Applied Analysis and Optimization
Volume 5, Number 2, 2021, 251–261

Yokohama Publishers
ISSN 2189-1664 Online Journal

A MORE COMPLETE VERSION OF A MINIMAX THEOREM

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Abstract. In this paper, we present a more complete version of the minimax theorem established in [7]. As a consequence, we get, for instance, the following result: Let X be a compact, not singleton subset of a normed space $(E, \|\cdot\|)$ and let Y be a convex subset of E such that $X \subseteq \overline{Y}$. Then, for every convex set $S \subseteq Y$ dense in Y, for every upper semicontinuous bounded function $\gamma : X \to \mathbf{R}$ and for every $\lambda > \frac{4\sup_X |\gamma|}{\mathrm{diam}(X)}$, there exists $y^* \in S$ such that the function $x \to \gamma(x) + \lambda\|x - y^*\|$ has at least two global maxima in X.

1. Introduction

Here and in what follows, X is a topological space and Y is a convex set in a real Hausdorff topological vector space. A function $h : X \to \mathbf{R}$ is said to be inf-compact if $h^{-1}(]-\infty, r])$ is compact for all $r \in \mathbf{R}$.

A function $k : Y \to \mathbf{R}$ is said to be quasi-concave (resp. quasi-convex)) $k^{-1}([r, +\infty[)$ (resp. if $k^{-1}(]-\infty, r])$ is convex for all $r \in \mathbf{R}$.

If S is a convex subset of Y, we denote by $\mathcal{A}_S$ the class of all functions $f : X \times Y \to \mathbf{R}$ such that, for each $y \in S$, the function $f(\cdot, y)$ is lower semicontinuous and inf-compact.

Moreover, we denote by $\mathcal{B}$ the class of all functions $f : X \times Y \to \mathbf{R}$ such that either, for each $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous, or, for each $x \in X$, the function $f(x, \cdot)$ is concave.

For any $f : X \times Y \to \mathbf{R}$, we set

$$\alpha_f = \sup_Y \inf_X f$$

and

$$\beta_f = \inf_X \sup_Y f\ .$$

Also, we denote by $\mathcal{C}_f$ the family of all sets $S \subseteq Y$ such that

$$\inf_X \sup_S f = \inf_X \sup_Y f$$

and by $\tilde{\mathcal{C}}_f$ the family of all sets $S \subseteq Y$ such that

$$\sup_{y \in S} f(x, y) = \sup_{y \in Y} f(x, y)$$

for all $x \in X$.

2020 *Mathematics Subject Classification.* 49J35, 49K35, 90C47, 35J92.

Key words and phrases. Strict minimax inequality, global extremum, multiplicity, farthest point, integral functional, Kirchhhoff-type equation, Neumann problem.

In particular, notice that $S \in \tilde{\mathcal{C}}_f$ provided, for each $x \in X$, there is a topology on Y for which S is dense and $f(x,\cdot)$ is lower semicontinuous.

Furthermore, we denote by τ_f the topology on Y generated by the family

$$\{\{y \in Y : f(x,y) < r\}\}_{x\in X, r\in \mathbf{R}} \ .$$

So, τ_f is the weakest topology on Y for which $f(x,\cdot)$ is upper semicontinuous for all $x \in X$. In [7], we established the following minimax result:

Theorem 1.1. *For every $g \in \mathcal{A}_Y \cap \mathcal{B}$, at least one of the following assertions holds:*
(j) $\sup_Y \inf_X g = \inf_X \sup_Y g$;
(jj) there exists $y^* \in Y$ such that the function $g(\cdot, y^*)$ has at least two global minima.

The relevance of Theorem 1.1 resides essentially in the fact that it is a flexible tool which can fruitfully be used to obtain meaningful results of various nature. This is clearly shown by a series of recent papers ([8]- [14]).

So, we believe that it is of interest to present a more complete form of Theorem 1.1: this is just the aim of this paper.

2. Main results

Here is the main abstract result (with the usual rules in $\overline{\mathbf{R}}$):

Theorem 2.1. *Let $f : X \times Y \to \mathbf{R}$. Assume that there is a function $\psi : Y \to \mathbf{R}$ such that $f + \psi \in \mathcal{B}$ and*

$$\alpha_{f+\psi} < \beta_{f+\psi} \ .$$

Then, for every convex set $S \in \mathcal{C}_{f+\psi}$, for every bounded function $\varphi : X \to \mathbf{R}$ and for every $\lambda > 0$ such that $\lambda f + \varphi \in \mathcal{A}_S$ and

$$\lambda > \frac{2\sup_X |\varphi|}{\beta_{f+\psi} - \alpha_{f+\psi}} \ , \tag{2.1}$$

there exists $y^* \in S$ such that the function $\lambda f(\cdot, y^*) + \varphi(\cdot)$ has at least two global minima.

Proof. Consider the function $g : X \times Y \to \mathbf{R}$ defined by

$$g(x,y) = \lambda(f(x,y) + \psi(y)) + \varphi(x)$$

for all $(x,y) \in X \times Y$. Since $S \in \mathcal{C}_{f+\psi}$, we have

$$\inf_X \sup_S (f+\psi) = \inf_X \sup_Y (f+\psi) \ . \tag{2.2}$$

So, taking (2.1) and (2.2) into account, we have

$$\sup_S \inf_X g \le \sup_Y \inf_X g \le \lambda\alpha_{f+\psi} + \sup_X |\varphi|$$

$$< \lambda\beta_{f+\psi} - \sup_X |\varphi| = \lambda \inf_X \sup_S (f+\psi) - \sup_X |\varphi| \le \inf_X \sup_S g \ . \tag{2.3}$$

Now, observe that $g \in \mathcal{A}_S$ since $\lambda f + \varphi \in \mathcal{A}_S$ and, at the same time, $g \in \mathcal{B}$ since $f + \psi \in \mathcal{B}$. As a consequence, we can apply Theorem 1.1 to the restriction of the function g to $X \times S$. Therefore, in view of (2.3), there exists $y^* \in S$ such that the function $g(\cdot, y^*)$ (and hence $\lambda f(\cdot, y^*) + \varphi(\cdot)$) has at least two global minima, as claimed. □

Remark 2.2. As the above proof shows, Theorem 2.1 is a direct consequence of Theorem 1.1. However, Theorem 2.1 has at least four advantages with respect to Theorem 1.1. Namely, suppose that, for a given function $g \in \mathcal{A}_Y$, we are interested in ensuring the validity of assertion (jj). Then, if we apply Theorem 1.1 in this regard, we have to show that $g \in \mathcal{B}$ and that assertion (j) does not hold. On the contrary, if we apply Theorem 2.1, we can ensure the validity of (jj) also in cases where either $g \notin \mathcal{B}$ or (j) holds true too. In addition, Theorem 2.1 is able to ensure the validity of (jj) even in a remarkably stronger way: not only extending it to suitable perturbations of g, but also offering an information on the location of y^*.

First, we wish to show how to obtain the very classical minimax theorems in [3] and [6] by means of Theorem 2.1.

Let V be a real vector space, $A \subseteq V$, $\varphi : A \to \mathbf{R}$. We say that φ is finitely lower semicontinuous if, for every finite-dimensional linear subspace $F \subseteq V$, the function $f_{|A\cap F}$ is lower semicontinuous in the Euclidean topology of F.

In the next result, the topology of X has no role.

Theorem 2.3. *Let X be a convex set in a real vector space and let $f \in \mathcal{B}$. Assume that there is a convex set $S \in \tilde{\mathcal{C}}_f$ such that $f(\cdot, y)$ is finitely lower semicontinous and convex for all $y \in S$. Finally, assume that, for some $x_0 \in X$, the function that $f(x_0, \cdot)$ is $\tau_f - \sup$-compact.*

Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f \ .$$

Proof. Arguing by contradiction, assume that

$$\sup_Y \inf_X f < \inf_X \sup_Y f \ .$$

Denote by $\mathcal{D}$ the family of all convex polytopes in X. Since $\mathcal{D}$ is a filtering cover of X and $f(x_0, \cdot)$ is $\tau_f -$ sup-compact, by Proposition 2.1 of [7], there exists $P \in \mathcal{D}$ such that

$$\sup_Y \inf_P f < \inf_P \sup_Y f \ .$$

Let $\|\cdot\|$ be the Euclidean norm on span(P). So, $\|\cdot\|^2$ is strictly convex. Now, fix λ so that

$$\lambda > \frac{2\sup_{x\in P}\|x\|^2}{\inf_P \sup_Y f - \sup_Y \inf_P f} \ .$$

Notice that, for each $y \in S$, the function $x \to \|x\|^2 + \lambda f(x, y)$ is inf-compact in P with respect to the Euclidean topology. This is due to the assumption that $f(\cdot, y)$ is finitely lower semicontinuous and to the compactness of P in the Euclidean topology. As a consequence, if we consider P equipped with the Euclidean topology, we can apply Theorem 2.1 to the restriction of f to $P \times Y$ (recall that $S \in \tilde{\mathcal{C}}_f$), taking $\varphi = \|\cdot\|^2$. Accordingly, there would exist $y^* \in S$ such that the function

$x \to \|x\|^2 + \lambda f(x, y^*)$ has at least two global minima in P. But, this is absurd since this function is strictly convex. □

Reasoning exactly as in the proof of Theorem 2.3 (even in a simplified way, since there is no need to consider the family $\mathcal{D}$), we also get

Theorem 2.4. *Let X be a compact convex set in a topological vector space such that there exists a lower semicontinuous, strictly convex, bounded function $\varphi : X \to \mathbf{R}$. Let $f \in \mathcal{B}$. Assume that there is a convex set $S \in \mathcal{C}_f$ such that $f(\cdot, y)$ is lower semicontinuous and convex for all $y \in S$.*

Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f \ .$$

We now revisit two applications of Theorem 1.1 in the light of Theorem 2.1.

The first one concerns the so called farthest points ([1]- [4]).

Theorem 2.5. *Let X be a non-singleton compact subset of a metric space (E, d). Moreover, let $h : Y \to E$ be such that $X \subseteq \overline{h(Y)}$ and let the function $(x, y) \to f(x, y) := -d(x, h(y))$ belong to $\mathcal{B}$.*

Then, for every convex set $S \in \mathcal{C}_f$, for every upper semicontinuous bounded function $\gamma : X \to \mathbf{R}$ and for every λ satisfying

$$\lambda > \frac{4 \sup_X |\gamma|}{\operatorname{diam}(X)} \ ,$$

there exists $y^* \in S$ such that the function $x \to \gamma(x) + \lambda d(x, h(y^*))$ has at least two global maxima in X.

Proof. Since $X \subseteq \overline{h(Y)}$, we have

$$\sup_{x \in X} \inf_{y \in Y} d(x, h(y)) = 0 \ . \tag{2.4}$$

Also, for each $x_1, x_2 \in X$, $y \in Y$, we have

$$\frac{d(x_1, x_2)}{2} \leq \max\{d(x_1, h(y)), d(x_2, h(y))\}$$

and so

$$\frac{\operatorname{diam}(X)}{2} \leq \inf_{y \in Y} \sup_{x \in X} d(x, h(y)) \ . \tag{2.5}$$

Hence, in view of (2.4) and (2.5), we have

$$\sup_Y \inf_X f \leq -\frac{\operatorname{diam}(X)}{2} < 0 = \inf_X \sup_Y f \ .$$

Now, the conclusion follows directly from Theorem 2.1 taking $\varphi = -\gamma$. □

Of course, the most natural corollary of Theorem 2.5 is as follows:

Corollary 2.6. *Let X be a non-singleton compact subset of a normed space $(E, \|\cdot\|)$ and let Y be a convex subset of E such that $X \subseteq \overline{Y}$.*

Then, for every convex set $S \subseteq Y$ dense in Y, for every upper semicontinuous bounded function $\gamma : X \to \mathbf{R}$ and for every $\lambda > \frac{4 \sup_X |\gamma|}{\operatorname{diam}(X)}$, there exists $y^* \in S$ such that the function $x \to \gamma(x) + \lambda\|x - y^*\|$ has at least two global maxima in X.

In turn, from Corollary 2.6, we clearly get

Corollary 2.7. *Let X be a compact subset of a normed space $(E, \|\cdot\|)$ and let Y be a convex subset of E such that $X \subseteq \overline{Y}$. Assume that there exist a sequence $\{S_n\}$ of convex subsets of Y dense in Y and a sequence $\{\gamma_n\}$ of upper semicontinuous bounded real-valued functions on X, with $\lim_{n\to\infty} \sup_X |\gamma_n| = 0$, such that, for each $n \in \mathbf{N}$ and for each $y \in S_n$, the function $x \to \gamma_n(x) + \|x - y\|$ has a unique global maximum in X.*

Then, X is a singleton.

Remark 2.8. Notice that Corollary 2.7 improves Theorem 1.1 of [14] which, in turn, extended a classical result by Klee ([5]) to normed spaces. More precisely, Theorem 1.1 of [14] agrees with the particular case of Corollary 2.7 in which each S_n is equal to $\operatorname{conv}(X)$ and each γ_n is equal to 0. The second application concerns the calculus of variations. We will use the same symbol $|\cdot|$ to denote the norm of $\mathbf{R}$ and the norm of $\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and let $p > 1$. On the Sobolev space $W^{1,p}(\Omega)$, we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}} .$$

If $n \geq p$, we denote by $\mathcal{E}$ the class of all continuous functions $\sigma : \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{\xi \in \mathbf{R}} \frac{|\sigma(\xi)|}{1 + |\xi|^q} < +\infty ,$$

where $0 < q < \frac{pn}{n-p}$ if $p < n$ and $0 < q < +\infty$ if $p = n$. While, when $n < p$, $\mathcal{E}$ stands for the class of all continuous functions $\sigma : \mathbf{R} \to \mathbf{R}$.

Recall that a function $h : \Omega \times \mathbf{R}^m \to \mathbf{R}$ is said to be a normal integrand ([15]) if it is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$-measurable and $h(x, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$. Here $\mathcal{L}(\Omega)$ and $\mathcal{B}(\mathbf{R}^m)$ denote the Lebesgue and the Borel σ-algebras of subsets of Ω and $\mathbf{R}^m$, respectively.

Recall that if h is a normal integrand then, for each measurable function $u : \Omega \to \mathbf{R}^m$, the composite function $x \to h(x, u(x))$ is measurable ([15]).

We denote by $\mathcal{F}$ the class of all normal integrands $h : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ such that $h(x, \xi, \cdot)$ is convex for all $(x, \xi) \in \Omega \times \mathbf{R}$ and there are $M \in L^1(\Omega)$, $b > 0$ such that

$$M(x) - b(|\xi| + |\eta|^p) \leq h(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$.

Let us also recall two results proved in [9].

Proposition 2.9. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary, let $p > 1$ and let $h : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ be a normal integrand such that, for some $c, d > 0$, one has*

$$c|\eta|^p - d \leq h(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ and

$$\lim_{|\xi| \to +\infty} \inf_{(x,\eta) \in \Omega \times \mathbf{R}^n} h(x, \xi, \eta) = +\infty \ .$$

Then, in $W^{1,p}(\Omega)$, one has

$$\lim_{\|u\| \to +\infty} \int_\Omega h(x, u(x), \nabla u(x)) dx = +\infty \ .$$

Proposition 2.10. *Let X, Y be two non-empty sets and $I : X \to \mathbf{R}$, $J : X \times Y \to \mathbf{R}$ two given functions. Assume that there are two sets $A, B \subset X$ such that:*
(a) $\sup_A I < \inf_B I$;
(b) $\sup_Y \inf_A J(x, y) \leq 0$;
(c) $\inf_B \sup_Y J(x, y) \geq 0$;
(d) $\inf_{X \setminus B} \sup_Y J(x, y) = +\infty$.
Then, one has

$$\sup_Y \inf_X (I + J) \leq \sup_A I < \inf_B I \leq \inf_X \sup_Y (I + J) \ .$$

Furthermore, let us also recall the following classical fact:

Proposition 2.11. *Let $A \subseteq \mathbf{R}^n$ be any open set and let $v \in L^1(A) \setminus \{0\}$.*
Then, one has

$$\sup_{\alpha \in C_0^\infty(A)} \int_A \alpha(x) v(x) dx = +\infty \ .$$

After these preliminaries, we can prove the following result:

Theorem 2.12. *Let $h, k \in \mathcal{F}$ and let $\sigma \in \mathcal{E}$ be a strictly monotone function. Assume that:*
(i) *there are $c, d > 0$ such that*

$$c|\eta|^p - d \leq h(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ and

$$\lim_{|\xi| \to +\infty} \frac{\inf_{(x,\eta) \in \Omega \times \mathbf{R}^n} h(x, \xi, \eta)}{|\sigma(\xi)| + 1} = +\infty \ ;$$

(ii) *for each $\xi \in \mathbf{R}$, the function $h(\cdot, \xi, 0)$ lies in $L^1(\Omega)$;*
(iii) *there are $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$, with $\xi_1 < \xi_2 < \xi_3$, such that*

$$\max\left\{ \int_\Omega h(x, \xi_1, 0) dx, \int_\Omega h(x, \xi_3, 0) dx \right\} < \int_\Omega h(x, \xi_2, 0) dx \ .$$

Then, for every sequentially weakly closed set $V \subseteq W^{1,p}(\Omega)$, containing the constants, for every convex set $T \subseteq L^\infty(\Omega)$ dense in $L^\infty(\Omega)$, for every non-decreasing, *continuous, bounded function* $\omega : U \to \mathbf{R}$*, where* $U := \{\int_\Omega k(x, u(x), \nabla u(x))dx : u \in W^{1,p}(\Omega)\}$*, and for every* λ *satisfying*

$$\lambda > \frac{2\sup_U |\omega|}{\int_\Omega h(x,\xi_2,0)dx - \max\left\{\int_\Omega h(x,\xi_1,0)dx, \int_\Omega h(x,\xi_3,0)dx\right\}} \ , \tag{2.6}$$

there exists $\gamma \in T$ *such that the restriction to* V *of the functional*

$$u \to \lambda \int_\Omega h(x,u(x),\nabla u(x))dx + \int_\Omega \gamma(x)\sigma(u(x))dx + \omega\left(\int_\Omega k(x,u(x),\nabla u(x))dx\right)$$

has at least two global minima. The same conclusion holds also with $T = C_0^\infty(\Omega)$*.*

Proof. Fix V, T, ω, λ as in the conclusion. Since $\sigma \in \mathcal{E}$, in view of the Rellich-Kondrachov theorem, for each $u \in W^{1,p}(\Omega)$, we have $\sigma \circ u \in L^1(\Omega)$ and, for each $\gamma \in L^\infty(\Omega)$, the functional $u \to \int_\Omega \gamma(x)\sigma(u(x))dx$ is sequentially weakly continuous. Moreover, since $h, k \in \mathcal{F}$ the functionals $u \to \int_\Omega h(x, u(x), \nabla u(x)dx$ and $u \to \int_\Omega k(x, u(x), \nabla u(x)dx$ (possibly taking the value $+\infty$) are sequentially weakly lower semicontinuous ([2], Theorem 4.6.8). Hence, since ω is non-decreasing and continuous, the functional $u \to \omega\left(\int_\Omega k(x, u(x), \nabla u(x)dx\right)$ is sequentially weakly lower semicontinuous too. Set

$$X = \left\{u \in V : \int_\Omega h(x,u(x),\nabla u(x))dx < +\infty\right\} \ .$$

By (ii), the constants belong to X. Fix $\gamma \in L^\infty(\Omega)$. By (i), there is $\delta > 0$ such that

$$h(x,\xi,\eta) - 2\|\gamma\|_{L^\infty(\Omega)}|\sigma(\xi)| \geq 0$$

for all $(x,\xi,\eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ with $|\xi| > \delta$. So, we have

$$\frac{c}{2}|\eta|^p - d - \|\gamma\|_{L^\infty(\Omega)} \sup_{|\xi|\leq\delta} |\sigma(\xi)| \leq h(x,\xi,\eta) + \gamma(x)\sigma(\xi)$$

for all $(x,\xi,\eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ and, of course,

$$\lim_{|\xi|\to+\infty} \inf_{(x,\eta)\in\Omega\times\mathbf{R}^n} (h(x,\xi,\eta) + \gamma(x)\sigma(\xi)) = +\infty \ .$$

Consequently, in view of Proposition 2.9, we have, in $W^{1,p}(\Omega)$,

$$\lim_{\|u\|\to+\infty} \left(\int_\Omega h(x,u(x),\nabla u(x))dx + \int_\Omega \gamma(x)\sigma(u(x))dx\right) = +\infty \ .$$

This implies that, for each $r \in \mathbf{R}$, the set

$$\left\{u \in V : \int_\Omega h(x,u(x),\nabla u(x))dx + \int_\Omega \gamma(x)\sigma(u(x))dx \leq r\right\}$$

is weakly compact by reflexivity and by Eberlein-Smulyan's theorem. Of course, we also have

$$\left\{u \in V : \int_\Omega h(x,u(x),\nabla u(x))dx + \int_\Omega \gamma(x)\sigma(u(x))dx \leq r\right\}$$
$$= \left\{u \in X : \int_\Omega h(x,u(x),\nabla u(x))dx + \int_\Omega \gamma(x)\sigma(u(x))dx \leq r\right\} \ .$$

Now, observe that, if we put

$$A = \{\xi_1, \xi_3\}$$

and

$$B = \{\xi_2\} \ ,$$

and define $I : X \to \mathbf{R}$, $J : X \times L^\infty(\Omega) \to \mathbf{R}$ by

$$I(u) = \int_\Omega h(x, u(x), \nabla u(x))dx \ ,$$

$$J(u, \gamma) = \int_\Omega \gamma(x)(\sigma(u(x)) - \sigma(\xi_2))dx$$

for all $u \in X$, $\gamma \in L^\infty(\Omega)$, we clearly have

$$\inf_{u \in B} \sup_{\gamma \in L^\infty(\Omega)} J(u, \gamma) = 0$$

and, by (iii),

$$\sup_A I < \inf_B I \ .$$

Since σ is strictly monotone, the numbers $\sigma(\xi_1) - \sigma(\xi_2)$ and $\sigma(\xi_3) - \sigma(\xi_2)$ have opposite signs. This clearly implies that

$$\sup_{\gamma \in L^\infty(\Omega)} \inf_{u \in A} J(u, \gamma) \leq 0 \ .$$

Furthermore, if $u \in X \setminus \{\xi_2\}$, again by strict monotonicity, $\sigma \circ u \neq \sigma(\xi_2)$, and so we have

$$\sup_{\gamma \in L^\infty(\Omega)} J(u, \gamma) = +\infty \ .$$

Therefore, the sets A, B and the functions I, J satisfy the assumptions of Proposition 2.10 and hence we have

$$\begin{aligned} \sup_{L^\infty(\Omega)} \inf_X (I + J) \ &\leq \ \max\left\{\int_\Omega h(x, \xi_1, 0)dx, \int_\Omega h(x, \xi_3, 0)dx\right\} \\ &< \ \int_\Omega h(x, \xi_2, 0)dx = \inf_X \sup_{L^\infty(\Omega)} (I + J) \ . \end{aligned} \tag{2.7}$$

Now, we can apply Theorem 2.1 considering X equipped with the weak topology and taking

$$Y = L^\infty(\Omega) \ ,$$

$$f = I + J \ ,$$

$$\psi = 0 \ ,$$

$$S = \frac{1}{\lambda} T$$

and

$$\varphi(u) = \omega\left(\int_\Omega k(x, u(x), \nabla u(x))dx\right) \ .$$

Notice that, in view of (2.7), inequality (2.1) holds thanks to (2.6), and the conclusion follows. When $T = C_0^\infty(\Omega)$ the same proof as above holds in view of Proposition 2.11. □

Remark 2.13. Notice that condition (iii) holds if and only if the function $\int_\Omega h(x, \cdot, 0)$ is not quasi-convex.

Remark 2.14. For $\omega = 0$, Theorem 2.12 reduces to Theorem 1.2 of [9].

We conclude presenting an application of Theorem 2.12 to the Neumann problem for a Kirchhoff-type equation.

Given $K : [0, +\infty[\to \mathbf{R}$ and a Carathéodory function $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$, consider the following Neumann problem

$$\begin{cases} -K\left(\int_\Omega |\nabla u(x)|^p dx\right) \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \psi(x, u) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$.

Let us recall that a weak solution of this problem is any $u \in W^{1,p}(\Omega)$ such that, for every $v \in W^{1,p}(\Omega)$, one has $\psi(\cdot, u(\cdot))v(\cdot) \in L^1(\Omega)$ and

$$K\left(\int_\Omega |\nabla u(x)|^p dx\right) \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx - \int_\Omega \psi(x, u(x)) v(x) dx = 0 \ .$$

Theorem 2.15. *Let $f, g : \mathbf{R} \to \mathbf{R}$ be two C^1 functions lying in $\mathcal{E}$ and satisfying the following conditions:*
(a_1) *the function g' has a constant sign and* $\operatorname{int}((g')^{-1}(0)) = \emptyset$ *;*
(a_2) $\lim_{|\xi| \to +\infty} \frac{f(\xi)}{|g(\xi)|+1} = +\infty$ *;*
(a_3) *there are $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$, with $\xi_1 < \xi_2 < \xi_3$, such that*

$$\max\{f(\xi_1), f(\xi_3)\} < f(\xi_2) \ .$$

Then, for every $a > 0$, for every $\beta \in L^\infty(\Omega)$, with $\inf_\Omega \beta > 0$, for every convex set $T \subseteq L^\infty(\Omega)$ dense in $L^\infty(\Omega)$, for every C^1, non-decreasing, bounded function $\chi : [0, +\infty[\to \mathbf{R}$, and for every λ satisfying

$$\lambda > \frac{2 \sup_{[0,+\infty[} |\chi|}{p(f(\xi_2) - \max\{f(\xi_1), f(\xi_3)\}) \int_\Omega \beta(x) dx}$$

there exists $\gamma \in T$ such that the problem

$$(P) \begin{cases} -\left(a + \chi'\left(\int_\Omega |\nabla u(x)|^p dx\right)\right) \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \gamma(x) g'(u) - \lambda \beta(x) f'(u) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has at least two weak solutions.

Proof. Fix a, β, T, χ and λ as in the conclusion. We are going to apply Theorem 2.12, defining h, k, σ by

$$h(x, \xi, \eta) = \frac{a}{p\lambda} |\eta|^p + \beta(x) f(\xi) \ ,$$

$$k(\eta) = |\eta|^p \ ,$$

$$\sigma(\xi) = -g(\xi)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$. It is immediate to realize that, by $(a_1) - (a_3)$, the above h, k, σ satisfy the assumptions of Theorem 2.12. Then, applying Theorem

2.12 with $\omega = \frac{1}{p}\chi$, we get the existence of $\gamma \in T$ such that the functional

$$u \to \lambda\left(\frac{a}{p\lambda}\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \beta(x)f(u(x))dx\right) - \int_\Omega \gamma(x)g(u(x))dx + \frac{1}{p}\chi\left(\int_\Omega \nabla u(x)|^p dx\right)$$

has at least two global minima in $W^{1,p}(\Omega)$. But, by classical results (recall that $f, g \in \mathcal{E}$), such a functional is C^1 and its critical points (and so, in particular, its global minima) are weak solutions of problem (P). The proof is complete. □

A challenging problem is as follows:

PROBLEM 1. - Does the conclusion of Theorem 2.15 hold with *three* instead of *two*?

Acknowledgement

The author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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Manuscript received March 31 2021
revised May 10 2021

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