ \forall\psi \in \mathbb{R}_+^{nm},\ \forall\mu \in \mathbb{R}_+^n,\ \forall\varepsilon \in \mathbb{R}_+^n.$$

Variational inequality (23) can be written as

$$\min_{X\in\mathcal{K}} V(x, g, f) = 0. \tag{27}$$

This equivalence is justified because we have

$$V(x, g, f) \geq 0 \text{ in } \mathcal{K} \text{ and } \min_{X\in\mathcal{K}} V(x, g, f) = V(x^*, g^*, f^*) = 0. \tag{28}$$

Its dual problem is

$$\max_{\Pi\in\mathbb{R}^{nm+6n}}\ \inf_{X\in\mathbb{R}^{nm+2n}} \mathcal{L}(X, \Pi), \tag{29}$$

where $\Pi = (\omega, \varphi, \vartheta, \lambda, \psi, \mu, \varepsilon)$. We recall that the problem of the strong duality between (27) and (29) is to find

$$\min_{X \in \mathcal{K}} \langle F(X^*), X - X^* \rangle = \max_{\Pi \in \mathbb{R}^{nm+6n}} \inf_{X \in \mathbb{R}^{nm+2n}} \mathcal{L}(X, \Pi). \tag{30}$$

We have the following result.

Theorem 3.1. *Problem (27) satisfies the Karush–Kuhn–Tucker conditions.*

Proof. Following Jahn (1996), we recall that KKT conditions for the existence of Lagrange multipliers can be rewritten as follows. Let X^* be the solution to variational inequality (27) and let us set

$$I_{a_i}(X^*) = \{i \in \{1, \dots, n\} : a_i = 0\};$$

$$I_{b_i}(X^*) = \{i \in \{1, \dots, n\} : b_i = 0\};$$

$$I_{q_i}(X^*) = \{i \in \{1, \dots, n\} : q_i = 0\};$$

$$I_{k_i}(X^*) = \{i \in \{1, \dots, n\} : k_i = 0\}.$$

$$I_{e_{ij}}(X^*) = \{(i, j) \in \{1, \dots, n\} \times \{j, \dots, m\} : e_{ij} = 0\};$$

$$I_{n_i}(X^*) = \{i \in \{1, \dots, n\} : n_i = 0\};$$

$$I_{h_i}(X^*) = \{i \in \{1, \dots, n\} : h_i = 0\}.$$

Then the existence of the Lagrange multipliers is guaranteed if there exists a vector $X \in \mathbb{R}_+^{nm+2n}$ such that the KKT conditions are verified, that is

$$\begin{aligned}
m_i - \sum_{j=1}^{m} x_{ij} - \sum_{j=1}^{m} \underline{x}_{ij} < 0, &\quad \forall i \in I_{a_i}(X^*); \\
\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} - u_i < 0, &\quad \forall i \in I_{b_i}(X^*); \\
\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i) - \sum_{j=1}^{m} \alpha_{ij} x_{ij} - \sum_{j=1}^{m} \alpha_{ij} \underline{x}_{ij} < 0, &\quad \forall i \in I_{q_i}(X^*); \\
g_i - \sum_{j=1}^{m} \gamma_j x_{ij} < 0, &\quad \forall i \in I_{k_i}(X^*); \\
-x_{ij} + \underline{x}_{ij} < 0, &\quad \forall i \in I_{e_{ij}}(X^*); \\
-g_i + \underline{g}_i < 0, &\quad \forall i \in I_{n_i}(X^*); \\
-f_i + \underline{f}_i < 0, &\quad \forall i \in I_{h_i}(X^*).
\end{aligned} \tag{31}$$

It is easy to verify that system (31) admits a solution (see Colajanni et al., 2018). □

As a consequence, we get the following result.

Theorem 3.2. *Let X^* be the solution to variational inequality (23), then the Lagrange multipliers, $\omega^* \in \mathbb{R}^n_+$, $\varphi^* \in \mathbb{R}^n_+$, $\vartheta^* \in \mathbb{R}^n_+$, $\lambda^* \in \mathbb{R}^n_+$, $\psi^* \in \mathbb{R}^{nm}_+$, $\mu^* \in \mathbb{R}^n_+$, and $\varepsilon^* \in \mathbb{R}^n_+$ associated with the constraints system (25) do exist.*

Moreover, as the KKT conditions imply that Assumption S (see Daniele et al., 2007; Giuffrè et al., 2015) is verified and (x^*, g^*, f^*) is a minimal solution to problem (28), assuming that Assumptions 2.1 are satisfied, by virtue of well-known theorems (see Jahn, 1996) we can state that the vector $(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi^*, \mu^*, \varepsilon^*)$ is a saddle point of the Lagrange function (26), namely:

$$\begin{aligned}\mathcal{L}(X^*, \omega, \varphi, \vartheta, \lambda, \psi, \mu, \varepsilon, \lambda) &\le \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi,^* \mu^*, \varepsilon^*)\\ &\le \mathcal{L}(X, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi,^* \mu^*, \varepsilon^*)\end{aligned} \tag{32}$$

$$\forall X \in \mathbb{R}^{nm+2n}_+,\ \forall \omega \in \mathbb{R}^n_+,\ \forall \varphi \in \mathbb{R}^n_+,\ \forall \vartheta \in \mathbb{R}^n_+,$$

$$\forall \lambda \in \mathbb{R}^n_+,\ \forall \psi \in \mathbb{R}^{nm}_+,\ \forall \mu \in \mathbb{R}^n_+,\ \forall \varepsilon \in \mathbb{R}^n_+,$$

and

$$\begin{aligned}&\omega_i^* a_i^* = 0,\ \varphi_i^* b_i^* = 0,\ \vartheta_i^* q_i^* = 0, \quad \forall i;\\ &\lambda_i^* h_i^* = 0,\ \mu_i^* n_i^* = 0,\ \varepsilon_i^* h_i^* = 0, \quad \forall i;\\ &\qquad \psi_{ij}^* x_{ij}^* = 0, \quad \forall i,\ \forall j.\end{aligned} \tag{33}$$

By virtue of these considerations, we can calculate the Lagrange multipliers $\omega^* \in \mathbb{R}^n_+$, $\varphi^* \in \mathbb{R}^n_+$, $\vartheta^* \in \mathbb{R}^n_+$, $\lambda^* \in \mathbb{R}^n_+$, $\psi^* \in \mathbb{R}^{nm}_+$, $\mu^* \in \mathbb{R}^n_+$, and $\varepsilon^* \in \mathbb{R}^n_+$ associated with the constraints and the solution X^* to variational inequality (23).

From the right-hand side of (32), it follows that $X^* \in \mathbb{R}^{nm+2n}_+$ is a minimal point of $\mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi^*, \mu^*, \varepsilon^*)$ in the whole space $\mathbb{R}^{nm+2n}$ and hence, for all $i = 1, \dots n$ and $j = 1, \dots, m$, we get

$$\begin{aligned}\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi^*, \mu^*, \varepsilon^*)}{\partial x_{ij}} &= \frac{\partial c^a_{ij}(x^*_{ij})}{\partial x_{ij}} + \frac{\partial c^t_{ij}(x^*_{ij})}{\partial x_{ij}} + \frac{\partial c^m_{ij}(x^*_{ij}, g^*_i)}{\partial x_{ij}}\\ &\quad - \omega_i^* + \varphi_i^* - \alpha_j \vartheta_i^* - \gamma_j \lambda_i^* - \psi_{ij}^* = 0,\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi^*, \mu^*, \varepsilon^*)}{\partial g_i} &= \frac{\partial c^T_i(g^*_i)}{\partial g_i} + \sum_{j=1}^m \frac{\partial c^m_{ij}(x^*_{ij}, g^*_i)}{\partial g_i}\\ &\quad + \lambda_i^* - \mu_i^* = 0,\end{aligned} \tag{35}$$

$$\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi^*, \mu^*, \varepsilon^*)}{\partial f_i} = \frac{\partial c^c_i(f^*_i)}{\partial f_i} + \vartheta_i^* \frac{\partial e_{i3}(f^*_i)}{\partial f_i} - \varepsilon_i^* = 0, \tag{36}$$

together with conditions (33).

It is easy to verify that conditions (33)–(36) are equivalent to variational inequality (21).

The importance of the Lagrange function consists in the fact that constraints are included in such a function and it allows us, when the strong duality holds, to express the solution to variational inequality by means of the system of equations derived from the KKT conditions. The existence and uniqueness of the solution to variational inequality is guaranteed by Theorems 2.1 and 2.3, respectively.

We now can interpret the meaning of some Lagrange multipliers. Let us consider, first, the case when $a_i^* < 0$ and $b_i^* < 0$, that is,

$$m_i < \sum_{j=1}^{m} x_{ij}^* + \sum_{j=1}^{m} \underline{x}_{ij} < u_i.$$

Then, from (33), we get $\omega_i^* = \varphi_i^* = 0$. Also, let us assume that $x_{ij}^* > \underline{x}_{ij}$, which implies, from (33), that $\psi_{ij}^* = 0$. Hence (34) becomes

$$\frac{\partial c_{ij}^a(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^t(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial x_{ij}} = \alpha_j \vartheta_i^* + \gamma_j \lambda_i^*. \tag{37}$$

If $\lambda_i^* > 0$ and $\vartheta_i^* > 0$, from (33), we get

$$\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m} \alpha_{ij} x_{ij}^* - \sum_{j=1}^{m} \alpha_{ij} \underline{x}_{ij} = 0;$$

$$g_i^* - \sum_{j=1}^{m} \gamma_j x_{ij}^* = 0,$$

which means that CO_2 emissions in city i is completely absorbed by the optimal green area and the number of employees reaches the maximum allowed number. Hence, from (37), summing up with respect to j, it follows that the total marginal cost associated with the expansion of green area increases. In this case, each additional unit of green area or employees is unnecessary. Similar considerations hold even if only one between λ_i^* and ϑ_i^* is positive.

On the contrary, if both $\lambda_i^* = 0$ and $\vartheta_i^* = 0$, then, from (33), we get

$$\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m} \alpha_{ij} x_{ij}^* - \sum_{j=1}^{m} \alpha_{ij} \underline{x}_{ij} < 0;$$

$$g_i^* - \sum_{j=1}^{m} \gamma_j x_{ij}^* < 0.$$

In this case, from (37), summing up with respect to j, it follows that, given g_i^* and f_i^*, the total cost associated with the expansion of green area, that is,

$$\sum_{j=1}^{m}\Big(c_{ij}^{a}(x_{ij})+c_{ij}^{t}(x_{ij})+c_{ij}^{m}(x_{ij},g_i)\Big),$$

reaches its minimum value in x_{ij}^*.

Finally, if $\varphi_i^* > 0$, from (33), we get

$$\sum_{j=1}^{m} x_{ij}^* + \sum_{j=1}^{m} \underline{x}_{ij} = u_i.$$

In this case, the optimal green area added to the pre-existing one reaches the maximum percentage of city i to be allocated to green areas. Moreover, it is clear that $\omega_i^* = 0$. If $\psi_{ij}^* = 0$, (34) becomes

$$\frac{\partial c_{ij}^{a}(x_{ij}^*)}{\partial x_{ij}}+\frac{\partial c_{ij}^{t}(x_{ij}^*)}{\partial x_{ij}}+\frac{\partial c_{ij}^{m}(x_{ij}^*,g_i^*)}{\partial x_{ij}} = -\varphi_i^* + \alpha_j\vartheta_i^* + \gamma_j\lambda_i^*. \tag{38}$$

If $\lambda_i^* = 0$ and $\vartheta_i^* = 0$, then, like above, we get

$$\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m}\alpha_{ij}x_{ij}^* - \sum_{j=1}^{m}\alpha_{ij}\underline{x}_{ij} < 0;$$

$$g_i^* - \sum_{j=1}^{m}\gamma_j x_{ij}^* < 0.$$

Hence (38) becomes

$$\frac{\partial c_{ij}^{a}(x_{ij}^*)}{\partial x_{ij}}+\frac{\partial c_{ij}^{t}(x_{ij}^*)}{\partial x_{ij}}+\frac{\partial c_{ij}^{m}(x_{ij}^*,g_i^*)}{\partial x_{ij}} = -\varphi_i^* < 0.$$

Summing with respect to j, we obtain that the total marginal costs associated with the expansion of green area decrease, which means that it is necessary to increase the green area. This is also evident if we note that the total emissions of city i are not completely absorbed by the optimal green area. However, the presence of constraint $\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \le u_i$ does not allow us for this increase. In this case, the total emissions cannot be completely absorbed by the green area of the city and therefore actions should be taken in this sense.

Analogously, from (33), we get

$$\vartheta_i^*\left(\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m}\alpha_{ij}x_{ij}^* - \sum_{j=1}^{m}\alpha_{ij}\underline{x}_{ij}\right) = 0 \text{ and } \varepsilon_i^*(-f_i^* + \underline{f}_i) = 0.$$

If we assume that the flow is positive, that is $f_i^* > \underline{f}_i$, then $\varepsilon_i^* = 0$. Hence (36) becomes

$$\frac{\partial c_i^c(f_i^*)}{\partial f_i} = -\vartheta_i^* \frac{\partial e_{i3}(f_i^*)}{\partial f_i}.$$

If $\vartheta_i^* > 0$, then the total CO_2 emissions of city i is completely absorbed by optimal green areas. As a consequence, $-\vartheta_i^* \frac{\partial e_{i3}(f_i)}{\partial f_i}$ represents the marginal cost associated to the imposing additional containment measures in the city i due to excessive flow f_i. On the contrary, if the total CO_2 emissions of city i are not completely absorbed by optimal green areas, then $\vartheta_i^* = 0$. In this case, (35) becomes

$$\frac{\partial c_i^c(f_i^*)}{\partial f_i} = 0,$$

which means that the total cost function c_i^c attains its minimum value in f_i^*.

Analogous considerations hold for (35).

4. Computational procedure

We now recall the Euler method (see Dupuis and Nagurney, 1993). Variational inequality (23) is solvable as follows. For every iteration τ, we calculate

$$X^{\tau+1} = P_{\mathcal{K}}(X^\tau - a_\tau F(X^\tau)), \tag{39}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ defined as

$$P_{\mathcal{K}} = \text{argmin}_{z \in \mathcal{K}} \|\xi - z\|$$

and F is the function entering variational inequality (23). In order to get the convergence of the iterative scheme, we need the sequence $\{a_\tau\}$ to be such that

$$\sum_{\tau=0}^{\infty} a_\tau = \infty, \; a_\tau > 0, \; a_\tau \to 0, \; \text{as } \tau \to \infty. \tag{40}$$

Now we describe the method.

Step 0: Initialization

Set $X^0 \in \mathcal{K}$. Let τ denote an iteration counter and set $\tau = 1$. Set the sequence a_τ such that condition (40) is satisfied.

Step 1: Computation

Calculate $X^\tau \in \mathcal{K}$ solving the following variational inequality subproblem:

$$\langle X^\tau + a_\tau F(X^{\tau-1}) - X^{\tau-1}, X - X^\tau \rangle \geq 0, \quad \forall X \in \mathcal{K}.$$

Step 2: Convergence

Fix a tolerance $\epsilon > 0$ and check whether $|X^\tau - X^{\tau+1}| \leq \epsilon$, then stop; otherwise,

Table 1
Types of green area considered in our model, for all city i

Type of green area	Level of maintenance	Capacity of a km^2 to absorb CO_2: α_{ij}
$j = 1$: Urban green area	High	$\alpha_{i1} = 545,000$ kg/yr
$j = 2$: Natural green area	Medium	$\alpha_{i2} = 569,070$ kg/yr

set $\tau := \tau + 1$, and go to Step 1.

The explicit formulas for the Euler method used in this model are as follows:

$$X^\tau = \max\{0, X^{\tau-1} - a_{\tau-1}F(X^{\tau-1})\}.$$

5. Numerical examples

In this section, we present some numerical examples using the model described in Section 2.

For the computation of the optimal solutions, we have applied the Euler method described in the previous section. The calculations were performed using the MATLAB program. The algorithm was implemented on a laptop with 1.8 GHz Intel Core i5 dual-core and 8 GB RAM, 1600 MHz DDR3. For the convergence of the method, a tolerance of $\epsilon = 10^{-4}$ was fixed. The method has been implemented with a constant step $\alpha = 0.1$.

We now present different scenarios consisting in considering different number of cities. Following Ministero dell'ambiente e della tutela del territorio e del mare, in every scenario we suppose that there are two types of different green areas: urban and natural green areas. We suppose that the coefficient that expresses the capacity of the green area of type j to absorb CO_2 in city i are the same in all cities and that they differ only in the type of green area. Their features are showed in Table 1.

Urban green areas require a high and ongoing level of maintenance and their allocation is in areas densely populated and with schools or sports areas. It includes botanical gardens, urban furniture areas, outdoor sports areas, school gardens, urban parks, equipped greenery, and uncultivated greenery. As its composition is varied, we can assume that urban green areas have a medium-high capacity to absorb CO_2.

Natural green areas require a medium level of maintenance and their allocation is in marginal areas furthest from inhabited zones with very limited and specific uses. Its composition provides areas dominated by the shrubby and arboreal component, refuge for biodiversity where the vegetation develops spontaneously. For this type of green areas, the capacity to absorb CO_2 is high.

In the following examples, the costs are in thousands of euros, the space of green area in units of km^2, and CO_2 emissions are expressed in kg/yr.

5.1. Example 1. Metropolitan city of Catania

The metropolitan city of Catania is a metropolitan city of 1,108,040 inhabitants and includes the 58 municipalities of the former regional province of Catania. It extends for 3570 km^2 and it is the

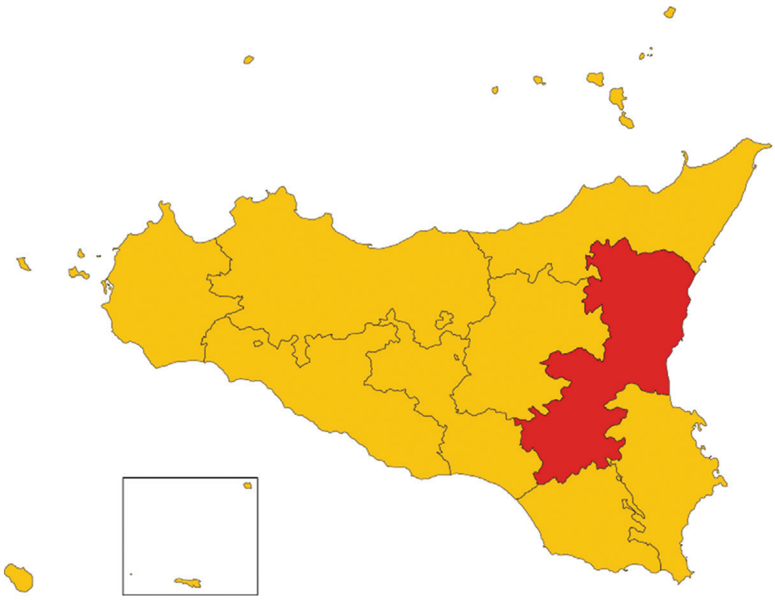

Fig 1. Metropolitan city of Catania in Sicily.

seventh metropolitan city in Italy by population. Its geographical position in Sicily can be seen in Fig. 1.

Italian law requires that every citizen has 9 m^2 of green area available. Considering that the population of Catania municipality is around 1,108,040 inhabitants, by law in this city at least 9.97 km^2 of green area are needed. Hence, we put $l_1 = 9.97$ km^2. Moreover, the existing green area of Catania is 636.11 km^2 (data obtained by ISPRA), that is, $\underline{x}_{11} + \underline{x}_{12} = 636.11$ km^2. So,

$$m_1 = \max\left\{l_1, \underline{x}_{11} + \underline{x}_{12}\right\} = 636.11 \text{ km}^2.$$

Specifically, $\underline{x}_{11} = 536.05$ km^2 and $\underline{x}_{12} = 100.06$ km^2.

According to ISPRA, CO_2 emissions due to population are $e_{11} = 637,635,890$ kg/yr. CO_2 emissions due to industrial activities are $e_{12} = 728,210,000$ kg/yr (data obtained by European Environment Agency). Finally, the amount of CO_2 emissions due to vehicles is proportional to urban circulation, that is,

$$e_{14}(f_i) = \alpha f_i, \tag{41}$$

where α represents the average emission of CO_2 in one year of a car, which we can estimate around 1,232 kg/yr and f_1 represents the average number of vehicles circulating in the territory in one year. According to ISPRA, the number of vehicles that circulate in Catania is 790,878. So we use this data as the existing flow. We set $\underline{f}_1 = 0.85 \times 790,878$. Thereby, we are assuming at least 15% of the vehicles in circulation can be replaced through the use of public transport or through sustainable mobility (cycle paths, car sharing services, or electric scooters, ...).

Moreover, we suppose that $\gamma_1 = 120$ and $\gamma_2 = 100$.

We assume the following cost functions are given:

$$
\begin{aligned}
c^a_{11}(x_{11}) &= 0.20x^2_{11} - 0.15x_{11} + 13\\
c^a_{12}(x_{12}) &= 0.30x^2_{12} - 0.25x_{12} + 13\\
c^T_1(g_1) &= 0.8g^2_1 - 0.6g_1 + 8\\
c^t_{11}(x_{11}) &= 1.20x^2_{11} - 1.10x_{11} + 8\\
c^t_{12}(x_{12}) &= 1.20x^2_{12} - 1.15x_{12} + 8\\
c^m_{11}(x_{11}, g_1) &= 0.80x^2_{11} - 0.70x_{11} + 0.8g^2_1 - 0.6g_1 + 27\\
c^m_{12}(x_{12}, g_1) &= 0.50x^2_{12} - 0.50x_{12} + 0.8g^2_1 - 0.6g_1 + 22.\\
c^c_1(f_1) &= 0.6f^2_1 - 0.5f_1 + 16.
\end{aligned}
$$

Solving the associated variational inequality, we get the following optimal solution:

$$
\begin{aligned}
&x^*_{11} = 587.27\ \text{km}^2, \quad x^*_{12} = 703.47\ \text{km}^2,\\
&g^*_1 = 140,000, \quad f^*_1 = 672,246.
\end{aligned}
$$

We can observe that the pre-existing green space is not sufficient to counteract the huge emissions deriving from industrial activities, anthropogenic activities, and transport. In fact, the optimal solution establishes that the optimal green space amounts to 36.15% of the total territory of the metropolitan city, a value that differs greatly from the existing green area, which amounts to only 19.96% of the total territory. Moreover, we observe that the existing flow exceeds the optimal one. Therefore, the local authorities would be forced to take containment measures, such as the movement of vehicles with alternate plates, the promotion for the purchase of zero emission vehicles, and the intensification of the public infrastructure.

5.2. Example 2. Eastern Sicily: Catania, Messina, Syracuse, and Ragusa

In this example, we consider Eastern Sicily. It is defined as that part of the Sicilian territory overlooking the Ionian coast of Sicily. It is composed by the metropolitan cities of Catania ($i = 1$) and Messina ($i = 2$) and the free consortia of Syracuse ($i = 3$) and Ragusa ($i = 4$). Eastern Sicily is shown in Fig. 2.

The data for this example, taken by ISPRA, are shown in Table 2.

According to ISPRA, the average flows of vehicles in Messina, Syracuse, and Ragusa are 411,283, 81,393, and 51,619, respectively. So we use that parameters as initial values. Moreover, we set $f_2 = 0.9 \times 411,283$, $f_3 = 0.87 \times 81,393$, and $f_4 = 0.9 \times 51,619$.

We assume the following cost functions are given:

$$
\begin{aligned}
c^a_{11}(x_{11}) &= 0.20x^2_{11} - 0.15x_{11} + 13\\
c^a_{12}(x_{12}) &= 0.30x^2_{12} - 0.25x_{12} + 13
\end{aligned}
$$

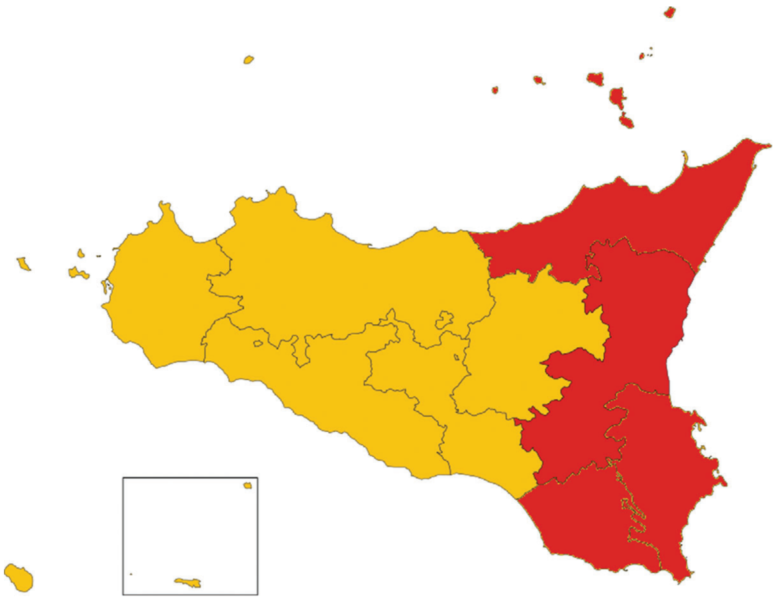

Fig 2. Eastern Sicily.

Table 2
Data for Example 2

City	Surface	Inhabitants	Emissions	Pre-existing green area
Catania	3570 km^2	1,108,040	e_{11}=637,635,890 kg/yr	$\underline{x}_{11} = 536$ km^2
			e_{12}=728,210,000 kg/yr	$\underline{x}_{12} = 100$ km^2
Messina	3266.12 km^2	627,251	e_{21}=360,959,667 kg/yr	$\underline{x}_{21} = 48.99$ km^2
			e_{22}=1,113,000,000 kg/yr	$\underline{x}_{22} = 2.29$ km^2
Syracuse	2124.13 km^2	397,341	e_{31}=228,654,421 kg/yr	$\underline{x}_{31} = 8.5$ km^2
			e_{32}=1,020,000,000 kg/yr	$\underline{x}_{32} = 120.9$ km^2
Ragusa	1623.89 km^2	320,893	e_{41}=184,661,545 kg/yr	$\underline{x}_{41} = 6.5$ km^2
			e_{42}=619,330,000 kg/yr	$\underline{x}_{42} = 105.55$ km^2

$$c^a_{21}(x_{21}) = 0.20x^2_{21} - 0.15x_{21} + 13$$
$$c^a_{22}(x_{22}) = 0.30x^2_{22} - 0.25x_{22} + 13$$
$$c^a_{31}(x_{31}) = 0.20x^2_{31} - 0.15x_{31} + 13$$
$$c^a_{32}(x_{32}) = 0.30x^2_{32} - 0.25x_{32} + 13$$
$$c^a_{41}(x_{41}) = 0.20x^2_{41} - 0.15x_{41} + 13$$
$$c^a_{42}(x_{42}) = 0.30x^2_{42} - 0.25x_{42} + 13$$
$$c^T_1(g_1) = 0.8g^2_1 - 0.6g_1 + 8$$
$$c^T_2(g_2) = 0.8g^2_2 - 0.6g_2 + 8$$

$$c_3^T(g_3) = 0.8g_3^2 - 0.6g_3 + 8$$
$$c_4^T(g_4) = 0.8g_4^2 - 0.6g_4 + 8$$
$$c_{11}^t(x_{11}) = 1.20x_{11}^2 - 1.10x_{11} + 8$$
$$c_{12}^t(x_{12}) = 1.20x_{12}^2 - 1.15x_{12} + 8$$
$$c_{21}^t(x_{21}) = 1.20x_{21}^2 - 1.10x_{21} + 8$$
$$c_{22}^t(x_{22}) = 1.20x_{22}^2 - 1.15x_{22} + 8$$
$$c_{31}^t(x_{31}) = 1.20x_{31}^2 - 1.10x_{31} + 8$$
$$c_{32}^t(x_{32}) = 1.20x_{32}^2 - 1.15x_{32} + 8$$
$$c_{41}^t(x_{41}) = 1.20x_{41}^2 - 1.10x_{41} + 8$$
$$c_{42}^t(x_{42}) = 1.20x_{42}^2 - 1.15x_{42} + 8$$

$$c_{11}^m(x_{11}, g_1) = 0.80x_{11}^2 - 0.70x_{11} + 0.8g_1^2 - 0.6g_1 + 27$$
$$c_{12}^m(x_{12}, g_1) = 0.50x_{12}^2 - 0.50x_{12} + 0.8g_1^2 - 0.6g_1 + 22$$
$$c_{21}^m(x_{21}, g_1) = 0.80x_{21}^2 - 0.70x_{21} + 0.8g_2^2 - 0.6g_2 + 27$$
$$c_{22}^m(x_{22}, g_1) = 0.50x_{22}^2 - 0.50x_{22} + 0.8g_2^2 - 0.6g_2 + 22$$
$$c_{31}^m(x_{31}, g_1) = 0.80x_{31}^2 - 0.70x_{31} + 0.8g_3^2 - 0.6g_3 + 27$$
$$c_{32}^m(x_{32}, g_1) = 0.50x_{32}^2 - 0.50x_{32} + 0.8g_3^2 - 0.6g_3 + 22$$
$$c_{41}^m(x_{41}, g_1) = 0.80x_{41}^2 - 0.70x_{41} + 0.8g_4^2 - 0.6g_4 + 27$$
$$c_{42}^m(x_{42}, g_1) = 0.50x_{42}^2 - 0.50x_{42} + 0.8g_4^2 - 0.6g_4 + 22$$
$$c_1^c(f_1) = 0.6f_1^2 - 0.5f_1 + 16$$
$$c_2^c(f_2) = 0.6f_2^2 - 0.5f_2 + 16$$
$$c_3^c(f_3) = 0.6f_3^2 - 0.5f_3 + 16$$
$$c_4^c(f_4) = 0.6f_4^2 - 0.5f_4 + 16.$$

These costs were built symmetrically with respect to each city.

Solving the associated variational inequality, we obtain the following optimal solution:

$$x_{11}^* = 587.27 \text{ km}^2,\ x_{12}^* = 703.47 \text{ km}^2,\ g_1^* = 140,000,\ f_1^* = 672,246;$$
$$x_{21}{}^* = 248.99 \text{ km}^2,\ x_{22}^* = 2.31 \text{ km}^2,\ g_2^* = 257,600,\ f_2^* = 370,154;$$
$$x_{31}{}^* = 408.5 \text{ km}^2,\ x_{32}^* = 1536 \text{ km}^2,\ g_3^* = 194,620,\ f_3^* = 70,811;$$
$$x_{41}{}^* = 93.5 \text{ km}^2,\ x_{42}^* = 225.98 \text{ km}^2,\ g_4^* = 320,790,\ f_4^* = 46,457.$$

As we can see from the optimal solution, in each of the four cities it is necessary to increase the percentage of green area to counteract the CO_2 emissions taken into consideration. In the case of Syracuse, the new green area should be 91% of the total area of the city, which is improbable. This result is due to the fact that Syracuse is a highly industrialized city, because of the presence of the petrochemical complex of Priolo Gargallo, Augusta, and Melilli, an establishment often at the center of controversies for the choice of the place, the huge emissions, the numerous accidents, and the increase in the number of tumor diseases. This result makes us understand how the CO_2 emissions are disproportionate compared to the quantity absorbed by the existing green area. In each city, moreover, the optimal flow is less than the initial flow, which means that local authorities would be forced to take containment measures, as discussed earlier, in order to reduce CO_2 emissions due to transport.

6. Conclusion

In this paper, we proposed an optimization model for the management of green areas in order to find the optimal green space to absorb CO_2 emissions of industrialized cities. We obtained a minimization problem and the related variational inequality. Furthermore, we studied the Lagrange theory to better understand the process that regulates the possible increase in green space. Also, we presented a computational procedure, based on the Euler method, to find the optimal variables of the model. Some concrete examples, set in Southern Italy, were presented and solved. The optimal solution showed the effectiveness of the model and the need for many local governments to improve the living conditions of the inhabitants, increasing, for instance, the green areas, especially in those zones with an intense industrial activity.

The studied model could be further extended and improved, by introducing also budget constraints to the local organizations or increasing the awareness of inhabitants and of the industries with respect to environment and life, encouraging, for instance, sustainable manufacturing by industries (see, for instance Liu et al., 2020) and requiring that every year a part of their revenue has to be destined to the improvement and maintenance of green areas.

The results in this paper add to the growing literature of operations research for green area managing.

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References

Azaria, L., Wibowo, A., Shidiq, I.P.A., Rokhmatuloh, 2018. Carbon sequestration capability analysis of urban green space using geospatial data. *E3S Web of Conferences* 73. https://doi.org/10.1051/e3sconf/20187303009.

Barbagallo, A., Daniele, P., Maugeri, A., 2012. Variational formulation for a general dynamic financial equilibrium problem: balance law and liability formula. *Nonlinear Analysis, Theory, Methods and Applications* 75, 3, 1104–1123.

Caruso, V., Daniele, P., 2018. A network model for minimizing the total organ transplant costs. *European Journal of Operational Research* 266, 2, 652–662.

Colajanni, G., Daniele, P., Giuffrè, S., Nagurney, A., 2018. Cybersecurity investments with nonlinear budget constraints and conservation laws: variational equilibrium, marginal expected utilities, and Lagrange multipliers. *International Transactions in Operational Research* 25, 1443–1464.

Daniele, P., 2001. Variational inequalities for static equilibrium market, Lagrangian function and duality. In F. Giannessi, A. Maugeri, P. M. Pardalos (eds) *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*. Kluwer Academic Publishers, Amsterdam, pp. 43–58.

Daniele, P., 2004. Time-dependent spatial price equilibrium problem: existence and stability results for the quantity formulation model. *Journal of Global Optimization* 28, 3-4, 283–295.

Daniele, P., 2006. *Dynamic Networks and Evolutionary Variational Inequalities*. Edward Elgar Publishing, Cheltenham.

Daniele, P., Giuffrè, S., 2015. Random variational inequalities and the random traffic equilibrium problem. *Journal of Optimization Theory and Applications* 167, 1, 363–381.

Daniele, P., Giuffrè, S., Idone, G., Maugeri, A., 2007. Infinite dimensional duality and applications. *Mathematische Annalen* 339, 1, 221–239.

Daniele, P., Maugeri, A., Nagurney, A., 2017. Cybersecurity investments with nonlinear budget constraints: analysis of the marginal expected utilities. In Daras, N.J., Rassias, T.M. (eds) *Operations Research, Engineering, and Cyber Security*, Vol. 113. Springer, Berlin, pp. 117–134.

Dupuis, P., Nagurney, A., 1993. Dynamical systems and variational inequalities. *Annals of Operations Research* 44, 9–42.

European Environment Agency, 2020. European pollutant release and transfer register (E-PRTR). Available at https://prtr.eea.europa.eu/#/home (accessed June 2020).

Gill, S.E., Handley, J.F., Ennos, A.R., Pauleit, S., 2007. Adapting cities for climate change: the role of the green infrastructure. *Built Environment* 33, 115–133.

Giuffrè, S., Maugeri, A., Puglisi, D., 2015. Lagrange multipliers in elastic-plastic torsion problem for nonlinear monotone operators. *Journal of Differential Equations* 259, 3, 817–837.

Gómez, F., Jabaloyes, J., Montero, L., De Vicente, V., Valcuende, M., 2011. Green areas, the most significant indicator of the sustainability of cities: research on their utility for urban planning. *Journal of Urban Planning and Development* 137, 3, 311–329.

Indrawati, E., Listyarini, S., Pardede, T., Warlina, L., 2014. Optimization model for predicting green areas in Jakarta to minimize impacts of climate change. *The Sustainable City IX* 1, 305–315.

Intergovernmental Panel on Climate Change, 2014. Climate Change, 2014. Synthesis Report: Summary for Policymakers. IPCC, Geneva, Switzerland. Available at http://www.ipcc.ch/pdf/assessment-report/ar5/syr/AR5_SYR_FINAL_SPM.pdf (accessed June 2020).

ISPRA, Istituto Superiore per la protezione e la Ricerca Ambientale. Available at http://www.ost.sinanet.isprambiente.it/Report_indicatorismry.php (accessed June 2020).

Jahn, J., 1996. *Introduction to the Theory of Nonlinear Optimization*. Springer, Berlin.

Kinderlehrer, D., Stampacchia, G., 1980. *An Introduction to Variational Inequalities and their Applications*. Academic Press, New York, NY.

Liu, G., Cao, H., Zhu, G., 2020. Competitive pricing and innovation investment strategies of green products considering firms' farsightedness and myopia. *International Transactions in Operational Research* 28, 2021, 839–871.

Ministero dell'ambiente e della tutela del territorio e del mare, Comitato per lo sviluppo delverde pubblico, 2017. *Linee guida per la gestione del verde urbano e prime indicazioni per una pianificazione sostenibile*.

Nagurney, A., Shukla, S., 2017. Multifirm models of cybersecurity investment competition vs. cooperation and network vulnerability. *European Journal of Operational Research* 269, 2, 588–600.

Toyasaki, F., Daniele, P., Wakolbinger, T., 2014. A variational inequality formulation of equilibrium models for end-of-life products with nonlinear constraints. *European Journal of Operational Research* 236, 1, 340–350.

UNFCCC, 1997. *Kyoto Protocol to the United Nations Framework Convention on Climate Change*. United Nations, New York, NY.

UNFCCC, 2015. *21st Conference of the Parties of the UNFCCC*. France, Paris.

United States Environmental Protection Agency. Overview of greenhouse gases: carbon dioxide emissions, 2019. Available at https://www.epa.gov/ghgemissions/overview-greenhouse-gases#carbon-dioxide.

Yoon, E.J., Bomi Kim, B., Lee, D.K., 2019. Multi-objective planning model for urban greening based on optimization algorithms. *Urban Forestry & Urban Greening* 40, 183–194.