

Research Article

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Singular quasilinear convective elliptic systems in \mathbb{R}^N <https://doi.org/10.1515/anona-2021-0208>

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Abstract: The existence of a positive entire weak solution to a singular quasi-linear elliptic system with convection terms is established, chiefly through perturbation techniques, fixed point arguments, and a priori estimates. Some regularity results are then employed to show that the obtained solution is actually strong.

Keywords: quasilinear elliptic system, gradient dependence, singular term, entire solution, strong solution

MSC: 35J47, 35J75, 35B08, 35D35

1 Introduction and main result

In this paper, we deal with the problem

$$\begin{cases} -\Delta_p u = f(x, u, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = g(x, u, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (\text{P})$$

where $N \geq 3$, $2 - \frac{1}{N} < p, q < N$, $\Delta_r z := \operatorname{div}(|\nabla z|^{r-2} \nabla z)$ denotes the r -Laplacian of z for $1 < r < +\infty$, while $f, g : \mathbb{R}^N \times (0, +\infty)^2 \times \mathbb{R}^{2N} \rightarrow (0, +\infty)$ are Carathéodory functions satisfying assumptions H_1 – H_3 below.

Problem (P) exhibits three interesting features:

- The reaction terms f and g can be singular at zero.
- f, g depend on the gradient of solutions.
- Equations are set in the whole space \mathbb{R}^N .

However, they give rise to some nontrivial difficulties, such as the loss of variational structure and the lack of compactness for Sobolev embedding. This work continues the study started in [32], whose setting was \mathbb{R}^N and convective terms did not appear, along the very recent papers [6, 23, 24, 31], which address analogous questions, but concerning a bounded domain.

Primarily, we need an appropriate functional framework where to treat the problem, mainly because the integrability properties of solutions and their gradients may differ at infinity, as the example in [20, p. 80] shows. Accordingly, one is led to employ the so-called Beppo Levi (or homogeneous Sobolev) spaces $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, systematically studied for the first time by Deny and Lions [15]. The monographs [20, 29, 38] provide an exhaustive introduction on the topic.

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Let $X := \mathcal{D}_0^{1,p}(\mathbb{R}^N) \times \mathcal{D}_0^{1,q}(\mathbb{R}^N)$ and let r' denote the conjugate exponent of $r > 1$. A pair $(u, v) \in X$ such that $u, v > 0$ a.e. in \mathbb{R}^N is called:

1) *distributional solution* to (P) if for every $(\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N)^2$ one has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_1 \, dx &= \int_{\mathbb{R}^N} f(\cdot, u, v, \nabla u, \nabla v) \varphi_1 \, dx, \\ \int_{\mathbb{R}^N} |\nabla v|^{q-2} \nabla v \nabla \varphi_2 \, dx &= \int_{\mathbb{R}^N} g(\cdot, u, v, \nabla u, \nabla v) \varphi_2 \, dx; \end{aligned} \tag{1.1}$$

2) *(weak) solution* of (P) when (1.1) holds for all $(\varphi_1, \varphi_2) \in X$;

3) *'strong' solution* to (P) if $|\nabla u|^{p-2} \nabla u, |\nabla v|^{q-2} \nabla v \in W_{loc}^{1,2}(\mathbb{R}^N)$ and the differential equations are satisfied a.e. in \mathbb{R}^N .

Obviously, both 2) and 3) force 1), whilst reverse implications turn out generally false; see also Remark 4.5. Moreover, as observed at p. 48 of [38], problems in unbounded domains may admit strong solutions that are not weak or vice-versa. So, the search for strong solutions appears of some interest in this context.

Roughly speaking, our technical approach proceeds as follows. We first solve an auxiliary problem (P^ε) , $\varepsilon > 0$, obtained by shifting variables of reactions, which avoids singularities. To do this, nonlinear regularity theory, a priori estimates, Moser's iteration, trapping region, and fixed point arguments are employed. Unfortunately, bounds from above alone do not allow to get a solution of (P): treating singular terms additionally requires some estimates from below. Theorem 3.1 in [14] ensures that solutions to (P^ε) turn out locally greater than a positive constant regardless of ε . Thus, under hypotheses H_1 – H_3 below, we can construct a sequence $\{(u_\varepsilon, v_\varepsilon)\} \subseteq X$ such that $(u_\varepsilon, v_\varepsilon)$ solves (P^ε) for all $\varepsilon > 0$ and whose weak limit as $\varepsilon \rightarrow 0^+$ is a distributional solution to (P); cf. Lemma 4.1. Next, a localization-regularization reasoning (see Lemma 4.2) shows that

$$(u, v) \text{ distributional solution} \Rightarrow (u, v) \text{ weak solution.}$$

Through a recent differentiability result [10, Theorem 2.1] one then has

$$(u, v) \text{ distributional solution} \Rightarrow (u, v) \text{ strong solution;}$$

cf. Lemma 4.3. Further, $(u, v) \in C_{loc}^{1,\alpha}(\mathbb{R}^N)$ once condition H_3 is slightly strengthened; see Remark 4.4.

Singular elliptic problems, either in bounded domains or in \mathbb{R}^N , have a long history, that traces back to [12, 28] and [8, 9, 13, 25, 27] for semi-linear equations. More recent results, involving also systems, can be found in [17, 32, 33, 36] and the references therein. A very fruitful approach has been developed in [3, 4]; see also [7] and [34]. Existence, regularity, and qualitative properties of the solutions have been investigated, e.g., in [1, 2, 11, 19, 30, 40].

Henceforth, the assumptions below will be posited. If $1 < r < N$ then, by definition, $r^* := \frac{Nr}{N-r}$. H₁(f) There exist $\alpha_1 \in (-1, 0]$, $\beta_1, \delta_1 \in [0, q-1]$, $\gamma_1 \in [0, p-1]$, $m_1, \hat{m}_1 > 0$, and $a_1 \in L_{loc}^{s_p}(\mathbb{R}^N)$, with $s_p > p'N$, such that

$$m_1 a_1(x) s_1^{\alpha_1} s_2^{\beta_1} \leq f(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \leq \hat{m}_1 a_1(x) \left(s_1^{\alpha_1} s_2^{\beta_1} + |\mathbf{t}_1|^{\gamma_1} + |\mathbf{t}_2|^{\delta_1} \right)$$

in $\mathbb{R}^N \times (0, +\infty)^2 \times \mathbb{R}^{2N}$. Moreover, $\text{ess inf}_{B_\rho} a_1 > 0$ for all $\rho > 0$.

H₁(g) There exist $\beta_2 \in (-1, 0]$, $\alpha_2, \gamma_2 \in [0, p-1]$, $\delta_2 \in [0, q-1]$, $m_2, \hat{m}_2 > 0$, and $a_2 \in L_{loc}^{s_q}(\mathbb{R}^N)$, with $s_q > q'N$, such that

$$m_2 a_2(x) s_1^{\alpha_2} s_2^{\beta_2} \leq g(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \leq \hat{m}_2 a_2(x) \left(s_1^{\alpha_2} s_2^{\beta_2} + |\mathbf{t}_1|^{\gamma_2} + |\mathbf{t}_2|^{\delta_2} \right)$$

in $\mathbb{R}^N \times (0, +\infty)^2 \times \mathbb{R}^{2N}$. Moreover, $\text{ess inf}_{B_\rho} a_2 > 0$ for all $\rho > 0$.

H₁(a) There exist $\zeta_1, \zeta_2 \in (N, +\infty]$ such that $a_i \in L^1(\mathbb{R}^N) \cap L^{\zeta_i}(\mathbb{R}^N)$, $i = 1, 2$, where

$$\frac{1}{\zeta_1} < 1 - \frac{p}{p^*} - \theta_1, \quad \frac{1}{\zeta_2} < 1 - \frac{q}{q^*} - \theta_2,$$

with

$$\theta_1 := \max \left\{ \frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < 1 - \frac{p}{p^*}, \quad \theta_2 := \max \left\{ \frac{\alpha_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < 1 - \frac{q}{q^*}.$$

H₂ If $\eta_1 := \max\{\beta_1, \delta_1\}$ and $\eta_2 := \max\{\alpha_2, \gamma_2\}$ then

$$\eta_1 \eta_2 < (p - 1 - \gamma_1)(q - 1 - \delta_2).$$

H₃ One has

$$\frac{1}{s_p} + \max\left\{\frac{\gamma_1}{p}, \frac{\delta_1}{q}\right\} \leq \frac{1}{2}, \quad \frac{1}{s_q} + \max\left\{\frac{\gamma_2}{p}, \frac{\delta_2}{q}\right\} \leq \frac{1}{2}.$$

Example 1.1. $H_1(a)$ is fulfilled once $a_1, a_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\max\left\{\frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q}\right\} < 1 - \frac{p}{p^*}, \quad \max\left\{\frac{\alpha_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q}\right\} < 1 - \frac{q}{q^*}.$$

In fact, it suffices to choose $\zeta_1 := \zeta_2 := +\infty$.

Remark 1.2. By interpolation (see, e.g., [32, Proposition 2.1]), condition $H_1(a)$ entails $a_i \in L^{\sigma_{i,j}}(\mathbb{R}^N)$, $i = 1, 2$, where:

(i) $\sigma_{1,j} := \frac{1}{1-t_j}$, $j = 1, 2, 3, 4$, with

$$t_1 = \frac{\alpha_1 + 1}{p^*} + \frac{\beta_1}{q^*}, \quad t_2 = \frac{1}{p^*} + \frac{\beta_1}{q^*}, \quad t_3 = \frac{1}{p^*} + \frac{\gamma_1}{p}, \quad t_4 = \frac{1}{p^*} + \frac{\delta_1}{q};$$

(ii) $\sigma_{2,j} := \frac{1}{1-t_j}$, $j = 1, 2, 3, 4$, with

$$t_1 = \frac{\beta_2 + 1}{q^*} + \frac{\alpha_2}{p^*}, \quad t_2 = \frac{1}{q^*} + \frac{\alpha_2}{p^*}, \quad t_3 = \frac{1}{q^*} + \frac{\gamma_2}{p}, \quad t_4 = \frac{1}{q^*} + \frac{\delta_2}{q}.$$

The aim of this paper is to prove the following

Theorem 1.3. Under hypotheses H_1 – H_3 , problem (P) admits a weak and strong solution $(u, v) \in X$.

2 Preliminaries

Let Z be a Hausdorff topological space and let $T : Z \rightarrow Z$ be continuous. Following [22, p. 2], the operator T is called compact when $\overline{T(Z)}$ turns out a compact subset of Z . If Z is a normed space, $\{z_n\} \subseteq Z$, and $z \in Z$ then $z_n \rightarrow z$ in Z means that the sequence $\{z_n\}$ strongly converges to z , while $z_n \rightharpoonup z$ stands for weak convergence. As usual, Z^* denotes the topological dual of Z and $Z^2 := Z \times Z$.

Hereafter, $N \geq 3$ is a fixed integer, $B(x, \rho)$ indicates the open ball in \mathbb{R}^N of radius $\rho > 0$ centered at $x \in \mathbb{R}^N$ and $B_\rho := B(0, \rho)$, while $|E|$ stands for the Lebesgue measure of E .

Let $Z := Z(\Omega)$ be a real-valued function space on a nonempty measurable set $\Omega \subseteq \mathbb{R}^N$. If $z_1, z_2 \in Z$ and $z_1(x) < z_2(x)$ a.e. in Ω then we simply write $z_1 < z_2$. The meaning of $z_1 \leq z_2$, etc. is analogous. Put

$$Z_+ := \{z \in Z : z > 0\}.$$

Given $\{z_n\} \subseteq Z$ and $z \in Z$, the symbol $z_n \uparrow z$ signifies that $\{z_n\}$ is monotone increasing and $z_n(x) \rightarrow z(x)$ for almost every $x \in \Omega$. Moreover,

$$z^\pm := \max\{\pm z, 0\}, \quad \text{supp } z := \overline{\{x \in \Omega : z(x) \neq 0\}}.$$

When $\Omega := \mathbb{R}^N$, we write $z \in Z_{\text{loc}}(\mathbb{R}^N)$ if for every nonempty compact subset K of \mathbb{R}^N the restriction $z|_K$ belongs to $Z(K)$. Similarly, a sequence $\{z_n\} \subseteq Z_{\text{loc}}(\mathbb{R}^N)$ is called bounded in $Z_{\text{loc}}(\mathbb{R}^N)$ once the same holds for $\{z_n|_K\}$ in $Z(K)$, with any K as above.

Let $1 < r < N$ and let $z : \mathbb{R}^N \rightarrow \mathbb{R}$ be a measurable function. Throughout the paper, $r' := \frac{r}{r-1}$, $r^* := \frac{Nr}{N-r}$,

$$\|z\|_r := \left(\int_{\mathbb{R}^N} |z(x)|^r dx\right)^{1/r}, \quad \|z\|_\infty := \text{ess sup}_{x \in \mathbb{R}^N} |z(x)|.$$

We now recall the notion and some relevant properties of Beppo Levi’s space $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, addressing the reader to [20, Chapter II] for a complete treatment. Put

$$\mathcal{D}^{1,r} := \left\{ z \in L^1_{\text{loc}}(\mathbb{R}^N) : |\nabla z| \in L^r(\mathbb{R}^N) \right\}$$

and denote by \mathcal{R} the equivalence relation that identifies two elements in $\mathcal{D}^{1,r}$ whose difference is a constant. The quotient set $\dot{\mathcal{D}}^{1,r}$, endowed with the norm

$$\|z\|_{1,r} := \left(\int_{\mathbb{R}^N} |\nabla z(x)|^r dx \right)^{1/r},$$

turns out complete. Write $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$ for the subspace of $\dot{\mathcal{D}}^{1,r}$ defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under $\|\cdot\|_{1,r}$, namely

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{1,r}}.$$

$\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, usually called Beppo Levi space, is reflexive and continuously embeds in $L^{r^*}(\mathbb{R}^N)$, i.e.,

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) \hookrightarrow L^{r^*}(\mathbb{R}^N). \tag{2.1}$$

Consequently, if $z \in \mathcal{D}_0^{1,r}(\mathbb{R}^N)$ then z vanishes at infinity, meaning that the set $\{x \in \mathbb{R}^N : |z(x)| \geq \varepsilon\}$ has finite measure for any $\varepsilon > 0$. In fact, by Chebichev’s inequality and (2.1), one has

$$|\{x \in \mathbb{R}^N : |z(x)| \geq \varepsilon\}| \leq \varepsilon^{-r^*} \|z\|_{r^*}^{r^*} \leq (c\varepsilon^{-1} \|z\|_{1,r})^{r^*} < +\infty,$$

where $c > 0$ is the best constant related to (2.1); see the seminal paper [39].

Hereafter, c , c_ε , and $c_\varepsilon(\cdot)$ will denote generic positive constants, which may change explicit value from line to line. Subscripts and/or arguments emphasize their dependence on a given variable.

To avoid cumbersome expressions, define

$$X := \mathcal{D}_0^{1,p}(\mathbb{R}^N) \times \mathcal{D}_0^{1,q}(\mathbb{R}^N), \quad \|(u, v)\| := \|u\|_{1,p} + \|v\|_{1,q} \quad \forall (u, v) \in X,$$

$$\mathcal{C}_+^1 := X_+ \cap C_{\text{loc}}^1(\mathbb{R}^N)^2, \quad \mathcal{C}_+^{1,\alpha} := X_+ \cap C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N)^2.$$

\mathcal{C}_+^1 and $\mathcal{C}_+^{1,\alpha}$ will be endowed with the topology induced by that of X .

The following a priori estimate will play a basic role in the sequel.

Lemma 2.1. *Let $2 - \frac{1}{N} < r < +\infty$, let $\zeta > N$, and let $h \in L^1(\mathbb{R}^N) \cap L^\zeta(\mathbb{R}^N)$. If $z \in \mathcal{D}_0^{1,r}(\mathbb{R}^N) \cap C_{\text{loc}}^1(\mathbb{R}^N)$ is a weak solution to $-\Delta_r z = h(x)$ in \mathbb{R}^N then there exists $c > 0$, independent of z , such that*

$$\|\nabla z\|_\infty^{r-1} \leq c \inf_{R>0} \left(R^{1-\frac{N}{\zeta}} \|h\|_\zeta + R^{-\frac{N}{r}} \|\nabla z\|_r^{r-1} \right).$$

Proof. Pick any $x \in \mathbb{R}^N$ and $R > 0$. Via [26, Theorem 1.1], when $r \geq 2$, or [18, Theorem 1.1], if $2 - \frac{1}{N} < r < 2$, with $\Omega := B(x, R)$ and $\mu := h dx$, as well as Hölder’s inequality, we easily get

$$\begin{aligned} |\nabla z(x)|^{r-1} &\leq c \left[\int_0^R \left(\rho^{-N} \int_{B(x,\rho)} |h| dy \right) d\rho + \left(R^{-N} \int_{B(x,R)} |\nabla z| dy \right)^{r-1} \right] \\ &\leq c \left(\|h\|_\zeta \int_0^R \rho^{-\frac{N}{\zeta}} d\rho + R^{-\frac{r-1}{r}N} \|\nabla z\|_r^{r-1} \right) \\ &\leq c \left(R^{1-\frac{N}{\zeta}} \|h\|_\zeta + R^{-\frac{N}{r}} \|\nabla z\|_r^{r-1} \right), \end{aligned}$$

where $c > 0$ does not depend on z , h , x , and R ; see [18, Remark 1.3]. Taking the infimum in $R > 0$ on the right and the supremum in $x \in \mathbb{R}^N$ on the left yields the conclusion. \square

3 The regularized system

3.1 ‘Freezing’ the right-hand side

Fix $w := (w_1, w_2) \in \mathcal{C}_+^1$, $\varepsilon > 0$ and define

$$f_{w,\varepsilon} := f(\cdot, w_1 + \varepsilon, w_2, \nabla w), \quad g_{w,\varepsilon} := g(\cdot, w_1, w_2 + \varepsilon, \nabla w),$$

where $\nabla w := (\nabla w_1, \nabla w_2)$. We first focus on the auxiliary problem

$$\begin{cases} -\Delta_p u = f_{w,\varepsilon}(x) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = g_{w,\varepsilon}(x) & \text{in } \mathbb{R}^N. \end{cases} \quad (\text{P}_w^\varepsilon)$$

Lemma 3.1. *If H_1 holds then (P_w^ε) admits a unique solution $(u, v) \in \mathcal{C}_+^{1,\alpha}$, for a suitable $\alpha \in (0, 1)$.*

Proof. Hypothesis H_1 and (2.1) guarantee that $(f_{w,\varepsilon}, g_{w,\varepsilon}) \in X^*$. Hence, by Minty-Browder’s Theorem [5, Theorem 5.16], problem (P_w^ε) possesses a unique solution $(u, v) \in X$. Thanks to H_1 again one has

$$(f_{w,\varepsilon}, g_{w,\varepsilon}) \in L_{\text{loc}}^{s_p}(\mathbb{R}^N) \times L_{\text{loc}}^{s_q}(\mathbb{R}^N).$$

Thus, standard results from nonlinear regularity theory [16, p. 830] entail $(u, v) \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N)^2$. Testing the first equation in (P_w^ε) with u^- we next obtain

$$-\|\nabla u^-\|_p^p = \int_{\mathbb{R}^N} f_{w,\varepsilon} u^- dx \geq 0,$$

because f is non-negative, which forces $u \geq 0$. Likewise, $v \geq 0$. The strong maximum principle [35, Theorem 1.1.1] finally yields $(u, v) \in X_+$. \square

Throughout this sub-section, (u, v) will denote the solution to (P_w^ε) given by Lemma 3.1.

Lemma 3.2. *Let H_1 be satisfied. Then there exists $L_\varepsilon > 0$ such that*

$$\begin{aligned} \|\nabla u\|_p^{p-1} &\leq L_\varepsilon (1 + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1}), \\ \|\nabla v\|_q^{q-1} &\leq L_\varepsilon (1 + \|\nabla w_1\|_p^{\eta_2} + \|\nabla w_2\|_q^{\delta_2}), \end{aligned}$$

where $\eta_1 := \max\{\beta_1, \delta_1\}$ and $\eta_2 := \max\{\alpha_2, \gamma_2\}$.

Proof. Test the first equation in (P_w^ε) with u and exploit $H_1(f)$, $H_1(a)$, besides (2.1), to achieve

$$\begin{aligned} \|\nabla u\|_p^p &= \int_{\mathbb{R}^N} f(\cdot, w_1 + \varepsilon, w_2, \nabla w_1, \nabla w_2) u dx \\ &\leq \hat{m}_1 \int_{\mathbb{R}^N} a_1 [(w_1 + \varepsilon)^{\alpha_1} w_2^{\beta_1} + |\nabla w_1|^{\gamma_1} + |\nabla w_2|^{\delta_1}] u dx \\ &\leq \hat{m}_1 \int_{\mathbb{R}^N} a_1 \max\{1, \varepsilon^{\alpha_1}\} (w_2^{\beta_1} + |\nabla w_1|^{\gamma_1} + |\nabla w_2|^{\delta_1}) u dx \\ &\leq c_\varepsilon \|u\|_p \cdot (\|w_2\|_q^{\beta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\delta_1}) \\ &\leq c_\varepsilon \|\nabla u\|_p (\|\nabla w_2\|_q^{\beta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\delta_1}) \\ &\leq L_\varepsilon \|\nabla u\|_p (1 + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1}), \end{aligned} \quad (3.1)$$

because

$$\|\nabla w_2\|_q^{\beta_1} + \|\nabla w_2\|_q^{\delta_1} \leq 2(1 + \|\nabla w_2\|_q^{\eta_1}). \quad (3.2)$$

This shows the first inequality. The other is verified similarly. \square

Lemma 3.3. *Under H_1 , there exists $M_\varepsilon := M_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q) > 0$ such that*

$$\max\{\|u\|_\infty, \|v\|_\infty\} \leq M_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q).$$

Proof. The proof can be made by adapting the one of Lemma 3.3 in [32]. So, we will briefly focus the key-points only. Fix any $\xi_1 \in [1, \frac{p^*}{p})$ such that

$$\frac{1}{\xi_1} < 1 - \frac{1}{\xi_1} - \theta_1, \tag{3.3}$$

where ξ_1 and θ_1 come from $H_1(a)$. Set $u_K := \min\{u, K\}$, $K > 1$, and test (P_w^ε) with $\varphi := u_K^{kp+1}$, $k \geq 0$. Fatou’s Lemma, Hölder’s inequality joined to $H_1(a)$, Sobolev’s embedding (2.1), and (3.2) produce

$$\begin{aligned} \frac{kp+1}{(k+1)p} \|u\|_{(k+1)p}^{(k+1)p} &\leq \frac{kp+1}{(k+1)p} \liminf_{K \rightarrow +\infty} \|u_K\|_{(k+1)p}^{(k+1)p} \\ &\leq c \max\{1, \varepsilon^{\alpha_1}\} \int_{\mathbb{R}^N} a_1(w_2^{\beta_1} + |\nabla w_1|^{\gamma_1} + |\nabla w_2|^{\delta_1}) u^{kp+1} dx \\ &\leq c_\varepsilon (\|\nabla w_2\|_q^{\beta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\delta_1}) \|u\|_{(kp+1)\xi_1}^{kp+1} \\ &\leq c_\varepsilon (1 + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1}) \|u\|_{(kp+1)\xi_1}^{kp+1}; \end{aligned}$$

cf. [32, pp. 1587–1588], but replacing ξ_1' with ξ_1 . Moreover,

$$(k+1)p^* > (kp+1)\xi_1 \quad \forall k \geq 0$$

as $\xi_1 < \frac{p^*}{p}$. Hence, Moser’s iteration can start, and we obtain $\|u\|_\infty \leq M_\varepsilon$, where

$$M_\varepsilon := c_\varepsilon (1 + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1})^\tau$$

for some $\tau > 0$; details can be read in [32, pp. 1588–1590], replacing ξ_1' with ξ_1 as above. A similar argument applies to v . □

Lemma 3.4. *If H_1 holds and $\max\{\|w_i\|_\infty, \|\nabla w_i\|_\infty\} < +\infty$, $i = 1, 2$, then*

$$\begin{aligned} \|\nabla u\|_\infty^{p-1} &\leq N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_2\|_\infty) (1 + \|\nabla w_1\|_\infty^{\gamma_1} + \|\nabla w_2\|_\infty^{\delta_1}), \\ \|\nabla v\|_\infty^{q-1} &\leq N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_1\|_\infty) (1 + \|\nabla w_1\|_\infty^{\gamma_2} + \|\nabla w_2\|_\infty^{\delta_2}) \end{aligned}$$

with suitable constants $N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_i\|_\infty) > 0$, $i = 1, 2$.

Proof. Lemma 3.1 ensures that $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \cap C_{loc}^1(\mathbb{R}^N)$, while H_1 entails $f_{w,\varepsilon} \in L^1(\mathbb{R}^N) \cap L^{\zeta_1}(\mathbb{R}^N)$. By Lemma 2.1, besides H_1 again, we thus have

$$\begin{aligned} \|\nabla u\|_\infty^{p-1} &\leq c(\|f_{w,\varepsilon}\|_{\zeta_1} + \|\nabla u\|_p^{p-1}) \\ &\leq c[\max\{1, \varepsilon^{\alpha_1}\} a_1\|_{\zeta_1} (\|w_2\|_\infty^{\beta_1} + \|\nabla w_1\|_\infty^{\gamma_1} + \|\nabla w_2\|_\infty^{\delta_1}) + \|\nabla u\|_p^{p-1}] \\ &\leq c_\varepsilon (\|w_2\|_\infty^{\beta_1} + \|\nabla w_1\|_\infty^{\gamma_1} + \|\nabla w_2\|_\infty^{\delta_1} + \|\nabla u\|_p^{p-1}). \end{aligned}$$

Now, using Lemma 3.2 yields

$$\begin{aligned} \|\nabla u\|_\infty^{p-1} &\leq c_\varepsilon (\|w_2\|_\infty^{\beta_1} + \|\nabla w_1\|_\infty^{\gamma_1} + \|\nabla w_2\|_\infty^{\delta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1} + 1) \\ &\leq N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_2\|_\infty) (1 + \|\nabla w_1\|_\infty^{\gamma_1} + \|\nabla w_2\|_\infty^{\delta_1}), \end{aligned}$$

where

$$N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_2\|_\infty) := c_\varepsilon (1 + \|w_2\|_\infty^{\beta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1}).$$

This shows the first inequality. The other is analogous. □

3.2 Regularizing the right-hand side

Let H_1 be satisfied. Given $\varepsilon > 0$, define

$$\mathcal{R}_\varepsilon := \{(w_1, w_2) \in \mathcal{C}_+^1 : \|\nabla w_1\|_p \leq A_1, \|\nabla w_2\|_q \leq A_2, \\ \|w_i\|_\infty \leq B_i, \|\nabla w_i\|_\infty \leq C_i, i = 1, 2\},$$

with $A_i, B_i, C_i > 0, i = 1, 2$, such that

$$\begin{cases} A_1^{p-1} & \geq L_\varepsilon(1 + A_1^{\gamma_1} + A_2^{\eta_1}), \\ A_2^{q-1} & \geq L_\varepsilon(1 + A_1^{\eta_2} + A_2^{\delta_2}), \\ B_1, B_2 & \geq M_\varepsilon(A_1, A_2), \\ C_1^{p-1} & \geq N_\varepsilon(A_1, A_2, B_2)(1 + C_1^{\gamma_1} + C_2^{\delta_1}), \\ C_2^{q-1} & \geq N_\varepsilon(A_1, A_2, B_1)(1 + C_1^{\gamma_2} + C_2^{\delta_2}), \end{cases} \quad (3.4)$$

and $L_\varepsilon, M_\varepsilon(\cdot, \cdot), N_\varepsilon(\cdot, \cdot, \cdot)$ stemming from Lemmas 3.2–3.4. Apropos, system (3.4) admits solutions. In fact, by H_1 , we can pick

$$1 < \sigma < \frac{(p-1)(q-1)}{\eta_1 \eta_2}. \quad (3.5)$$

If $A_1 := K^{\frac{1}{\eta_2}}$ and $A_2 := K^{\frac{\sigma}{q-1}}$ then the first two inequalities of (3.4) become

$$K^{\frac{p-1}{\eta_2}} \geq L_\varepsilon(1 + K^{\frac{\gamma_1}{\eta_2}} + K^{\frac{\sigma \eta_1}{q-1}}), \quad K^\sigma \geq L_\varepsilon(1 + K + K^{\frac{\sigma \delta_2}{q-1}}),$$

which, due to (3.5), are true for any sufficiently large $K > 0$. Next, choose

$$B_1 := B_2 := M_\varepsilon(K^{\frac{1}{\eta_2}}, K^{\frac{\sigma}{q-1}}).$$

With A_i, B_i as above, set $C_1 := H^{\frac{1}{\eta_2}}$ and $C_2 := H^{\frac{\sigma}{q-1}}$. The last two inequalities in (3.4) rewrite as

$$H^{\frac{p-1}{\eta_2}} \geq N_\varepsilon(A_1, A_2, B_2)(1 + H^{\frac{\gamma_1}{\eta_2}} + H^{\frac{\sigma \delta_1}{q-1}}), \\ H^\sigma \geq N_\varepsilon(A_1, A_2, B_1)(1 + H^{\frac{\gamma_2}{\eta_2}} + H^{\frac{\sigma \delta_2}{q-1}}).$$

Thanks to (3.5) again, they hold for every $H > 0$ big enough.

On the *trapping region* \mathcal{R}_ε we will consider the topology induced by that of X . Let us now investigate the regularized problem

$$\begin{cases} -\Delta_p u = f(x, u + \varepsilon, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = g(x, u, v + \varepsilon, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (\text{P}^\varepsilon)$$

where $\varepsilon \geq 0$. Evidently, (P^ε) reduces to (P) once $\varepsilon = 0$.

Lemma 3.5. *Under H_1 , for every $\varepsilon > 0$ problem (P^ε) possesses a solution $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}_+^{1,\alpha}$.*

Proof. Fix $\varepsilon > 0$ and define, provided $w \in \mathcal{R}_\varepsilon$,

$$T_\varepsilon(w) := (u, v), \quad \text{with } (u, v) \text{ being the unique solution to } (\text{P}_w^\varepsilon);$$

cf. Lemma 3.1. From Lemmas 3.2–3.4, besides (3.4), it follows $T_\varepsilon(\mathcal{R}_\varepsilon) \subseteq \mathcal{R}_\varepsilon$.

Claim 1. $T_\varepsilon(\mathcal{R}_\varepsilon)$ is relatively compact in X .

To see this, pick $\{w_n\} \subseteq \mathcal{R}_\varepsilon$, put

$$w_n := (w_{1,n}, w_{2,n}), \quad (u_n, v_n) := T_\varepsilon(w_n), \quad n \in \mathbb{N},$$

and understand any convergence up to sub-sequences. Since $\{T_\varepsilon(w_n)\} \subseteq \mathcal{R}_\varepsilon$ while X is reflexive, $\{(u_n, v_n)\}$ weakly converges to a point $(u, v) \in X$. Taking any $\rho > 0$, if $Y_\rho := L^p(B_\rho) \times L^q(B_\rho)$, then one has

$$X \hookrightarrow W^{1,p}(B_\rho) \times W^{1,q}(B_\rho) \hookrightarrow Y_\rho. \quad (3.6)$$

Actually, the first embedding in (3.6) is continuous by (2.1) and the continuity of the restriction map $L^r(\mathbb{R}^N) \rightarrow L^r(B_\rho)$, while the other one is compact due to Rellich-Kondrakov's theorem [5, Theorem 9.16]. Thus, $X \hookrightarrow Y_\rho$ compactly, which yields $(u_n, v_n) \rightarrow (u, v)$ in Y_ρ . Let us next verify that

$$(u_n(x), v_n(x)) \rightarrow (u(x), v(x)) \text{ for almost every } x \in \mathbb{R}^N. \tag{3.7}$$

In fact, $(u_n, v_n) \rightarrow (u, v)$ in Y_1 yields a sub-sequence $\{(u_n^{(1)}, v_n^{(1)})\}$ of $\{(u_n, v_n)\}$ such that

$$(u_n^{(1)}(x), v_n^{(1)}(x)) \rightarrow (u(x), v(x)) \text{ for almost all } x \in B_1.$$

Since $(u_n^{(1)}, v_n^{(1)}) \rightarrow (u, v)$ in Y_2 , we can extract a sub-sequence $\{(u_n^{(2)}, v_n^{(2)})\}$ from $\{(u_n^{(1)}, v_n^{(1)})\}$ fulfilling

$$(u_n^{(2)}(x), v_n^{(2)}(x)) \rightarrow (u(x), v(x)) \text{ for almost every } x \in B_2.$$

By induction, to each $k \geq 2$ there corresponds a sub-sequence $\{(u_n^{(k)}, v_n^{(k)})\}$ of $\{(u_n^{(k-1)}, v_n^{(k-1)})\}$ such that

$$(u_n^{(k)}(x), v_n^{(k)}(x)) \rightarrow (u(x), v(x)) \text{ for almost all } x \in B_k.$$

Now, Cantor's diagonal procedure leads to $(u_n^{(n)}, v_n^{(n)}) \rightarrow (u, v)$ a.e. in \mathbb{R}^N , because $\bigcup_{k=1}^\infty B_k = \mathbb{R}^N$, and (3.7) follows.

Through $H_1(f)$, besides the inclusion $\{w_n\} \subseteq \mathcal{R}_\varepsilon$, we get

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx \\ &= \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n) (u_n - u) dx \\ &\leq \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n) |u_n - u| dx \\ &\leq c_\varepsilon \int_{\mathbb{R}^N} a_1 |u_n - u| dx \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.8}$$

with $c_\varepsilon := \hat{m}_1(\varepsilon^{\alpha_1} B_2^{\beta_1} + C_1^{\gamma_1} + C_2^{\delta_1})$. Using $T_\varepsilon(\mathcal{R}_\varepsilon) \subseteq \mathcal{R}_\varepsilon$ and (3.7) one has

$$a_1 |u_n - u| \leq 2B_1 a_1 \in L^1(\mathbb{R}^N), \quad n \in \mathbb{N}.$$

So, by (3.7)–(3.8), Lebesgue's Theorem entails

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx \leq c_\varepsilon \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} a_1 |u_n - u| dx = 0.$$

Now, recall (cf., e.g., [32, Proposition 2.2]) that the operator $(-\Delta_p, \mathcal{D}_0^{1,p}(\mathbb{R}^N))$ is of type $(S)_+$ to achieve $u_n \rightarrow u$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. A similar reasoning applies to $\{v_n\}$.

Claim 2. $T_\varepsilon : \mathcal{R}_\varepsilon \rightarrow \mathcal{R}_\varepsilon$ is continuous.

Let $\{w_n\} \subseteq \mathcal{R}_\varepsilon$ and $w \in \mathcal{R}_\varepsilon$ satisfy $w_n \rightarrow w$ in X . Thanks to (2.1), Theorem 4.9 of [5] provides

$$w_n(x) \rightarrow w(x) \text{ and } \nabla w_n(x) \rightarrow \nabla w(x) \text{ for almost every } x \in \mathbb{R}^N. \tag{3.9}$$

Moreover, if $(u_n, v_n) := T_\varepsilon(w_n)$, $n \in \mathbb{N}$, then there exists a point $(u, v) \in X$ such that $(u_n, v_n) \rightarrow (u, v)$ in X ; see the proof of Claim 1. Arguing as before, we obtain

$$u_n(x) \rightarrow u(x) \text{ and } \nabla u_n(x) \rightarrow \nabla u(x) \text{ for almost every } x \in \mathbb{R}^N. \tag{3.10}$$

Since $\|\nabla u_n\|_p \leq A_1$ whatever n , the sequence $\{|\nabla u_n|^{p-2} \nabla u_n\} \subseteq L^{p'}(\mathbb{R}^N)$ turns out bounded. Due to (3.10) and [5, Exercise 4.16], this yields

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx, \quad \varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N). \tag{3.11}$$

On the other hand,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n) \varphi \, dx = \int_{\mathbb{R}^N} f(\cdot, w_1 + \varepsilon, w_2, \nabla w) \varphi \, dx \tag{3.12}$$

by Lebesgue’s Theorem jointly with (3.9) and the inequality

$$f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n) |\varphi| \leq c_\varepsilon a_1 |\varphi| \in L^1(\mathbb{R}^N) \quad \forall n \in \mathbb{N},$$

which easily arises from $H_1(f)$ besides the choice of \mathcal{R}_ε . Finally,

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx = \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n) \varphi \, dx, \quad n \in \mathbb{N}, \tag{3.13}$$

because (u_n, v_n) solves $(P_{w_n}^\varepsilon)$. Gathering (3.11)–(3.13) together we have

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f(\cdot, w_1 + \varepsilon, w_2, \nabla w) \varphi \, dx \quad \forall \varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).$$

The same is evidently true for v . So, (u, v) turns out a solution to (P_w^ε) . Uniqueness forces $(u, v) = T_\varepsilon(w)$, whence $T_\varepsilon(w_n) \rightarrow T_\varepsilon(w)$.

Now, Theorem 3.2 in [22, p. 119] can be applied, and T_ε admits a fixed point $(u_\varepsilon, v_\varepsilon) \in \mathcal{R}_\varepsilon$. By definition of T_ε , the pair $(u_\varepsilon, v_\varepsilon)$ solves problem (P^ε) , while Lemma 3.1 gives $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}_+^{1,\alpha}$. \square

Lemma 3.6. *If H_1 – H_2 hold then there exists a constant $L > 0$, independent of $\varepsilon \geq 0$, such that $\|(u, v)\| \leq L$ for every solution $(u, v) \in X_+$ to (P^ε) .*

Proof. Pick $\varepsilon \geq 0$ and suppose $(u, v) \in X_+$ solves (P^ε) . Via H_1 and (2.1) one has

$$\begin{aligned} \|\nabla u\|_p^p &= \int_{\mathbb{R}^N} f(\cdot, u + \varepsilon, v, \nabla u, \nabla v) u \, dx \\ &\leq \hat{m}_1 \int_{\mathbb{R}^N} a_1 [(u + \varepsilon)^{\alpha_1} v^{\beta_1} + |\nabla u|^{\gamma_1} + |\nabla v|^{\delta_1}] u \, dx \\ &\leq \hat{m}_1 \int_{\mathbb{R}^N} a_1 (u^{\alpha_1+1} v^{\beta_1} + |\nabla u|^{\gamma_1} u + |\nabla v|^{\delta_1} u) \, dx \\ &\leq c \left(\|u\|_{p^*}^{\alpha_1+1} \|v\|_q^{\beta_1} + \|\nabla u\|_p^{\gamma_1} \|u\|_{p^*} + \|\nabla v\|_q^{\delta_1} \|u\|_{p^*} \right) \\ &\leq c \left(\|\nabla u\|_p^{\alpha_1+1} \|\nabla v\|_q^{\beta_1} + \|\nabla u\|_p^{\gamma_1+1} + \|\nabla v\|_q^{\delta_1} \|\nabla u\|_p \right) \\ &\leq c \max\{1, \|\nabla u\|_p^{\gamma_1+1}\} \max\{1, \|\nabla v\|_q^{\delta_1}\}. \end{aligned} \tag{3.14}$$

Likewise,

$$\|\nabla v\|_q^q \leq c \max\{1, \|\nabla v\|_q^{\delta_2+1}\} \max\{1, \|\nabla u\|_p^{\eta_2}\}. \tag{3.15}$$

It should be noted that the constant c does not depend on (u, v) and ε . If either $\|\nabla v\|_q \leq 1$ or $\|\nabla u\|_p \leq 1$ then (3.14)–(3.15) directly lead to the conclusion, because $\gamma_1 + 1 < p$ and $\delta_2 + 1 < q$; see H_1 . Hence, we may assume $\min\{\|\nabla u\|_p, \|\nabla v\|_q\} > 1$. Dividing (3.14)–(3.15) by $\|\nabla u\|_p^{\gamma_1+1}$ and $\|\nabla v\|_q^{\delta_2+1}$, respectively, yields

$$\|\nabla u\|_p^{p-\gamma_1-1} \leq c \|\nabla v\|_q^{\eta_1}, \quad \|\nabla v\|_q^{q-\delta_2-1} \leq c \|\nabla u\|_p^{\eta_2}.$$

This clearly entails

$$\|\nabla u\|_p^{p-\gamma_1-1} \leq c \|\nabla u\|_p^{\frac{\eta_1 \eta_2}{q-\delta_2-1}}, \quad \|\nabla v\|_q^{q-\delta_2-1} \leq c \|\nabla v\|_q^{\frac{\eta_1 \eta_2}{p-\gamma_1-1}}.$$

The conclusion now follows from H_2 . \square

Lemma 3.7. *Let H_1 – H_2 be satisfied. Then there exists $M > 0$, independent of $\varepsilon \geq 0$, such that*

$$\max\{\|u\|_\infty, \|v\|_\infty\} \leq M$$

for every solution $(u, v) \in X_+$ to (P^ε) .

Proof. The proof is similar to that one of Lemma 3.3. With the same notation, fix $\varepsilon \geq 0$, suppose $(u, v) \in X_+$ solves (P^ε) , and define $\Omega_1 := \{x \in \mathbb{R}^N : u(x) \geq 1\}$. Moreover, given $z \in L^r(\mathbb{R}^N)$, $r > 1$, write $\|z\|_r$ in place of $\|z\|_{L^r(\Omega_1)}$ when no confusion can arise. Exploiting $H_1(f)$ one has

$$\int_{\Omega_1} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \leq \hat{m}_1 \int_{\Omega_1} a_1 (v^{\beta_1} + |\nabla u|^{\gamma_1} + |\nabla v|^{\delta_1}) \varphi \, dx$$

for all $\varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)_+$; cf. [32, Lemma 3.2]. If $\varphi := u_K^{kp+1}$, $k \geq 0$, then Fatou's Lemma, Hölder's inequality combined with $H_1(a)$, Sobolev's embedding (2.1), and Lemma 3.6 produce

$$\begin{aligned} \frac{kp+1}{(k+1)^p} \|u\|_{(k+1)^p}^{(k+1)p} &\leq \frac{kp+1}{(k+1)^p} \liminf_{K \rightarrow +\infty} \|u_K\|_{(k+1)^p}^{(k+1)p} \\ &\leq c \int_{\Omega_1} a_1 (v^{\beta_1} + |\nabla u|^{\gamma_1} + |\nabla v|^{\delta_1}) u^{kp+1} \, dx \\ &\leq c (\|\nabla v\|_q^{\beta_1} + \|\nabla u\|_p^{\gamma_1} + \|\nabla v\|_q^{\delta_1}) \|u\|_{(kp+1)\xi_1}^{kp+1} \\ &\leq c \|u\|_{(kp+1)\xi_1}^{kp+1}, \end{aligned}$$

where $\xi_1 \in [1, \frac{p}{p-1}]$ fulfills (3.3) while c does not depend on (u, v) and ε . We now proceed exactly as in the proof of Lemma 3.3, getting $\|u\|_\infty \leq M$. The other inequality is analogous. \square

Lemma 3.8. *Assume H_1 – H_2 . Then to every $\rho > 0$ there corresponds $\sigma_\rho > 0$ such that*

$$\min \left\{ \operatorname{ess\,inf}_{B_\rho} u, \operatorname{ess\,inf}_{B_\rho} v \right\} \geq \sigma_\rho \quad (3.16)$$

for all $(u, v) \in X_+$ distributional solution of (P^ε) , with $0 \leq \varepsilon \leq 1$.

Proof. Fix $\rho > 0$. Conditions $H_1(f)$ – $H_1(g)$, besides Lemma 3.7, entail

$$\begin{aligned} f(\cdot, u + \varepsilon, v, \nabla u, \nabla v) &\geq m_1 \left(\operatorname{ess\,inf}_{B_\rho} a_1 \right) (M+1)^{\alpha_1} v^{\beta_1}, \\ g(\cdot, u, v + \varepsilon, \nabla u, \nabla v) &\geq m_2 \left(\operatorname{ess\,inf}_{B_\rho} a_2 \right) (M+1)^{\beta_2} u^{\alpha_2} \end{aligned}$$

a.e. in B_ρ . From [14, Theorem 3.1] it thus follows

$$\begin{aligned} \left(\operatorname{ess\,inf}_{B_\rho} u \right)^{p-1} &\geq \frac{c_\rho}{|B_\rho|} \int_{B_\rho} v^{\beta_1} \, dx \geq c_\rho \left(\operatorname{ess\,inf}_{B_\rho} v \right)^{\beta_1}, \\ \left(\operatorname{ess\,inf}_{B_\rho} v \right)^{q-1} &\geq \frac{c_\rho}{|B_\rho|} \int_{B_\rho} u^{\alpha_2} \, dx \geq c_\rho \left(\operatorname{ess\,inf}_{B_\rho} u \right)^{\alpha_2}, \end{aligned}$$

which easily give

$$\operatorname{ess\,inf}_{B_\rho} u \leq c_\rho \left(\operatorname{ess\,inf}_{B_\rho} u \right)^{\frac{(p-1)(q-1)}{\alpha_2 \beta_1}}, \quad \operatorname{ess\,inf}_{B_\rho} v \leq c_\rho \left(\operatorname{ess\,inf}_{B_\rho} v \right)^{\frac{(p-1)(q-1)}{\alpha_2 \beta_1}}.$$

Now, (3.16) is a simple consequence of H_1 , because $\alpha_2 \beta_1 < (p-1)(q-1)$. \square

4 Proof of the main result

Lemma 4.1. *Under H_1 – H_3 , problem (P) possesses a distributional solution $(u, v) \in X_+$.*

Proof. Let $\varepsilon_n := \frac{1}{n}$, $n \in \mathbb{N}$. Lemma 3.5 furnishes a sequence $\{(u_n, v_n)\} \subseteq \mathcal{C}_+^1$ such that (u_n, v_n) solves (P^{ε_n}) for all $n \in \mathbb{N}$. Since X is reflexive, by Lemma 3.6 one has $(u_n, v_n) \rightharpoonup (u, v)$ in X , where a sub-sequence is

considered when necessary. As before (cf. the proof of Lemma 3.5), this forces (3.7). Moreover, $(u, v) \in X_+$ because, thanks to Lemma 3.8, to each $\rho > 0$ there corresponds $\sigma_\rho > 0$ satisfying

$$\min \left\{ \inf_{B_\rho} u_n, \inf_{B_\rho} v_n \right\} \geq \sigma_\rho \quad \forall n \in \mathbb{N}. \tag{4.1}$$

Claim. For every $\rho > 0$, and along a sub-sequence if necessary, one has

$$(u_n, v_n) \rightarrow (u, v) \text{ in } W^{1,p}(B_\rho) \times W^{1,q}(B_\rho). \tag{4.2}$$

Likewise the proof of (3.9), this will force

$$(\nabla u_n, \nabla v_n) \rightarrow (\nabla u, \nabla v) \text{ a.e. in } \mathbb{R}^N. \tag{4.3}$$

Let $\rho > 0$. Hypothesis H_1 , (4.1), Lemma 3.7, and H_3 yield

$$\begin{aligned} & f(\cdot, u_n + 1/n, v_n, \nabla u_n, \nabla v_n) \\ & \leq \hat{m}_1 a_1 \left[(u_n + 1/n)^{\alpha_1} v_n^{\beta_1} + |\nabla u_n|^{\gamma_1} + |\nabla v_n|^{\delta_1} \right] \quad \text{in } B_{2\rho} \\ & \leq \hat{m}_1 \left(\sigma_{2\rho}^{\alpha_1} M^{\beta_1} + |\nabla u_n|^{\gamma_1} + |\nabla v_n|^{\delta_1} \right) a_1 \in L^2(B_{2\rho}) \end{aligned} \tag{4.4}$$

whatever n . So, [10, Theorem 2.1] combined with Lemma 3.6 ensures that $\{|\nabla u_n|^{p-2} \nabla u_n\}$ turns out bounded in $W^{1,2}(B_\rho)$. Since $p > 2 - \frac{1}{N}$, by Rellich-Kondrakov’s theorem [5, Theorem 9.16], the embedding $W^{1,2}(B_\rho) \hookrightarrow L^{p'}(B_\rho)$ is compact. Thus, up to sub-sequences,

$$|\nabla u_n|^{p-2} \nabla u_n \rightarrow U \text{ in } L^{p'}(B_\rho). \tag{4.5}$$

Next, observe that the linear operator

$$z \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \mapsto \nabla z|_{B_\rho} \in L^p(B_\rho)$$

turns out well-defined and continuous in the strong topologies. Therefore,

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^p(B_\rho); \tag{4.6}$$

cf. [5, Theorem 3.10]. Gathering [5, Proposition 3.5] and (4.5)–(4.6) together gives

$$\lim_{n \rightarrow \infty} \int_{B_\rho} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx = 0.$$

Since $(-\Delta_p, W^{1,p}(B_\rho))$ enjoys the $(S)_+$ -property, we easily achieve $u_n \rightarrow u$ in $W^{1,p}(B_\rho)$. A similar conclusion holds for $\{v_n\}$, which shows (4.2).

Now, to verify that (u, v) is a distributional solution of (P), pick any $(\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N)^2$ and choose $\rho > 0$ fulfilling

$$\text{supp } \varphi_1 \cup \text{supp } \varphi_2 \subseteq B_\rho.$$

By (4.2), [5, Theorem 4.9] furnishes $(h, k) \in L^p(B_\rho) \times L^q(B_\rho)$ such that

$$|\nabla u_n| \leq h, \quad |\nabla v_n| \leq k \quad \text{a.e. in } B_\rho \text{ and for all } n \in \mathbb{N},$$

whence

$$f(\cdot, u_n + 1/n, v_n, \nabla u_n, \nabla v_n) |\varphi_1| \leq c_\rho (1 + h^{\gamma_1} + k^{\delta_1}) a_1 |\varphi_1| \in L^1(\mathbb{R}^N), \quad n \in \mathbb{N},$$

through (4.4). So, thanks to (3.7) and (4.3), Lebesgue’s Theorem entails

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(\cdot, u_n + 1/n, v_n, \nabla u_n, \nabla v_n) \varphi_1 dx = \int_{\mathbb{R}^N} f(\cdot, u, v, \nabla u, \nabla v) \varphi_1 dx.$$

On account of (4.5) and (4.3), we then get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi_1 dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_1 dx.$$

Recalling that each (u_n, v_n) weakly solves (P^{ε_n}) produces

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_1 dx = \int_{\mathbb{R}^N} f(\cdot, u, v, \nabla u, \nabla v) \varphi_1 dx.$$

Likewise,

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi_2 dx = \int_{\mathbb{R}^N} g(\cdot, u, v, \nabla u, \nabla v) \varphi_2 dx,$$

and the assertion follows. □

Lemma 4.2. *Let H_1 – H_2 be satisfied and let $(u, v) \in X_+$ be a distributional solution to problem (P). Then (u, v) weakly solves (P).*

Proof. We evidently have, for any $\varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$,

$$\varphi = \varphi^+ - \varphi^-. \tag{4.7}$$

Due to the nature of φ^+ , a localization-regularization procedure will be necessary. With this aim, fix $\theta \in C^\infty([0, +\infty))$ such that

$$\theta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{when } t \geq 2, \end{cases} \quad \theta \text{ is decreasing in } (1, 2) \tag{4.8}$$

and a sequence $\{\rho_k\} \subseteq C_0^\infty(\mathbb{R}^N)$ of standard mollifiers [5, p. 108]. Define, for every $n, k \in \mathbb{N}$,

$$\theta_n(\cdot) := \theta(|\cdot|/n) \in C_0^\infty(\mathbb{R}^N), \quad \varphi_n := \theta_n \varphi^+ \in \mathcal{D}_0^{1,p}(\mathbb{R}^N),$$

$$\psi_{k,n} := \rho_k * \varphi_n \in C_0^\infty(\mathbb{R}^N).$$

Using (4.8) we easily get $\varphi_n \uparrow \varphi^+$. Moreover, $\lim_{k \rightarrow \infty} \psi_{k,n} = \varphi_n$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$, which entails

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \psi_{k,n} dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_n dx, \quad n \in \mathbb{N}. \tag{4.9}$$

If, to shorten notation, $\hat{f} := f(\cdot, u, v, \nabla u, \nabla v)$ then the linear functional

$$\psi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \mapsto \int_{B_{2n+2}} \hat{f} \psi dx$$

turns out continuous. In fact, Lemmas 3.7–3.8, Hölder’s inequality combined with $H_1(a)$, and (2.1) produce

$$\int_{B_{2n+2}} a_1 u^{\alpha_1} v^{\beta_1} |\psi| dx \leq \sigma_{2n+2}^{\alpha_1} M^{\beta_1} \|a_1\|_{(p^*)'} \|\psi\|_{p^*} \leq c_n \|\nabla \psi\|_p.$$

Now, the assertion follows from $H_1(f)$, because convection terms can be estimated as already made in (3.14). Observe next that

$$\text{supp } \psi_{k,n} \subseteq \overline{\text{supp } \rho_k + \text{supp } \varphi_n} \subseteq \overline{B_1 + B_{2n}} \subseteq B_{2n+2} \quad \forall n, k \in \mathbb{N};$$

see [5, Proposition 4.18]. Hence,

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \hat{f} \psi_{k,n} dx &= \lim_{k \rightarrow \infty} \int_{B_{2n+2}} \hat{f} \psi_{k,n} dx \\ &= \int_{B_{2n+2}} \hat{f} \varphi_n dx = \int_{\mathbb{R}^N} \hat{f} \varphi_n dx. \end{aligned} \tag{4.10}$$

On the other hand, the hypothesis $(u, v) \in X_+$ distributional solution to (P) evidently forces

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \psi_{k,n} dx = \int_{\mathbb{R}^N} \hat{f} \psi_{k,n} dx, \quad k, n \in \mathbb{N}.$$

Letting $k \rightarrow +\infty$ and exploiting (4.9)–(4.10) we thus achieve

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_n dx = \int_{\mathbb{R}^N} \hat{f} \varphi_n dx \quad \forall n \in \mathbb{N}. \tag{4.11}$$

Claim. $\varphi_n \rightarrow \varphi^+$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$.

In fact, for every $n \in \mathbb{N}$ one has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \varphi_n - \nabla \varphi^+|^p dx &= \int_{\mathbb{R}^N} |\varphi^+ \nabla \theta_n + \theta_n \nabla \varphi^+ - \nabla \varphi^+|^p dx \\ &\leq c \left(\int_{\mathbb{R}^N} (1 - \theta_n)^p |\nabla \varphi^+|^p dx + \int_{B_{2n} \setminus B_n} |\nabla \theta_n|^p (\varphi^+)^p dx \right) \\ &\leq c \int_{\mathbb{R}^N} (1 - \theta_n)^p |\nabla \varphi^+|^p dx \\ &\quad + c \left(\int_{B_{2n} \setminus B_n} |\nabla \theta_n|^{\frac{pp^*}{p^*-p}} dx \right)^{1-\frac{p}{p^*}} \left(\int_{B_{2n} \setminus B_n} (\varphi^+)^{p^*} dx \right)^{\frac{p}{p^*}} \\ &= c \int_{\mathbb{R}^N} (1 - \theta_n)^p |\nabla \varphi^+|^p dx + c \|\nabla \theta_n\|_N^p \left(\int_{B_{2n} \setminus B_n} (\varphi^+)^{p^*} dx \right)^{\frac{p}{p^*}}. \end{aligned} \tag{4.12}$$

Recall that $\varphi^+ \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. By (4.8), Lebesgue’s Theorem yields

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (1 - \theta_n)^p |\nabla \varphi^+|^p dx = 0 \tag{4.13}$$

while, on account of (2.1),

$$\lim_{n \rightarrow \infty} \int_{B_{2n} \setminus B_n} (\varphi^+)^{p^*} dx = 0. \tag{4.14}$$

Since, due to (4.8) again,

$$\int_{\mathbb{R}^N} |\nabla \theta_n|^N dx = \frac{1}{n^N} \int_{\mathbb{R}^N} \left| \theta' \left(\frac{|x|}{n} \right) \right|^N dx = \int_{\mathbb{R}^N} |\theta'(|x|)|^N dx < +\infty \quad \forall n \in \mathbb{N},$$

gathering (4.12)–(4.14) together shows the claim.

Consequently,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_n dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi^+ dx. \tag{4.15}$$

From $\varphi_n \uparrow \varphi^+$ and $\hat{f} \geq 0$ it then follows

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \hat{f} \varphi_n dx = \int_{\mathbb{R}^N} \hat{f} \varphi^+ dx \tag{4.16}$$

by Beppo Levi’s Theorem. Through (4.11), (4.15)–(4.16) we thus arrive at

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi^+ dx = \int_{\mathbb{R}^N} \hat{f} \varphi^+ dx.$$

Likewise, one has

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi^- dx = \int_{\mathbb{R}^N} \hat{f} \varphi^- dx,$$

whence (cf. (4.7))

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_{\mathbb{R}^N} f(\cdot, u, v, \nabla u, \nabla v) \varphi dx \quad \forall \varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).$$

An analogous argument applies to the second equation in (P). □

Lemma 4.3. *Let H_1 – H_3 be satisfied and let $(u, v) \in X_+$ be a distributional solution of (P). Then (u, v) strongly solves (P).*

Proof. Reasoning as before (see (4.4)) provides $f(\cdot, u, v, \nabla u, \nabla v) \in L^2_{\text{loc}}(\mathbb{R}^N)$. Thanks to [10, Theorem 2.1], this implies $|\nabla u|^{p-2}\nabla u \in W^{1,2}_{\text{loc}}(\mathbb{R}^N)$. Moreover,

$$-\Delta_p u(x) = f(x, u(x), v(x), \nabla u(x), \nabla v(x)) \quad \text{a.e. in } \mathbb{R}^N$$

because of [5, Corollary 4.24]. Similarly about v and the other equation. \square

Proof of Theorem 1.3.

Lemmas 4.1–4.3 directly give the conclusion.

Remark 4.4. If H_3 is replaced by the stronger condition

\underline{H}'_3 One has

$$\frac{1}{s_p} + \max \left\{ \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < \frac{1}{p'N}, \quad \frac{1}{s_q} + \max \left\{ \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < \frac{1}{q'N}$$

then any distributional solution $(u, v) \in X_+$ to (P) actually lies in $\mathcal{C}^{1,\alpha}_+$, with $\alpha \in (0, 1)$. To show this, pick $\hat{s}_p, \hat{s}_q > 0$ such that

$$\frac{1}{s_p} + \max \left\{ \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} \leq \frac{1}{\hat{s}_p} < \frac{1}{p'N}, \quad \frac{1}{s_q} + \max \left\{ \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} \leq \frac{1}{\hat{s}_q} < \frac{1}{q'N}.$$

As in the proof of (4.4), for every $\rho > 0$ one has

$$\begin{aligned} f(\cdot, u, v, \nabla u, \nabla v) &\leq c_\rho a_1 (1 + |\nabla u|^{\gamma_1} + |\nabla v|^{\delta_1}) \in L^{\hat{s}_p}(B_\rho), \\ g(\cdot, u, v, \nabla u, \nabla v) &\leq c_\rho a_2 (1 + |\nabla u|^{\gamma_2} + |\nabla v|^{\delta_2}) \in L^{\hat{s}_q}(B_\rho). \end{aligned}$$

Hence, known nonlinear regularity results [16, p. 830] entail $(u, v) \in \mathcal{C}^{1,\alpha}_+$.

Remark 4.5. Unfortunately, we were not able to find in the literature a definition of strong solution for elliptic equations driven by *non-linear* operators in divergence form. The one adopted here represents a quite natural extension of the semi-linear case $p = 2$, where it is asked that the solution $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N)$ and satisfies the differential equation a.e. in \mathbb{R}^N ; cf. [21, p. 219] and [38, pp. 7–8]. We cannot expect $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^N)$ for some $q > 1$, as the example of [10, Remark 2.7] shows. Nevertheless, if $(u, v) \in \mathcal{C}^1_+$ is a distributional solution to (P) then $u, v \in W^{2,2}_{\text{loc}}(\mathbb{R}^N)$ once $1 < p, q < 3$; see [37, p. 2]. On the other hand, each strong solution turns out distributional. So, our notion of strong solution should be read as a distributional solution with an extra differentiability property on the fields $|\nabla u|^{p-2}\nabla u$ and $|\nabla v|^{q-2}\nabla v$.

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