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Inverse Derivative Operator and Umbral Methods for the Harmonic Numbers and Telescopic Series Study

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Abstract: The rules associated with differintegral operators of positive and negative real order offer an elegant tool to deal with integral operators, viewed as derivatives of order-1. Although it is well known that the integration is the inverse of the derivative operation, the afore mentioned rules offer a new mean to get either explicit iteration of the integration by parts and a general formula to get the primitive of any infinitely differentiable function. We show that the method provides an unexpected link with generalized telescoping series, yields new useful tools for the relevant treatment and allows a practically unexhausted tool to derive identities involving Harmonic numbers and the associated generalized forms. It is also shown that the same problems can be treated using methods of umbral algebraic nature. They are the natural complements to the integro-differential point of view, we foresee the possibility of embedding the two procedures.

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1. Introduction

New concepts and techniques emerged in the past within the framework of special functions have had a positive feedback in other and more abstract field of Mathematics. The techniques associated with Umbral calculus have opened new and unspected avenues in Analysis and simplified the technicalities of calculations, awkwardly tedious when performed with conventional computational means.

The Authors of this paper have largely benefitted from the techniques suggested by Umbral methods and have embedded them with other means associated, e.g. with algebraic operational procedure, to get new results and to reformulate previous, apparently extraneous topics, within a unifying point of view.

This paper follows the same stream. We embed Umbral methods and formal integration techniques to explore the field of number series by getting a non conventional treatment of problems usually treated with completely different means. We will recover new and old result, but the novelty of the paper mainly relies on the novelty and generality of the method itself.

We develop a new framework to treat problems in Number Theory, Combinatorial Analysis, Telescopic Series, Harmonic Numbers . . . .The method we foresee gathers together integro-differential and umbral means. The paper consists of two parts, the first is devoted to the formalism of Negative Derivative Operator and to the relevant use for the previously quoted fields of research. In the second part we discuss the application of Umbral Methods and how they interface to the integro-differential counterpart.

Elementary problems in Calculus reveal unexpected new features if viewed from a broader perspective which employs e.g. operational methods. The use of the negative derivative operator formalism [1,2] has provided an efficient tool to compute the primitive of a given function, or of products of functions as well. The underlying formalism allows the handling of integrals and derivatives on the same footing. Within this framework the primitive of the product of two functions is nothing but a restatement of the Leibniz formula [3,4] as it has been proved in ref. [5], namely

Definition 1. *$\forall x \in Dom\{f(x), g(x)\}$ we state*

$$\hat{D}_x^{-1}(g(x)f(x)) = \sum_{s=0}^{\infty}\binom{-1}{s} g^{(-1-s)}(x) f^{(s)}(x) \tag{1}$$

where

$$\hat{D}_x^{-1} s(x) = \int s(x)dx \tag{2}$$

is the negative derivative operator, *$f^{(\pm s)}(x)$ denotes the s^{th}- (positive/negative)-derivative of the function, and*

$$\binom{-1}{s} = (-1)^s. \tag{3}$$

For the operation of definite integration, we set

$${}_{\alpha}\hat{D}_x^{-1} s(x) = \int_{\alpha}^{x} s(\xi)d\xi = S(x) - S(\alpha). \tag{4}$$

By exploiting the Definition 1 we state what follows.

Proposition 1. *The integral of a function $f \in C^{\infty}$ can be written in terms of the series*

$$F(x) := \int f(x)dx = \sum_{s=0}^{\infty}(-1)^s \frac{x^{s+1}}{(s+1)!} f^{(s)}(x), \quad \forall x \in Dom\{f(x)\}, \tag{5}$$

where $f^{(s)}(x)$ denotes the s^{th}-derivative of the integrand function.

Proof. We rewrite eq. (5) according to Definition 1 as $\hat{D}_x^{-1}(g(x)f(x)) = \sum_{s=0}^{\infty}\binom{-1}{s} g^{(-1-s)}(x) f^{(s)}(x)$. By assuming $g(x) = 1$ we get

$$g^{(-1-s)}(x) = \hat{D}_x^{(-1-s)} 1 = \frac{x^{s+1}}{(s+1)!} \tag{6}$$

thus eventually ending up with

$$\int f(x)dx = \hat{D}_x^{-1}(1 \cdot f(x)) = \sum_{s=0}^{\infty}(-1)^s \frac{x^{s+1}}{(s+1)!} f^{(s)}(x).$$

□

For further comments the reader is addressed to refs. [1,2,5].

Example 1. *Let us now consider the primitive of the unit function and write*

$$\int 1\, d\xi = \int \xi\, \xi^{-1} d\xi \tag{7}$$

The use of the eq. (1) *and of the identifications* $g(\xi) = \xi$ *and* $f(\xi) = \xi^{-1}$ *yields*

$$\int_0^x \xi\, \xi^{-1} d\xi = \sum_{s=0}^{\infty} (-1)^s \frac{x^{s+2}}{(s+2)!} \left((-1)^s \frac{s!}{x^{s+1}} \right) = x \sum_{s=0}^{\infty} \frac{1}{(s+1)(s+2)} \tag{8}$$

and being $\int_0^x 1\, d\xi = x$, *we deduce the identity*

$$\sum_{s=0}^{\infty} \frac{1}{(s+1)(s+2)} = 1 \tag{9}$$

which is true, since its l.h.s., as shown below, reduces to the Telescopic series [6].

Example 2. *The same procedure applied to* $g(\xi) = \xi^2$ *and* $f(\xi) = \xi^{-1}$ *provides the further identity [7]*

$$\sum_{s=0}^{\infty} \frac{1}{(s+1)(s+2)(s+3)} = \frac{1}{4}. \tag{10}$$

The extension of the procedure to the case $g(\xi) = \xi^m$, $f(\xi) = \xi^{-1}$, *leads to the following series [7]*

$$\sum_{s=0}^{\infty} \frac{1}{(s+1)(s+2)\ldots(s+m+1)} = \sum_{s=0}^{\infty} \frac{s!}{(s+m+1)!} = \frac{1}{m!\, m} \tag{11}$$

which is a well known result (see e.g. formula 5.1.24.7 *of ref. [8]).*

Example 3. *By using furthermore, in eq.* (1), *the identification* $g(x) = x^{\alpha}$ *and* $f(x) = x^{-\alpha}$, $\forall \alpha \in \mathbb{R}$, *the inverse derivative Leibniz rule yields*

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+\alpha+2)\Gamma(1-s-\alpha)} = \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \tag{12}$$

which, after using the properties of the Gamma function "$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$" *yields the telescopic sum*

$$\sum_{s=0}^{n} \frac{1}{(s+\alpha+1)} \frac{1}{(s+\alpha)} = \frac{1}{\alpha}. \tag{13}$$

The previous results are quite surprising, they are associated with generalized forms of telescopic series and can be embedded in an even wider context, as proved in the forthcoming section.

2. Combinatorial Identities and Leibniz Formula

We exploit the formalism outlined in the previous section to foresee a systematic procedure to get identities of combinatorial type.

Example 4. *We consider eq.* (1) *with* $g(x) = 1$ *and* $f(x) = x^n$ *which yields*

$$\frac{x^{n+1}}{n+1} = \int x^n dx = \sum_{s=0}^{n} (-1)^s \frac{x^{s+1}}{(s+1)!} \frac{n!}{(n-s)!} x^{n-s} = x^{n+1} \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+1} \tag{14}$$

which implies

$$\sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+1} = \frac{1}{n+1}. \tag{15}$$

Example 5. *If we otherwise assume that* $g(x) = x$ *and* $f(x) = x^n$, *the following identity is straightforwardly obtained*

$$\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{(s+1)(s+2)}=\frac{1}{n+2} \tag{16}$$

indeed

$$\frac{x^{n+2}}{n+2}=\hat{D}_x^{-1}(x\cdot x^n)=\sum_{s=0}^{n}(-1)^s\frac{x^{s+2}}{(s+2)!}\frac{n!}{(n-s)!}x^{n-s}=x^{n+2}\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{(s+1)(s+2)}. \tag{17}$$

If we go on with $g(x)=x^2$ and $f(x)=x^n$, we end up with

$$\frac{x^{n+3}}{n+3}=\sum_{s=0}^{n}(-1)^s\,2\,\frac{x^{s+3}}{(s+3)!}\frac{n!}{(n-s)!}x^{n-s}=2\,x^{n+3}\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{(s+1)(s+2)(s+3)} \tag{18}$$

then

$$\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{(s+1)(s+2)(s+3)}=\frac{1}{2\,(n+3)}. \tag{19}$$

Remark 1. *We can now manipulate eqs.* (15)-(16) *to get a further result useful for the development of our discussion. It is evident that, by the use of the partial fractional expansion, we can write eq.* (17) *as*

$$\frac{1}{n+2}=\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{(s+1)(s+2)}=\sum_{s=0}^{n}\binom{n}{s}(-1)^s\left(\frac{1}{s+1}-\frac{1}{s+2}\right)=\frac{1}{n+1}-\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{s+2} \tag{20}$$

which easily yields the identity

$$\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{s+2}=\frac{1}{(n+1)(n+2)},\qquad \forall n\in\mathbb{N}. \tag{21}$$

The following expansion in terms of partial fractions

$$\frac{2}{(s+1)(s+2)(s+3)}=\frac{1}{s+1}-\frac{2}{s+2}+\frac{1}{s+3} \tag{22}$$

can be exploited to end up with

$$\sum_{n=0}^{n}\binom{n}{s}\frac{(-1)^s}{s+3}=\frac{2}{(n+1)(n+2)(n+3)}. \tag{23}$$

Finally, by iterating the procedure, we end up with the proof of the following Theorem for two general formulae.

Theorem 1. $\forall n,m\in\mathbb{N}$

$$\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s\,s!}{(s+m+1)!}=\frac{1}{m!\,(n+m+1)}. \tag{24}$$

and (see also eq. (52) *later)*

$$\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{s+m+1}=\frac{m!}{(n+1)(n+2)\dots(n+m+1)}=\frac{n!m!}{(n+m+1)!} \tag{25}$$

For a deeper discussion on Theorem 1 and 2 (see later) see Refs. [9–11] where the results have already been derived in a different context.

The results of this section have indicated how methods of integro-differential nature are suitable to derive combinatorial identities and make progresses in the Theory of Telescopic series. In the following sections we will treat in deeper details these points and will show how further interesting results in these directions can be obtained.

3. Inverse Derivatives and Generalized Telescopic Series

A systematic investigation of the series quoted in eq. (11) has been undertaken in ref. [7]. Here we adopt an analogous point of view and frame the results obtained so far within a different context. We use the following notation to indicate Telescopic series (T stands for telescopic)

Example 6.

$$ {}_2T_2 := \sum_{s=0}^{\infty} \frac{1}{(s+1)(s+2)} = \sum_{s=0}^{\infty} \left(\frac{1}{s+1} - \frac{1}{s+2} \right), \tag{26} $$

$$ {}_3T_3 := \sum_{s=0}^{\infty} \frac{1}{(s+1)(s+2)(s+3)} = \frac{1}{2} \sum_{s=0}^{\infty} \left(\frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3} \right), \tag{27} $$

where we have used the convention

$$ \begin{aligned} {}_mT_2 &:= \sum_{s=0}^{\infty} \frac{1}{(s+m-1)(s+m)}, \quad m > 1 \\ {}_mT_3 &:= \sum_{s=0}^{\infty} \frac{1}{(s+m-2)(s+m-1)(s+m)}, \quad m > 2. \end{aligned} \tag{28} $$

The eq. (27) can be eventually written as

$$ {}_3T_3 = \frac{1}{2} \left({}_2T_2 - {}_3T_2 \right) \tag{29} $$

and being ${}_mT_2 = \dfrac{1}{(m-1)}$*, we end up with*

$$ \begin{aligned} {}_2T_2 &= 1, \\ {}_3T_3 &= \frac{1}{4}. \end{aligned} \tag{30} $$

Remark 2. *The T notation allows to write the identity in eq. (11) as*

$$ {}_{m+1}T_{m+1} = \frac{1}{m!\, m}. \tag{31} $$

Regarding the first of eqs. (30), the The relevant proof can also be achieved by the use of an alternative and more general procedure.

Remark 3. *We use the Laplace transform method to write*

$$ \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+2} = \int_0^{\infty} \left(1 - e^{-\sigma}\right)^n e^{-2\sigma} d\sigma, \tag{32} $$

on account of eq. (21), after summing over n, we end up with

$$ \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \int_0^{\infty} e^{-\sigma} d\sigma = 1 \tag{33} $$

which is a restatement of eq. (9) on the basis of a totally different mean. The same procedure can be applied to get the proof for the other identities (see Sec. 5 for further comments).

Definition 2. *$\forall n, m \in \mathbb{N}$, we introduce the notation (see eq. (25))*

$$ {}_m b_n := \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j+m+1}, \tag{34}$$

to define the ${}_m b_n$ which will be called Binomial Harmonic Numbers BHN.

Remark 4. *The iteration of the previous formalism yields (see also the forthcoming sections) the following identities. The BHN ${}_1 b_n$ and ${}_2 b_n$ have been given in eqs. (21)-(23) and the generalized form ${}_m b_n$ in eq. (25). The use of the integral representation yields*

$$ {}_2 b_n = \int_0^\infty \left(1 - e^{-\sigma}\right)^n e^{-3\sigma} d\sigma \tag{35}$$

which can be exploited to find

$$ \sum_{n=0}^{\infty} {}_2 b_n = \int_0^\infty e^{-2\sigma} d\sigma = \frac{1}{2} \tag{36}$$

and, more in general,

$$ \sum_{n=0}^{\infty} {}_m b_n = \frac{1}{m}, \tag{37}$$

namely the derivation of eq. (11) from a different perspective.

4. Umbral Methods and Binomial Harmonic Numbers

The Harmonic numbers are defined[1] as [12–15]

$$ h_n := \sum_{s=0}^{n} \frac{1}{s+1}, \quad \forall n \in \mathbb{N}, \tag{38}$$

whose properties have been shown to be usefully studied using methods borrowed from umbral formalism. This point of view has been developed in a number of previous papers [16–19] in which the associated algebraic procedures have been shown to be particularly effective. They have allowed the study of the relevant properties by straightforward means, paved the way to the introduction of generalized forms along with the tools to develop the underlying theoretical background. The methods suggested in [18] gave rise to a series of speculations and conjectures then proved on more solid basis in subsequent researches [20,21].

In this section we will use the umbral formalism to go deeper into the properties of the BHN. The use of the umbral notation adopted in [3,17,18] allows to manipulate complicate expressions and obtain remarkable results.

Definition 3. *We impose that the umbral operator $\hat{a}$, acting on the vacuum φ_0, provides the position*

$$ \hat{a}^s \varphi_0 := \frac{1}{s+1}, \quad \forall s \in \mathbb{N}. \tag{39}$$

1 In ref. [16,19] we used a slight different definition which here corresponds, strictly speaking, to the upper limit of the sum equal to $n-1$.

Property 1. *The operator $\hat{a}$ satisfies the composition rule*

$$\hat{a}^s \hat{a}^p = \hat{a}^{s+p}, \quad \forall s, p \in \mathbb{N}. \tag{40}$$

Then we can state the following identity.

Proposition 2. $\forall n \in \mathbb{N}$

$${}_0b_n = (1-\hat{a})^n \varphi_0. \tag{41}$$

Proof. $\forall n \in \mathbb{N}$, by the use of the Newton binomial, of the Definition 3 and of the Property 1, we can write

$$(1-\hat{a})^n \varphi_0 = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \, \hat{a}^s \varphi_0 = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+1} = {}_0b_n$$

□

Corollary 1. *The use of eq. (15) allows to write*

$$(1-\hat{a})^n \varphi_0 = {}_0b_n = \frac{1}{n+1}. \tag{42}$$

Remark 5. *We note that the umbral operator $\hat{a}$ acting on the same vacuum φ_0 yields the same algebraic result*

$$(1-\hat{a})^n \varphi_0 = \hat{a}^n \varphi_0. \tag{43}$$

indeed, for the Proposition 2, Corollary 1 and Definition 3, we get

$$(1-\hat{a})^n \varphi_0 = \frac{1}{n+1} = \hat{a}^n \varphi_0.$$

Even though unenecessary, we note that eq. (43) does not imply that $(1-\hat{a}) = \hat{a}$. It is not indeed an identity between operators but between algebraic quantities after the action of the umbral operators on the vacuum[2]*.*

We iterate the method in the following way.

Proposition 3. $\forall n \in \mathbb{N}$

$$\hat{a}(1-\hat{a})^n \varphi_0 = {}_1b_n = \frac{1}{(n+1)(n+2)}. \tag{44}$$

Proof. It is evident that

$$(1-\hat{a})^{n+1} \varphi_0 = \frac{1}{(n+1)+1}. \tag{45}$$

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[2] As shown in [17], the differential realization of the umbral operator $\hat{a}$ is a shift operator $\hat{a} = e^{\partial_z}$ and the corresponding vacuum is $\varphi(z) = \frac{1}{z+1}$. The action on the vacuum of the operator $\hat{a}^n, \forall n \in \mathbb{N}$, is accordingly expressed through $\hat{a}^n \varphi_0 = e^{n\partial_z} \left. \frac{1}{z+1} \right|_{z=0} = \frac{1}{n+1}$ and we also get

$$(1-\hat{a})^n \varphi_0 = \left(1-e^{\partial_z}\right)^n \varphi(z)\Big|_{z=0} = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \hat{a}^s \varphi(z)\Big|_{z=0} = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+1}.$$

We can now clarify the remark after the identity (43) which in differenetial terms reads $\left(1-e^{\partial_z}\right)^n \varphi(z)\Big|_{z=0} = e^{n\partial_z} \varphi(z)\big|_{z=0}$ which is true for $z=0$ but not in general, it is indeed easily checked that $\left(1-e^{\partial_z}\right)^n \varphi(z) \neq e^{n\partial_z} \varphi(z)$.

We split the power so that

$$(1-\hat{a})^{n+1}\varphi_0 = (1-\hat{a})(1-\hat{a})^n\varphi_0 = [(1-\hat{a})^n - \hat{a}(1-\hat{a})^n]\,\varphi_0 \tag{46}$$

and calculate the term $\hat{a}(1-\hat{a})^n\varphi_0$ by using the Definition 3 and the Property 1,

$$\hat{a}(1-\hat{a})^n\varphi_0 = \hat{a}\sum_{s=0}^{n}\binom{n}{s}(-1)^s\,\hat{a}^s\varphi_0 = \sum_{s=0}^{n}\binom{n}{s}(-1)^s\,\hat{a}^{s+1}\varphi_0 = \sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{(s+1)+1} = {}_1b_n. \tag{47}$$

Furthermore, from eqs. (45) and (46), we can establish the recurrence

$${}_0b_n - {}_1b_n = \frac{1}{n+2} \tag{48}$$

which can be exploited to get the explicit form of ${}_1b_n$, namely

$${}_1b_n = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)},$$

□

Corollary 2. *The extension of the procedure yields, for ${}_2b_n$, the recurrence*

$${}_0b_n - 2\,{}_1b_n + {}_2b_n = \frac{1}{n+3} \tag{49}$$

indeed

$$\frac{1}{(n+2)+1} = (1-\hat{a})^{n+2}\varphi_0 = (1-\hat{a})(1-\hat{a})^{n+1}\varphi_0 = \left[(1-\hat{a})^{n+1} - \hat{a}(1-\hat{a})^{n+1}\right]\varphi_0 = \\ = \left\{(1-\hat{a})^{n+1} - \hat{a}\left[(1-\hat{a})^n - \hat{a}(1-\hat{a})^n\right]\right\}\varphi_0 = {}_0b_n - {}_1b_n - {}_1b_n + {}_2b_n$$

which, once solved for ${}_2b_n$, yields

$${}_2b_n = \frac{2}{(n+1)(n+2)(n+3)} \tag{50}$$

which confirms the results in eq. (23). Eq. (49) can be viewed as the umbral version of the partial fraction expansion exploited in the previous sections.

Theorem 2. *All the BHN can be defined as binomial convolution of the lower order case ${}_rb_s$*

$${}_mb_n = \sum_{s=0}^{n}\binom{n}{s}(-1)^s\sum_{r=0}^{m}\binom{m}{r}(-1)^r\,{}_rb_s \tag{51}$$

where

$${}_rb_s = \hat{a}^r(1-\hat{a})^s\varphi_0 = \frac{s!r!}{(s+r+1)!}. \tag{52}$$

Proof. The proof is obtained by the iteration of the previous procedure ny noting that

$$
{}_m b_n = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+m+1} = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \, (1-\hat{a})^m (1-\hat{a})^s \varphi_0 =
$$
$$
= \sum_{s=0}^{n} \binom{n}{s} (-1)^s \sum_{r=0}^{m} \binom{m}{r} (-1)^r \, \hat{a}^r (1-\hat{a})^s \varphi_0.
$$

□

As a further example we consider the complementary forms of BHN.

Definition 4. *We introduce the Complementary Binomial Harmonic Number (CBHN) defined as*

$$
{}_m b_n^+ := \sum_{s=0}^{n} \binom{n}{s} \frac{1}{s+m+1}, \qquad \forall m, n \in \mathbb{N}. \tag{53}
$$

Example 7. *The CBHN can be constructed recursively from the identity*

$$
{}_0 b_n^+ = \frac{2^{n+1}-1}{n+1}, \tag{54}
$$

easily proved by induction and further derived in eq. (74).

According to the same procedure as before we set

$$
{}_0 b_n^+ = (1+\hat{a})^n \varphi_0 \tag{55}
$$

which yields

$$
\frac{2^{n+2}-1}{n+2} = {}_0 b_{n+1}^+ = (1+\hat{a})^{n+1} \varphi_0 = {}_0 b_n^+ + {}_1 b_n^+ \tag{56}
$$

and, once solved for ${}_1 b_n^+$*, gets*

$$
{}_1 b_n^+ = \frac{2^{n+1} n + 1}{(n+1)(n+2)}. \tag{57}
$$

The same method provides for ${}_2 b_n^+$

$$
\frac{2^{n+3}-1}{n+3} = {}_0 b_{n+2}^+ = (1+\hat{a})^{n+2} \varphi_0 = {}_0 b_n^+ + 2 {}_1 b_n^+ + {}_2 b_n^+ \tag{58}
$$

thus finding

$$
{}_2 b_n^+ = \frac{2\left(2^n (n^2+n+2) - 1\right)}{(n+1)(n+2)(n+3)}, \tag{59}
$$

$$
{}_3 b_n^+ = \frac{2\left(2^n n \, (n^2+3n+8) + 3\right)}{(n+1)(n+2)(n+3)(n+4)}, \tag{60}
$$

$$
{}_4 b_n^+ = \frac{2\left(2^n (n^4+6n^3+23n^2+18n+24) - 12\right)}{(n+1)(n+2)(n+3)(n+4)(n+5)} \tag{61}
$$

and by iteration we can obtain other expressions for ${}_m b_n^+$ *(with* $m > 4$*), a syntehetic expression in terms of* ${}_2F_1$ *hypergeometric function can also be obtained but is no more informative than eq.* (53)*, namely*

$$
{}_m b_n^+ = \frac{{}_2F_1(-n, m+1, m+2, -1)}{m+1} \tag{62}
$$

and the associated convolution finite sum

$$\sum_{r=0}^{m}\binom{m}{r}{}_rb_n^+ = \sum_{r=0}^{m}\binom{m}{r}\sum_{s=0}^{n}\binom{n}{s}\frac{1}{s+r+1} = \frac{2^{n+m+1}-1}{n+m+1} \tag{63}$$

In the previous section we have derived the identity shown in the forthcoming section which will be exploited to introduce an extension of the 2-order BHN.

Definition 5. *We introduce the 2-order BHN*

$${}_mb_n^{(2)} := \sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{2s+m+1}, \quad \forall m,n \in \mathbb{N}. \tag{64}$$

Example 8. *According to our umbral notation, the case* $m = 0$ *can be written as*

$${}_0b_n^{(2)} = (1-\hat{a}^2)^n\varphi_0 \tag{65}$$

which, according to the algebraic Properties 1 of $\hat{a}$ *and the coefficients (52), allows to write*

$${}_0b_n^{(2)} = (1+\hat{a})^n(1-\hat{a})^n\varphi_0 = \sum_{s=0}^{n}\binom{n}{s}\hat{a}^s\,(1-\hat{a})^n\varphi_0 = \sum_{s=0}^{n}\binom{n}{s}\,{}_sb_n. \tag{66}$$

Along with eq. (52), the last identity yields the following further result

$${}_0b_n^{(2)} = n!^2\sum_{s=0}^{n}\frac{1}{(n-s)!(n+s+1)!} = n!^2\frac{2^{2n}}{(2n+1)!} \tag{67}$$

easily generalized to

$${}_mb_n^{(2)} = \hat{a}^m(1-\hat{a}^2)^n\varphi_0 = \sum_{s=0}^{n}\binom{n}{s}\,{}_{s+m}b_n = n!^2\sum_{s=0}^{n}\frac{(s+m)!}{(n-s)!s!(s+m+n+1)!}. \tag{68}$$

Eqs. (67) and (68) are not satisfactory, in the sense that we are looking for a result not in the form of a series, albeit truncated.

The umbral technique can be used as a complementary procedure to frame the previous results within a different context.

5. Combinatorial Identities, Integral Representations and Special Functions

5.1. Combinatorial Identities and Integral Representations

The procedure we have just put forward has indicated that the formulae derived in Sec. 1 are a fairly straightforward mean to get old and new identities appearing in Combinatorics, Theory of Telescoping series and of the associated generalized forms. We have been indeed able to recover the results of ref. [7] as well as other disseminated in the mathematical literature, by following a unifying and straightforward procedure.

Before entering the discussion of a different topics, let us consider further examples.

Example 9. *We start from the manipulation of the integral*

$$\frac{(1+x)^{n+1}-1}{n+1}=\int_0^x (1+\xi)^n d\xi=\sum_{s=0}^{n}(-1)^s\frac{\xi^{s+1}}{(s+1)!}\frac{n!}{(n-s)!}(1+\xi)^{n-s}\Bigg|_0^x= \\ =\sum_{s=0}^{n}(-1)^s\frac{x^{s+1}}{s+1}\binom{n}{s}\,(1+x)^{n-s} \tag{69}$$

which, after keeping e.g. $x=1$, *yields*

$$\sum_{s=0}^{n}\binom{n}{s}\,\frac{(-1)^s}{s+1}\,2^{n-s}=\frac{2^{n+1}-1}{n+1}. \tag{70}$$

The identity in eq. (69) can be further handled and generalized along the same lines, we discussed in the previous sections, thus getting for example

$$\sum_{s=0}^{n}(-1)^s\frac{x^{s+2}}{(s+1)(s+2)}\binom{n}{s}\,(1+x)^{n-s}=\frac{(nx+x-1)(1+x)^{n+1}+1}{(n+1)(n+2)}. \tag{71}$$

After breaking the l.h.s. in partial sums we also find, for $x=1$,

$$\sum_{s=0}^{n}\binom{n}{s}\,\frac{(-1)^s}{(s+2)}\,2^{n-s}=\frac{2^{n+2}-(n+3)}{(n+1)(n+2)}. \tag{72}$$

Example 10. *The slightly generalized form of eq. (69)*

$$\frac{(a+bx)^{n+1}-a^{n+1}}{b\,(n+1)}=\int_0^x (a+b\xi)^n d\xi=\sum_{s=0}^{n}(-1)^s\frac{x^{s+1}}{(s+1)!}\frac{n!}{(n-s)!}(a+bx)^{n-s}b^s \tag{73}$$

can be exploited to derive further identities. By keeping for example $a=2, b=1, x=-1$ *we find*

$$\sum_{s=0}^{n}\binom{n}{s}\,\frac{1}{s+1}=\frac{2^{n+1}-1}{n+1}. \tag{74}$$

Example 11. *A further identity of pivotal importance, for the present purposes, is provided by*

$$\sum_{s=0}^{n}\binom{n}{s}\,\frac{(-1)^s}{2s+1}=\int_0^1(1-\xi^2)^n d\xi=\int_0^1(1-\xi^2)^n d\xi=\frac{1}{2}\,B\left(\frac{1}{2},\,n+1\right)=\left(\frac{3}{2}+n\right)\,B\left(\frac{3}{2},n+1\right) \tag{75}$$

where $B(x,y)$ *being the Euler Beta function and, by expoliting the B-function properties, we obtain*

$$\sum_{s=0}^{n}\binom{n}{s}\,\frac{(-1)^s}{2s+1}=\frac{2^{2n}n!^2}{(2n+1)!} \tag{76}$$

indeed

$$\frac{1}{2}\,B\left(\frac{1}{2},\,n+1\right)=\frac{1}{2}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}=\frac{1}{2}\sqrt{\pi}\,n!\frac{4^{n+1}(n+1)!}{(2(n+1)!)\sqrt{\pi}}=\frac{2^{2n}n!^2}{(2n+1)!}. \tag{77}$$

Furthermore, by setting $x=1$ *and using the transformation* $\xi^2=t$, *the result in eq. (75) is achieved. In more general terms we also get*

$$\sum_{s=0}^{n}\binom{n}{s}\,\frac{(-1)^s}{\alpha s+\beta}=\int_0^1(1-\xi^\alpha)^n\,\xi^{\beta-1}d\xi=\frac{1}{\alpha}\,B\left(\frac{\beta}{\alpha},\,n+1\right). \tag{78}$$

Example 12. *Furthermore, by using the Laplace transform method, we get the integral representation of the two equivalent series*

$$\begin{aligned} &\sum_{s=0}^{n}(-1)^s \frac{x^s}{(s+2)}\binom{n}{s}(1+x)^{n-s} = \sum_{s=0}^{n}\binom{n}{s}\frac{x^s}{(s+1)(s+2)} = \phi_n(x), \\ &\phi_n(x) = \int_0^\infty \left[1+(1-e^{-\sigma})x\right]^n e^{-2\sigma} d\sigma, \end{aligned} \tag{79}$$

whose sum is

$$\sum_{s=0}^{n}(-1)^s \frac{x^s}{(s+2)}\binom{n}{s}(1+x)^{n-s} = \sum_{s=0}^{n}\binom{n}{s}\frac{x^s}{(s+1)(s+2)} = \frac{(x+1)^{n+2}-(nx+2x+1)}{x^2(n+1)(n+2)}. \tag{80}$$

The integral representation can also be exploited to show that the sum $\sum_{n=0}^{\infty}\phi_n(x)$ *is actually converging for* $x<0$ *et* $0<\mid x\mid<1$.

Example 13. *The umbral method developed in the section can be further exploited by noting that the previous procedure allows the derivation of the identities*

$$(1-\hat{a}^2)^n\varphi_0 = {}_0b_n^{(2)} = n!\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}. \tag{81}$$

We can take advantage from them by applying the paradigm outlined in the introductory section. Accordingly we find

$$(1-\hat{a}^2)^{n+1}\varphi_0 = (n+1)!\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{5}{2}\right)}. \tag{82}$$

By following the technique of eq. (46) with eq. (52), we note that

$$(1-\hat{a}^2)^{n+1}\varphi_0 = (1-\hat{a}^2)(1-\hat{a}^2)^n\varphi_0 = {}_0b_n^{(2)} - {}_2b_n^{(2)}, \tag{83}$$

then we can obtain the identity

$${}_2b_n^{(2)} = \sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{2s+3} = n!\Gamma\left(\frac{3}{2}\right)\frac{\Gamma\left(n+\frac{5}{2}\right)-(n+1)\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{5}{2}\right)} = \frac{n!}{2}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{5}{2}\right)}. \tag{84}$$

Example 14. *We can generalize the method to any real power*

$${}_mb_n^{(k)} = \sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^s}{ks+m+1} = \frac{n!}{k}\frac{\Gamma\left(\dfrac{m+1}{k}\right)}{\Gamma\left(\dfrac{kn+m+k+1}{k}\right)}, \quad \forall k\in\mathbb{R}. \tag{85}$$

5.2. Leibniz Rule, Umbral Methods and Special Functions

By considering now the two variable Hermite polynomial $H_n(x,y)$ [5]

$$H_n(x,y) = n!\sum_{r=0}^{\lfloor \frac{n}{2}\rfloor}\frac{x^{n-2r}y^r}{(n-2r)!r!}, \quad \forall x,y\in\mathbb{R}, \forall n\in\mathbb{N}, \tag{86}$$

we can provide the following example.

Example 15. *By reminding the Hermite polynomials property*

$$\partial_x^s H_n(x,y) = \frac{n!}{(n-s)!} H_{n-s}(x,y), \tag{87}$$

it is worth considering the integral

$$\begin{aligned} \frac{H_{n+1}(x,y) - H_{n+1}(0,y)}{n+1} &= \int_0^x H_n(\xi,y)\, d\xi = \sum_{s=0}^{n} (-1)^s \frac{x^{s+1}}{(s+1)!} \frac{n!}{(n-s)!} H_{n-s}(x,y) = \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{x^{s+1}}{(s+1)} H_{n-s}(x,y) \end{aligned} \tag{88}$$

which is by no means surprising since the Hermite polynomials satisfy a monic type derivative [22].

Furthermore, the use of the umbral notation [17,23]

$$\begin{aligned} & H_n(x,y) = (x + {}_y\hat{h})^n \theta_0, \qquad \forall x,y \in \mathbb{R}, \forall n \in \mathbb{N}, \\ & {}_y\hat{h}^r \theta_0 := \theta_r = \frac{y^{\frac{r}{2}} r!}{\Gamma\left(\frac{r}{2}+1\right)} \left|\cos\left(r\frac{\pi}{2}\right)\right| = \begin{cases} 0 & r = 2s+1 \\ y^s \dfrac{(2s)!}{s!} & r = 2s \end{cases} \qquad \forall s \in \mathbb{Z}. \end{aligned} \tag{89}$$

allows to writes

$$\sum_{s=0}^{n} (-1)^s \frac{x^{s+1}}{(s+1)!} \frac{n!}{(n-s)!} H_{n-s}(x,y) = x \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{x^s}{s+1} (x + {}_y\hat{h})^{n-s} \theta_0 \tag{90}$$

and also the identities

$$\begin{aligned} x \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{x^s}{s+1} (x + {}_y\hat{h})^{n-s} \theta_0 &= x \int_0^\infty \left[x(1-e^{-\sigma}) + {}_y\hat{h} \right]^n e^{-\sigma} d\sigma = \\ &= x \int_0^\infty H_n\left(x\,(1-e^{-\sigma}), y\right) e^{-\sigma} d\sigma \end{aligned} \tag{91}$$

which lead to the unexpected integral representation

$$\int_0^\infty H_n\left(x\,(1-e^{-\sigma}), y\right) e^{-\sigma} d\sigma = \frac{H_{n+1}(x,y) - H_{n+1}(0,y)}{(n+1)\, x}. \tag{92}$$

The identities we have discussed so far are just a few examples of the possibilities offered by the negative derivative formalism. A forthcoming investigation will provide a more carefully discussion in this respect.

In the final section we will show how a formalism of umbral nature can be exploited to provide a useful complement for the treatment outlined in the previous sections.

6. Final Comments

This paper has been aimed at developing a self-contained treatment of the Theory of Combinatorial identities and of generalized Harmonic numbers by embedding them within the context of a twofold complementary formalism.

The results we have obtained have so wide implications and cannot be comprised in the space of an article. We like however to note that the negative derivative formalism is essentially a reformulation of the Leibniz rule. The ordinary formula is also a very useful tool to derive combinatorial identities.

Example 16. *We note that, since* $\partial_x^n x^{3n} = \dfrac{(3n)!}{(2n)!}x^{2n}$*, comparing with* $\partial_x^n \left(x^n x^{2n}\right)$ *and after using the Leibniz rule, we get*

$$\sum_{s=0}^{n} \binom{n}{s}\binom{2n}{s} = \frac{(3n)!}{(2n)!\; n!} \tag{93}$$

which is easily generalized to (a particular case of the Chu-Vandermonde identity)

$$\sum_{s=0}^{n} \binom{kn}{s}\binom{n}{s} = \frac{((k+1)\; n)!}{(kn)!\; n!} = \binom{(k+1)n}{n}. \tag{94}$$

The umbral formalism deserves further comments.

Definition 6. *We introduce Binomial Convoluted Harmonic Number (BCHN) as*

$$_m\beta_n := \sum_{s=0}^{n} \binom{n}{s} (-1)^s \, h_{m+s}, \quad \forall m, n \in \mathbb{N}, \tag{95}$$

where h_{m+s} *are the Harmonic numbers cited in eq. (38).*

The relevant properties can be studied by the use of the methods outlined in Section 4.

Example 17. *According to the umbral notation in [16,19],* $\hat{h}^r \psi_0 := h_r$*, we can set*

$$_m\beta_n := \hat{h}^m (1-\hat{h})^n \psi_0 \tag{96}$$

by the way

$$\hat{h}^m (1-\hat{h})^n \psi_0 = \hat{h}^m \sum_{r=0}^{n} \binom{n}{r} (-1)^r \, \hat{h}^r \psi_0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r \, \hat{h}^{m+r} \psi_0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r \, h_{m+r} = {}_m\beta_n.$$

We prove that

$$_0\beta_n = -\frac{1}{n(n+1)} \tag{97}$$

indeed

$$\begin{aligned} _0\beta_n &= \sum_{s=0}^{n} \binom{n}{s} (-1)^s \sum_{r=0}^{s} \frac{1}{r+1} = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \sum_{r=0}^{s} \int_0^1 x^r dx = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \int_0^1 \frac{x^{s+1}-1}{x-1} dx = \\ &= -\int_0^1 \frac{x(1-x)^n}{x-1} dx = -\frac{1}{n(n+1)} \end{aligned} \tag{98}$$

and it is evident that, once we know $_0\beta_n$*, we can infer recursively all the* $_m\beta_n$*.*

$$_0\beta_{n+1} = -\frac{1}{(n+1)(n+2)}, \tag{99}$$

from eq. (96) we find

$$ {}_0\beta_{n+1} = (1-\hat{h})(1-\hat{h})^n \psi_0 \tag{100} $$

which yields the recurrence

$$ {}_0\beta_{n+1} = {}_0\beta_n - {}_1\beta_n \tag{101} $$

and which allows the computation of the BCHN for $m = 1$

$$ {}_1\beta_n = -\frac{2}{n(n+1)(n+2)}. \tag{102} $$

The extension of the procedure yields, for $m = 2$,

$$ {}_2\beta_n = -\frac{6}{n(n+1)(n+2)(n+3)} \tag{103} $$

which eventually suggests that

$$ {}_m\beta_n = -\frac{(m+1)!}{n(n+1)(n+2)\dots(n+m+1)} = -\frac{(n-1)!(m+1)!}{(n+m+1)!}. \tag{104} $$

This last comment completes the paper, the results of which are summarized in the following Tables.

The article has addressed a number of points either in calculus and Number Theory. We believe that its most noticeable achievement is having established a clear link between the formalism of negative derivatives and the properties of harmonic numbers. Albeit some of the results we have discussed have been obtained in previous authoritative papers, our efforts have been directed towards opening an alternative research avenue, eventually connecting apparently separated fields. In a forthcoming publication we will extend the methodology to fractional integration and to the many variable cases.

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Author Contributions

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Table 1. Combinatorial Identities.

Binomial Harmonic Numbers	**Umbral Images**	**Complementary BHN**
${}_m b_n = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+m+1}$	$\hat{a}^n \varphi_0 = \frac{1}{n+1}$	${}_m b_n^+ = \sum_{s=0}^{n} \binom{n}{s} \frac{1}{s+m+1}$
${}_0 b_n = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+1} = \frac{1}{n+1}$	$(1 \mp \hat{a})^n \varphi_0$	${}_0 b_n^+ = \frac{2^{n+1}-1}{n+1}$
${}_1 b_n = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+2} = \frac{1}{(n+1)(n+2)}$	$\hat{a}(1 \mp \hat{a})^n \varphi_0$	${}_1 b_n^+ = \frac{2^{n+1}n+1}{(n+1)(n+2)}$
${}_2 b_n = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{s+3} = \frac{2}{(n+1)(n+2)(n+3)}$	$\hat{a}^2(1 \mp \hat{a})^n \varphi_0$	${}_2 b_n^+ = \frac{2\left(2^n(n^2+n+2)-1\right)}{(n+1)(n+2)(n+3)}$
...	...	...
${}_m b_n = \frac{n!m!}{(n+m+1)!}$	$\hat{a}^m(1 \mp \hat{a})^n \varphi_0$	${}_m b_n^+ = \frac{{}_2F_1(-n, m+1, m+2, -1)}{m+1}$
High Order BHN	**U.I.**	**Recursivity**
${}_m b_n^{(k)} = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{ks+m+1}$		
${}_0 b_n^{(2)} = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{2s+1} = \frac{n!\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}$	$(1-\hat{a}^2)^n \varphi_0$	$= \sum_{s=0}^{n} \binom{n}{s} {}_s b_n \Rightarrow \sum_{s=0}^{n} \frac{1}{(n-s)!(s+n+1)!} = \frac{2^{2n}}{(2n+1)!}$
${}_1 b_n^{(2)} = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{2s+2} = \frac{1}{2(n+1)}$	$\hat{a}(1-\hat{a}^2)^n \varphi_0$	$= \sum_{s=0}^{n} \binom{n}{s} {}_{s+1} b_n \Rightarrow \sum_{s=0}^{n} \frac{(s+1)}{(n-s)!(s+n+2)!} = \frac{1}{2n!(n+1)!}$
${}_2 b_n^{(2)} = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{2s+3} = \frac{n!\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(n+\frac{5}{2}\right)}$	$\hat{a}^2(1-\hat{a}^2)^n \varphi_0$	$= \sum_{s=0}^{n} \binom{n}{s} {}_{s+2} b_n \Rightarrow \sum_{s=0}^{n} \frac{(s+1)(s+2)}{(n-s)!(s+n+3)!} = \frac{2^{2n}}{(2n+3)(2n+1)!}$
...	...	...
${}_m b_n^{(2)} = \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{2s+m+1} = \frac{n!\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{2n+m+3}{2}\right)}$	$\hat{a}^m(1-\hat{a}^2)^n \varphi_0$	$= \sum_{s=0}^{n} \binom{n}{s} {}_{s+m} b_n \Rightarrow \sum_{s=0}^{n} \frac{(s+m)!}{(n-s)!s!(s+m+n+1)!} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2n!\Gamma\left(\frac{2n+m+3}{2}\right)}$
...	...	...
${}_m b_n^{(k)} = \frac{n!\Gamma\left(\frac{m+1}{k}\right)}{k\Gamma\left(\frac{kn+m+k+1}{k}\right)}$	$\hat{a}^m(1-\hat{a}^k)^n \varphi_0$	$= \sum_{s=0}^{n} \binom{n}{s} {}_{\frac{k}{2}s+m} b_n^{\left(\frac{k}{2}\right)} \Rightarrow \sum_{s=0}^{n} \frac{\Gamma\left(s+2\frac{m+1}{k}\right)}{(n-s)!s!\frac{k}{2}\Gamma\left(n+s+1+2\frac{m+1}{k}\right)} = \frac{\Gamma\left(\frac{m+1}{k}\right)}{kn!\Gamma\left(\frac{kn+m+k+1}{k}\right)}$

Table 2. Combinatorial Identities.

BHN Combinatorics	**Integral Representations**	**Sum**
$\frac{1}{n+1} = {}_0b_n$	${}_0b_n = \int_0^\infty (1-e^{-\sigma})^n e^{-\sigma} d\sigma = \frac{1}{n+1}$	$\sum_{n=0}^\infty {}_0b_n = \infty$
$\frac{1}{n+2} = {}_0b_n - {}_1b_n$	${}_1b_n = \int_0^\infty (1-e^{-\sigma})^n e^{-2\sigma} d\sigma = \frac{1}{(n+1)(n+2)}$	$\sum_{n=0}^\infty {}_1b_n = 1$
$\frac{1}{n+3} = {}_0b_n - 2{}_1b_n + {}_2b_n$	${}_2b_n = \int_0^\infty (1-e^{-\sigma})^n e^{-3\sigma} d\sigma = \frac{2}{(n+1)(n+2)(n+3)}$	$\sum_{n=0}^\infty {}_2b_n = \frac{1}{2}$
...	...	...
$\frac{1}{n+m+1} = \sum_{r=0}^m \binom{m}{r}(-1)^r {}_rb_n$	${}_mb_n = \int_0^\infty (1-e^{-\sigma})^n e^{-m+1\sigma} d\sigma = \frac{n!m!}{(n+m+1)!}$	$\sum_{n=0}^\infty {}_mb_n = \frac{1}{m}$
Binomial TS	**Integral Representations**	**Telescopic Series**
	BTS; **TS**	${}_mT_k = \sum_{s=0}^\infty \frac{(s+m-k)!}{(s+m)!}$
$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s}{s+1} = \frac{1}{n+1}$	$\int x^n\ dx$	...
$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s}{(s+1)(s+2)} = \frac{1}{n+2}$	$\int x^{n+1}\ dx;\ \int_0^x \xi\xi^{-1}\ d\xi$	${}_2T_2 = \sum_{s=0}^\infty \frac{1}{(s+1)(s+2)} = 1$
$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s}{(s+1)(s+2)(s+3)} = \frac{1}{2(n+3)}$	$\int x^{n+2}\ dx;\ \int_0^x \xi^2\xi^{-1}\ d\xi$	${}_3T_3 = \sum_{s=0}^\infty \frac{1}{(s+1)(s+2)(s+3)} = \frac{1}{4}$
...	...	...
$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s\ s!}{(s+m+1)!} = \frac{1}{m!(m+n+1)}$	$\int x^{n+m}\ dx;\ \int_0^x \xi^m\xi^{-1}\ d\xi$	${}_{m+1}T_{m+1} = \sum_{s=0}^\infty \frac{s!}{(s+m+1)!} = \frac{1}{m!m}$

Table 3. Combinatorial Identities

Integral Representations	**Further Results**
$\int_0^x (1+\xi)^n d\xi \quad \Rightarrow\|_{x=1}$	$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s 2^{n-s}}{s+1} = \frac{2^{n+1}-1}{n+1}$
$\int_0^x (1+\xi)^n \xi d\xi \quad \Rightarrow\|_{x=1}$	$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s 2^{n-s}}{s+2} = \frac{2^{n+2}-(n+3)}{(n+1)(n+2)}$
$\int_0^x (a+b\xi)^n d\xi \quad \Rightarrow\|_{x=-1,a=2,b=1}$	$\sum_{s=0}^n \binom{n}{s} \frac{1}{s+1} = \frac{2^{n+1}-1}{n+1}$
$\int_0^1 (1-\xi^2)^n d\xi \quad \Rightarrow$	$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s}{2s+1} = \frac{2^{2n} n!^2}{(2n+1)!} = \frac{1}{2} B\left(\frac{1}{2}, n+1\right) = \left(\frac{3}{2}+n\right) B\left(\frac{3}{2}, n+1\right)$
$\int_0^1 (1-\xi^\alpha)^n \xi^{\beta-1} d\xi \quad \Rightarrow$	$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s}{\alpha s+\beta} = \frac{1}{\alpha} B\left(\frac{\beta}{\alpha}, n+1\right)$
$\int_0^x H_n(\xi, y) d\xi \quad \Rightarrow$	$\sum_{s=0}^n \binom{n}{s} \frac{(-1)^s x^{s+1}}{s+1} H_{n-s}(x,y) = \frac{H_{n+1}(x,y) - H_{n+1}(0,y)}{n+1} \Rightarrow$ $\Rightarrow \frac{H_{n+1}(x,y) - H_{n+1}(0,y)}{(n+1)x} = \int_0^\infty H_n(x\left(1-e^{-\sigma}\right), y) e^{-\sigma} d\sigma$
Binomial Convoluted HN	**Umbral Image**
${}_m\beta_n = \sum_{s=0}^n \binom{n}{s} (-1)^s h_{m+s}$	$\hat{h}^r \psi_0 = h_r = \sum_{s=0}^r \frac{1}{s+1}$
${}_0\beta_n = -\frac{1}{n(n+1)}$	$(1-\hat{h})^n \psi_0$
${}_1\beta_n = -\frac{2}{n(n+1)(n+2)}$	$\hat{h}(1-\hat{h})^n \psi_0$
${}_2\beta_n = -\frac{6}{n(n+1)(n+2)(n+3)}$	$\hat{h}^2(1-\hat{h})^n \psi_0$
...	...
${}_m\beta_n = -\frac{(n-1)!(m+1)!}{(n+m+1)!}$	$\hat{h}^m(1-\hat{h})^n \psi_0$