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# Tripled Fixed Points and Existence Study to a Tripled Impulsive Fractional Differential System via Measures of Noncompactness

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**Abstract:** In this paper, a tripled fractional differential system is introduced as three associated impulsive equations. The existence investigation of the solution is based on contraction principle and measures of noncompactness in terms of tripled fixed point and modulus of continuity. Our results are valid for both Kuratowski and Hausdorff measures of noncompactness. As an application, we apply the obtained results to a control problem.

**Keywords:** tripled system; impulsive conditions; caputo derivative; tripled fixed point; measures of noncompactness



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## 1. Introduction

Fractional differential equations are considered as the most appropriate models for many applicable phenomena (see in [1,2] and references therein). This opens the research gate to study analytic solutions and their behaviors in a theoretic sense such as existence, uniqueness, stability, controllability, etc. [3–21].

The investigations of existence problems of fractional differential equations have diverse topics ranging from the shape of initial and boundary conditions including impulsive conditions, throughout various types of the used fractional derivatives, and reaching to different forms fixed point theorems.

The fixed point theorems are essential resources for solving many existence problems of solutions of differential and integral equations. In the meantime, the standard Banach principle for contractions can be used not only for establishing the existence of a solution, but also to investigate the uniqueness of this solution. The main assumption for using Banach's fixed point theorem [22] is the contraction principle applied to the operator equation. Another famous theorem is the Schauder's fixed point theorem [23], which mainly utilizes the relative compactness of the image of the solution operator. The application of these two important theorems can be observed in different fractional modeling problems such as the investigation of existence-uniqueness criteria for the generalized Navier system [24], the nonsingular 4D-memristor-based circuit model [25], the SARS-CoV-2 virus model [26], the hearing loss model caused by mump virus [27], the Ebola model [28], the Langevin problem [29], the fractional BVP of the hexasilinane graph [30], etc.

In connection with bounded relatively compact subsets, measures of noncompactness are considered very applicable tools to investigate existence problems by imposing weaker conditions [31–33]. Therefore, there exists a correlation between the measure of noncompactness and the Schauder theorem. Darbo [34] used this idea to prove a generalization of

Schauder fixed point theorem, and then many researchers presented extended results of Darbo’s fixed point criterion [35].

Coupled systems are introduced as two associated differential equations that may be solved simultaneously [36–38]. The main notion of the existence of coupled fixed points to be a solution of coupled systems are considered recently by many researchers [39–43]. In [31], the authors considered coupled fractional systems using the idea of the measure of compactness.

Recently, many researchers introduced a tripled system and a tripled fixed point [44–47]. In [45], the authors used tripled fixed points and the measure of noncompactness to investigate the existence of a solution in relation to a functional tripled system via fractional operators.

We design and discuss in this article a tripled impulsive nonlinear system formulated as

$$\begin{cases} {}^c D_a^{\kappa_m} x_m(t) = f_m(t, x(t)), t \in J', \\ x_m(a) = \Phi_m x, x'_m(a) = \Theta_m x, \\ \Delta x_m|_{t=t_k} = I_{m,k}(x(t_k)), \Delta x'_m|_{t=t_k} = \bar{I}_{m,k}(x(t_k)), \end{cases} \tag{1}$$

where  $J = [a, b]$ ,  $J' = J - \{t_1, t_2, \dots, t_p\}$ ,  $a = t_0 < t_1 < \dots < t_p < t_{p+1} = b$ ,  ${}^c D_a^{\kappa_m}$ ,  $m = 1, 2, 3$ , are the Caputo fractional derivatives such that  $\kappa_m \in (1, 2]$ ,  $f_m : J \times \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $x(t) = (x_1(t), x_2(t), x_3(t))$ ,  $I_{m,k}, \bar{I}_{m,k} : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $k = 1, 2, \dots, p$ , are given functions,  $\Phi_m, \Theta_m$  are given operators,  $\Delta x_m|_{t=t_k} = x_m(t_k^+) - x_m(t_k^-)$ ,  $\Delta x'_m|_{t=t_k} = x'_m(t_k^+) - x'_m(t_k^-)$ , and

$$x_m(t_k^+) = \lim_{\hbar \rightarrow 0^+} x_m(t_k + \hbar), \quad x_m(t_k^-) = \lim_{\hbar \rightarrow 0^-} x_m(t_k + \hbar),$$

represent the right and left limits of the function  $x_m(t)$  at the given points  $t = t_k$ , respectively. The main contribution and novelty of the present manuscript are that we here discuss the existence notion with the help of a combination of extended fixed point theorems for tripled fixed points in relation to a tripled impulsive system for the first time. Darbo’s criterion and measure of noncompactness are used for tripled fixed points. Note that our findings will be valid for both Kuratowski and Hausdorff measures [33–35].

The structure of the present research is organized as follows. In Section 2, some useful preliminaries and lemmas are recalled to facilitate the proof of main theorems. In Section 3, we prove and verify the existence results via different fixed point theorems. To conclude, we introduce an example to examine the results.

### 2. Basic Notions

For convenience, we present firstly some preliminaries concerning with fractional calculus. For more details on this topic, see the monograph [48].

**Definition 1** ([48]). *A real-valued function  $f$  is said to be  $\kappa$ th-fractional integrable at  $t$  in the sense of Riemann–Liouville (RL) if the integral*

$$\int_a^t \frac{(t-s)^{\kappa-1} f(s)}{\Gamma(\kappa)} ds,$$

*exists for  $\kappa > 0$ . If  $f$  is  $\kappa$ th-fractional RL-integrable for every  $t \in J$ , then it is said to be  $\kappa$ th-fractional RL-integrable. The  $\kappa$ th-fractional RL-integral of  $f$  is denoted by  $I_a^\kappa f(t)$ , and conventionally, we let  $I_a^0 f = f$ , and  $I_a^\kappa f(a) = 0$ .*

It is obvious that if  $f$  is  $\kappa$ th-fractional integrable, then it is  $\beta$ th-fractional integrable for any  $0 < \beta \leq \kappa$ . We also notice that if  $f$  is  $\kappa$ th-fractional integrable on  $[a, b]$ , then it satisfies that  $I_0^\kappa f(t_k^+) = I_0^\kappa f(t_k^-)$ ,  $k = 1, 2, \dots, p$ . However, if  $f$  is continuous, then it is  $\kappa$ th-fractional integrable.

**Definition 2** ([48]). A real-valued function  $f$  is said to have a Caputo derivative on  $J$  of order  $\kappa \in (n - 1, n)$ ,  $n \in \mathbb{N}$ , if the  $n$ th derivative  $f^{(n)}$  of  $f$  is  $(n - \kappa)$ th-fractional integrable on  $J$ . It is denoted by  ${}^c D_a^\kappa f$ , and then we write  ${}^c D_a^\kappa f = I_a^{n-\kappa} f^{(n)}$ .

The continuity of the  $n$ th derivative  $f^{(n)}$  ensures the continuity of  ${}^c D_a^\kappa f$ . We notice that the  $n$ th derivative of any function of the form of  $c_i(t - a)^i$ ,  $i \in \{0, \dots, n - 1\}$  is zero, then the Caputo derivative on  $J$  of order  $\kappa \in (n - 1, n]$  for such functions is zero.

**Lemma 1** ([48]). Assume that  $f$  has a Caputo derivative on  $[a, b]$  of order  $\kappa \in (n - 1, n]$ ; then,  $D_a^\kappa I_a^\kappa f = f$  and

$$I_a^\kappa {}^c D_a^\kappa f(t) = f(t) + \sum_{i=0}^{n-1} c_i(t - a)^i, \tag{2}$$

in which  $c_i \in \mathbb{R}$  and  $t \geq a$ .

**Definition 3** ([44]). A triplet  $(\varkappa_1, \varkappa_2, \varkappa_3) \in Y$  for  $Y := X^3$  is termed as a tripled fixed point of a mapping  $F : Y \rightarrow X$  whenever  $F(\varkappa_1, \varkappa_2, \varkappa_3) = \varkappa_1$ ,  $F(\varkappa_2, \varkappa_1, \varkappa_3) = \varkappa_2$ , and  $F(\varkappa_3, \varkappa_2, \varkappa_1) = \varkappa_3$ .

Here, in our fundamental theorem,  $X$  is a Banach space. In general, Definition 3 can be applied for any space  $X$  which has some primitive algebraic structures such as partially ordered space [49].

Define  $\Psi : X^3 \rightarrow X^3$  so that

$$\Psi(\varkappa_1, \varkappa_2, \varkappa_3) = (F(\varkappa_1, \varkappa_2, \varkappa_3), F(\varkappa_2, \varkappa_1, \varkappa_3), F(\varkappa_3, \varkappa_2, \varkappa_1)).$$

Then,  $(\varkappa_1, \varkappa_2, \varkappa_3)$  is a tripled fixed point of  $F$  iff  $(\varkappa_1, \varkappa_2, \varkappa_3)$  is a fixed point of  $\Psi$ , i.e.,  $\Psi(\varkappa_1, \varkappa_2, \varkappa_3) = (\varkappa_1, \varkappa_2, \varkappa_3)$ .

Next, we recall some preliminaries about a measure of noncompactness.

**Definition 4** ([34]). Let  $X$  be a Banach space and  $\mathcal{B}_X$  the collection of bounded sets in  $X$ . A measure  $\mu : \mathcal{B}_X \rightarrow [0, \infty)$  is termed as the measure of noncompactness if for any  $V, V_1, V_2 \in \mathcal{B}_X$ , it fulfills the following:

- (M1) (Regularity)  $\ker \mu = \{V \in \mathcal{B}_X : \mu(V) = 0\}$  is nonempty subset of the category of relatively compact sets in  $\mathcal{B}_X$ ;
- (M2) (Monotonicity)  $V_1 \subset V_2$  implies  $\mu(V_1) \leq \mu(V_2)$ ;
- (M3) (Invariance)  $\mu(\text{Conv } V) = \mu(\bar{V}) = \mu(V)$ , where  $\text{Conv } V$ , and  $\bar{V}$  are, respectively, the closed convex hull and closure of  $V$ ;
- (M4) (Semi-homogeneity)  $\mu(cV) \leq |c|\mu(V)$ , for  $c \in \mathbb{R}$ ;
- (M5) (Sublinearity)  $\mu(V_1 + V_2) \leq \mu(V_1) + \mu(V_2)$ ;
- (M6)  $\mu(cV_1 + (1 - c)V_2) \leq c\mu(V_1) + (1 - c)\mu(V_2)$ ;
- (M7) If  $(V_j)$  is a decreasing sequence of subsets in  $\mathcal{B}_X$  with  $\lim_{j \rightarrow \infty} \mu(V_j) = 0$ ,  $\bigcap_{j=1}^\infty V_j$  is nonempty.

For more properties and details, the reader may refer to the works in [34,35]. Kuratowski and Hausdorff measures of noncompactness are two famous measures of this type which are defined, respectively, as

$$\alpha(V) = \inf\{\mathfrak{R} > 0 : V \text{ can be covered by a finite number of sets of diameter } \leq \mathfrak{R}\},$$

and

$$\chi(V) = \inf\{\mathfrak{R} > 0 : V \text{ can be covered by a finite number of balls of radius } \leq \mathfrak{R}\},$$

which are equivalent to regular measures, since  $\chi(V) \leq \alpha(V) \leq 2\chi(V)$ . In fact all regular measures are equivalent [34]. The diameter  $diam V = \sup\{\|a_1 - a_2\|, \text{ for all } a_1, a_2 \in V\}$  and the norm  $\|V\| = \{\|x\| : x \in V\}$  of a set  $A$  are nonregular measures with kernels of singleton sets and  $\{0\}$ , respectively.

In the space of continuous mappings given on  $J$ , the modulus of continuity of  $x \in C(J)$  is a function  $\omega(x, \cdot) : [0, \infty) \rightarrow [0, \infty)$  such that

$$\omega(x, \mathfrak{A}) = \sup\{|x(s) - x(t)| : s, t \in J, |s - t| \leq \mathfrak{A}\},$$

and the modulus of continuity of a set  $V \subseteq C(J)$  is defined as

$$\omega(V, \mathfrak{A}) = \sup\{\omega(x, \mathfrak{A}) : x \in V\}.$$

Define a measure of noncompactness  $\omega_0$  as

$$\omega_0(V) = \lim_{\mathfrak{A} \rightarrow 0} \omega(V, \mathfrak{A}),$$

where it satisfies that  $\omega_0(V) = 2\chi(V)$  [34].

Schauder fixed point theorem is one of the well-known applications on existence problems but it focuses on compact operators.

**Theorem 1 ([23]).** *Let  $\Omega \neq \emptyset$  be a convex closed set with boundedness property in a Banach space  $X$ . Then  $\exists x \in \Omega$  for every continuous compact mapping  $F : \Omega \rightarrow \Omega$  so that  $x = Fx$ .*

Therefore, if  $\Omega$  satisfies the hypotheses of Theorem 1, and  $F$  is continuous whose image embedded in  $\Omega$  and the set  $F\Omega$  is equicontinuous, then by Arzela Ascoli theorem  $F$  is compact, i.e.,  $F\Omega$  is relatively compact. This means that  $F\Omega \in \ker \mu$ , or  $\mu(F\Omega) = 0$ , where  $\mu$  is an arbitrary regular measure.

A useful extension of Darbo’s fixed point criterion is given in the next step.

**Theorem 2 ([35]).** *Let  $\Omega \neq \emptyset$  be a convex closed set with boundedness property in a Banach space  $X$  and  $F : \Omega \rightarrow \Omega$  be continuous which satisfies*

$$\mu(F(V)) \leq \sigma(\mu(V)), \tag{3}$$

$\forall V \subseteq \Omega$ , in which  $\sigma : [0, \infty) \rightarrow [0, \infty)$  is increasing with  $\lim_{j \rightarrow \infty} \sigma^j(t) = 0, \forall t \in [0, \infty)$ , and  $\mu$  is an arbitrary measure of noncompactness. Then,  $\exists x \in \Omega$  so that  $x = Fx$ .

Assuming  $\sigma(t) = kt$  for  $t \geq 0$ , and  $k < 1$ ,

$$\lim_{j \rightarrow \infty} \sigma^j(t) = \lim_{j \rightarrow \infty} k^j t = 0.$$

Moreover, the condition (3) becomes  $\mu(F(V)) \leq k\mu(V)$  which is called Darbo’s condition or  $\mu$ -contraction and the theorem will be the same original Darbo’s fixed point result [34].

The next result concerns with an integral solution of the corresponding linear system of (1).

**Lemma 2.** Let  $f_m$  be  $\kappa_m$ th-fractional integrable with  $I_0^\kappa f_m(t_k^+) = I_0^\kappa f_m(t_k^-)$ ,  $0 < \kappa \leq \kappa_m$ ,  $k = 1, 2, \dots, p$ , and also  $x_m$  be differentiable of the Caputo type on  $J$  of order  $\kappa_m$  where  $\kappa_m \in (1, 2]$ ,  $m = 1, 2, 3$ . In this case, the solution of the impulsive fractional differential system

$$\begin{cases} {}^c D_0^{\kappa_m} x_m(t) = f_m(t), t \in J', \\ x_m(a) = a_m, x'_m(a) = b_m, \\ \Delta x_m|_{t=t_k} = I_{m,k}(x(t_k)), \Delta x'_m|_{t=t_k} = \bar{I}_{m,k}(x(t_k)), \end{cases} \tag{4}$$

is equivalent to

$$x_m(t) = \begin{cases} a_m + b_m t + \int_0^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s) ds, t \in J_0, \\ a_m + b_m t + \int_0^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s) ds \\ + \sum_{i=1}^k I_{m,i}(x(t_i)) + \sum_{i=1}^k (t-t_i) \bar{I}_{m,i}(x(t_i)), t \in J_k. \end{cases} \tag{5}$$

**Proof.** The given conditions imply that  $I_a^{\kappa_m} D_a^{\kappa_m} x_m$  exists and satisfies the identity (2). Taking the fractional integral  $I_a^{\kappa_m}$  to both sides of the differential Equation (4), and using Lemma 1, we obtain

$$x_m(t) = a_{k,m} + b_{k,m} t + I_a^{\kappa_m} f_m(t), t \in J_k, \tag{6}$$

and

$$x'_m(t) = b_{k,m} + I_a^{\kappa_m-1} f_m(t), t \in J_k,$$

where  $I_a^{\kappa_m} f_m(t) = \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s) ds + \int_{t_k}^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s) ds$ , and  $a_{k,m}, b_{k,m}, k = 1, 2, \dots, p, m = 1, 2, 3$ , are constants to be determined. Applying the boundary conditions in (4), if  $t \in J_0$ , we have  $a_{0,m} = a_m$ , and  $b_{0,m} = b_m$ . Then, the solution (6) becomes

$$x_m(t) = a_m + b_m t + I_a^{\kappa_m} f_m(t).$$

Next, if  $t \in J_k, k = 1, 2, \dots, p$ , then

$$\begin{aligned} a_{k,m} &= a_{k-1,m} - (b_{k,m} - b_{k-1,m})t_k + I_{m,k}(x(t_k)), \\ b_{k,m} &= b_{k-1,m} + \bar{I}_{m,k}(x(t_k)). \end{aligned}$$

Solving these recursions leads to

$$b_{k,m} = b_m + \sum_{i=1}^k \bar{I}_{m,i}(x(t_i)),$$

and

$$a_{k,m} = a_m - \sum_{i=1}^k t_i \bar{I}_{m,i}(x(t_i)) + \sum_{i=1}^k I_{m,i}(x(t_i)).$$

Substituting these constants in (6), we obtain

$$\begin{aligned} x_m(t) &= a_m + b_m t + I_a^{\kappa_m} f_m(t) \\ &+ \sum_{i=1}^k (t-t_i) \bar{I}_{m,i}(x(t_i)) + \sum_{i=1}^k I_{m,i}(x(t_i)). \end{aligned}$$

This is equivalent to the solution (5). On another side, as  $x_m$  has  $\kappa_m$ th-Caputo derivative, using Lemma 1, it is easy to deduce (4). This finishes the proof.  $\square$

If  $x$  has a continuous second derivative and  $f$  is continuous on  $J$ , then the result of Lemma 2 are valid, as  $I_a^{\kappa_m} f_m$  and  $D_a^{\kappa_m} x_m$  are continuous.

### 3. Results on the Existence Criterion

In this place, we discuss the existence and uniqueness problems for the impulsive tripled system (1).

A Banach space  $C(J)$  of all real-valued continuous mappings is endowed with the supremum norm. Consider the space  $PC(J)$  defined by

$$PC(J) = \left\{ \zeta : J \rightarrow \mathbb{R} \mid \zeta \in C(J'), \zeta(t_k^+) \text{ and } \zeta(t_k^-) \text{ exist with } \zeta(t_k^-) = \zeta(t_k) \right\},$$

endowed with the norm  $\|\zeta\| = \sup\{|\zeta(t)|, t \in J\}$ , and  $k = 1, 2, \dots, p$ . Let  $X = PC(J)$ , and  $Y := X^3$  be the usual tripled product which becomes a Banach space with  $\|x\| = \max\{\|x_1\|, \|x_2\|, \|x_3\|\}$  for any  $x = (x_1, x_2, x_3) \in Y$ . Define the operators  $\Psi : Y \rightarrow Y$ , and  $\Psi_m : Y \rightarrow X, m = 1, 2, 3$  such that

$$\Psi(x_1, x_2, x_3) = (\Psi_1(x_1, x_2, x_3), \Psi_2(x_2, x_1, x_3), \Psi_3(x_3, x_2, x_1))$$

and  $\Psi_m$  satisfying  $\Psi_1(x_1, x_2, x_3) = x_1, \Psi_2(x_2, x_1, x_3) = x_2$ , and  $\Psi_3(x_3, x_2, x_1) = x_3$ . Without loss of generality we use a common notation  $x = (x_1, x_2, x_3)$  for three cases of domain  $\Psi_m$ . Using Lemma 2,  $\Psi_m$  satisfies the corresponding integral solution (5) of the system (1) given by

$$\Psi_m x(t) = \begin{cases} \Phi_m x + t\Theta_m x + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s, x(s)) ds, t \in J_0, \\ \Phi_m x + t\Theta_m x + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s, x(s)) ds \\ + \sum_{i=1}^k I_{m,i}(x(t_i)) + \sum_{i=1}^k (t-t_i)\bar{I}_{m,i}(x(t_i)), t \in J_k, k = 1, 2, \dots, p. \end{cases} \tag{7}$$

We need the following assumptions:

**Hypothesis 1 (H1).**  $f_m$  is a Carathéodory function, that is  $(s, x) \mapsto f_m(s, x)$  is continuous in  $x$  and strongly measurable in  $s$ . There is a nondecreasing  $\kappa_m$ th-fractional integrable function  $\psi_{f_m} : J \rightarrow [0, \infty)$ , and

$$|f_m(t, x) - f_m(t, y)| \leq \psi_{f_m}(t) \|x - y\|,$$

$\forall t \in J$  and  $x, y \in \mathbb{R}^3, m = 1, 2, 3$ . Moreover, let  $\sup_{t \in J} I_a^{\kappa_m} \psi_{f_m}(t) \leq L_{\psi_m}$ , for  $m = 1, 2, 3$ , and  $L_{\psi} = \max\{L_{\psi_1}, L_{\psi_2}, L_{\psi_3}\}$ .

**Hypothesis 2 (H2).**  $I_{m,i}$  and  $\bar{I}_{m,i}$  are continuous functions that maps zero vector into zero value, and there exist constants  $L_{I_{m,i}}, L_{\bar{I}_{m,i}} > 0$ , provided that  $\forall x, y \in \mathbb{R}^3, i = 1, 2, \dots, p$ ,

$$|I_{m,i}(x) - I_{m,i}(y)| \leq L_{I_{m,i}} \|x - y\|, |\bar{I}_{m,i}(x) - \bar{I}_{m,i}(y)| \leq L_{\bar{I}_{m,i}} \|x - y\|.$$

Moreover, let  $L_{I_i} = \max\{L_{I_{1,i}}, L_{I_{2,i}}, L_{I_{3,i}}\}$ , and  $L_{\bar{I}_i} = \max\{L_{\bar{I}_{1,i}}, L_{\bar{I}_{2,i}}, L_{\bar{I}_{3,i}}\}$ .

**Hypothesis 3 (H3).** For continuous maps  $\Phi_m, \Theta_m : \mathbb{R}^3 \rightarrow \mathbb{R}, \exists L_{\Phi_m}, L_{\Theta_m} > 0$  such that for all  $x, y \in \mathbb{R}^3, m = 1, 2, 3,$

$$\begin{cases} |\Phi_m(x) - \Phi_m(y)| \leq L_{\Phi_m} \|x - y\|, \\ |\Theta_m(x) - \Theta_m(y)| \leq L_{\Theta_m} \|x - y\|. \end{cases}$$

Moreover, let  $L_{\Phi} = \max\{L_{\Phi_1}, L_{\Phi_2}, L_{\Phi_3}\},$  and  $L_{\Theta} = \max\{L_{\Theta_1}, L_{\Theta_2}, L_{\Theta_3}\}.$

**Theorem 3.** Let (H1)–(H3) be held. Then, there is a unique solution for the impulsive tripled system (1) whenever

$$L_{\Phi} + L_{\Theta}b + L_{\psi} + \sum_{i=1}^p L_{I_i} + \sum_{i=1}^p (t_{i+1} - t_i)L_{\bar{I}_i} < 1. \tag{8}$$

**Proof.** Take  $X^3 =: Y.$  Let  $\|\Phi_m 0\| = N_{\Phi_m} \geq 0, \|\Theta_m 0\| = N_{\Theta_m} \geq 0,$  and  $\max_{t \in J} \|f_m(t, 0)\| = N_{f_m} \geq 0.$  Define a subset  $\mathfrak{B}_r = \{x \in Y : \|x\| \leq r\}$  of  $Y$  via

$$r \geq \frac{N_{\Phi} + N_{\Theta}b + N_f N_0}{1 - L_{\Phi} + L_{\Theta}b + L_{\psi} + \sum_{i=1}^p L_{I_i} + \sum_{i=1}^p (t_{i+1} - t_i)L_{\bar{I}_i}}, \tag{9}$$

where  $N_{\Phi} = \max\{N_{\Phi_1}, N_{\Phi_2}, N_{\Phi_3}\}, N_{\Theta} = \max\{N_{\Theta_1}, N_{\Theta_2}, N_{\Theta_3}\},$  and

$$N_0 = \max\left\{ \frac{(b - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)}, m = 1, 2, 3 \right\}.$$

In this case, for  $t \in J_0,$

$$\begin{aligned} |\Psi_m x(t)| &\leq \|\Phi_m x\| + t\|\Theta_m x\| + \int_a^t \frac{(t - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \|f_m(s, x(s))\| ds \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m} \psi_{f_m}(t) \right) \|x\| \\ &\quad + \left( N_{\Phi_m} + N_{\Theta_m} t + \frac{N_{f_m}(t - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t_{k+1} + L_{\psi_m} \right) \|x\| \\ &\quad + \left( N_{\Phi_m} + N_{\Theta_m} t_{k+1} + \frac{N_{f_m}(t_{k+1} - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right). \end{aligned}$$

Similarly, for  $t \in J_k$  we write ( $k = 1, 2, \dots, p$ ),

$$\begin{aligned} |\Psi_m x(t)| &\leq \|\Phi_m x\| + t\|\Theta_m x\| + \int_a^t \frac{(t - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \|f_m(s, x(s))\| ds \\ &\quad + \sum_{i=1}^k \|I_{m,i}(x(t_i))\| + \sum_{i=1}^k (t - t_i) \|\bar{I}_{m,i}(x(t_i))\| \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m} \psi_{f_m}(t) + \sum_{i=1}^k L_{I_{m,i}} + \sum_{i=1}^k (t - t_i)L_{\bar{I}_{m,i}} \right) \|x\| \\ &\quad + \left( N_{\Phi_m} + N_{\Theta_m} t + \frac{N_{f_m}(t - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \end{aligned}$$

$$\leq \left( L_{\Phi_m} + L_{\Theta_m} t_{k+1} + L_{\psi_m} + \sum_{i=1}^k L_{I_{m,i}} + \sum_{i=1}^k (t_{i+1} - t_i) L_{\bar{I}_{m,i}} \right) \|x\| + \left( N_{\Phi_m} + N_{\Theta_m} t_{k+1} + \frac{N_{f_m} (t_{k+1} - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right).$$

As  $\sum_{i=1}^k L_{I_{m,i}} + \sum_{i=1}^k (t - t_i) L_{\bar{I}_{m,i}} \geq 0$ , then for  $x \in \mathfrak{B}_r$ , we obtain

$$\|\Psi x\| \leq \left( L_{\Phi} + L_{\Theta} b + L_{\psi} + \sum_{i=1}^p L_{I_i} + \sum_{i=1}^p (t_{i+1} - t_i) L_{\bar{I}_i} \right) r + N_{\Phi} + N_{\Theta} b + N_f N_0.$$

In virtue of (9), we deduce that  $\Psi x \in \mathfrak{B}_r$ . Next, the contractivity of  $\Psi$  is checked. Let  $x, y \in Y$ , we have

$$\begin{aligned} |\Psi_m x(t) - \Psi_m y(t)| &\leq \|\Phi_m x - \Phi_m y\| + t \|\Theta_m x - \Theta_m y\| \\ &\quad + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \|f_m(s, x(s)) - f_m(s, y(s))\| ds \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m} \psi_{f_m}(t) \right) \|x - y\| \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t_1 + L_{\psi_m} \right) \|x - y\|, t \in J_0, \end{aligned}$$

and for  $t \in J_k$ , we get

$$\begin{aligned} |\Psi_m x(t) - \Psi_m y(t)| &\leq \|\Phi_m x - \Phi_m y\| + t \|\Theta_m x - \Theta_m y\| \\ &\quad + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \|f_m(s, x(s)) - f_m(s, y(s))\| ds \\ &\quad + \sum_{i=1}^k \|I_{m,i}(x(t_i)) - I_{m,i}(y(t_i))\| \\ &\quad + \sum_{i=1}^k (t - t_i) \|\bar{I}_{m,i}(x(t_i)) - \bar{I}_{m,i}(y(t_i))\| \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t + I_a^{\kappa_m} \psi_{f_m}(t) + \sum_{i=1}^k L_{I_{m,i}} + \sum_{i=1}^k (t - t_i) L_{\bar{I}_{m,i}} \right) \|x - y\| \\ &\leq \left( L_{\Phi_m} + L_{\Theta_m} t_{k+1} + L_{\psi_m} + \sum_{i=1}^k L_{I_{m,i}} + \sum_{i=1}^k (t_{i+1} - t_i) L_{\bar{I}_{m,i}} \right) \|x - y\|. \end{aligned}$$

In virtue of condition (8), and the estimate

$$\|\Psi x - \Psi y\| \leq \left( L_{\Phi} + L_{\Theta} b + L_{\psi} + \sum_{i=1}^p L_{I_i} + \sum_{i=1}^p (t_{i+1} - t_i) L_{\bar{I}_i} \right) \|x - y\|,$$

it is figured out that  $\Psi$  is a contraction. Therefore, by the Banach contraction criterion, a unique fixed point and so a unique solution is found for  $\Psi$ , and the impulsive tripled system (1), respectively. The uniqueness proof is ended here.  $\square$



The next result is an investigation of the existence of a tripled fixed point to the operator  $\Psi : X^3 \rightarrow X^3$  that leads to a solution for the mentioned impulsive system (1) using the Schauder Theorem 1. These hypotheses are needed to establish the result and are given as follows:

- (A1)  $f_m : J \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is Carathéodory, and there exist nondecreasing  $\kappa$ th-fractional integrable function  $v : J \rightarrow [0, \infty)$  and nondecreasing continuous  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying  $\varphi(s) < s$  for all  $s > 0$  equipped with

$$|f_m(t, x) - f_m(t, y)| \leq v_f(t)\varphi(\|x - y\|), t \in J,$$

$\forall x, y \in \mathbb{R}^3, \kappa$  is chosen such that  $\sup_{t \in J} I_a^\kappa v_f(t) \leq C_v$ . Moreover, let  $\sup_{t \in J} |f_m(t, 0)| \leq C_f$  for any  $m = 1, 2, 3$ .

- (A2)  $I_{m,i}$  and  $\bar{I}_{m,i}$  are continuous functions, and  $\exists C_I, C_{\bar{I}} > 0$ , such that for all  $x, y \in \mathbb{R}^3, i = 1, 2, \dots, p$ , we have

$$|I_{m,i}(x) - I_{m,i}(y)| \leq C_I \|x - y\|, |\bar{I}_{m,i}(x) - \bar{I}_{m,i}(y)| \leq C_{\bar{I}} \|x - y\|.$$

Moreover, let  $I_{m,i}(0) = \bar{I}_{m,i}(0) = 0$ .

- (A3)  $\Phi_m$  and  $\Theta_m$  are continuous operators for  $m = 1, 2, 3$ , and there exist constants  $C_\Phi, C_\Theta > 0$  s.t.  $\forall x, y \in X^3$ ,

$$\begin{cases} \|\Phi_m(x) - \Phi_m(y)\| \leq C_\Phi \|x - y\|, \\ \|\Theta_m(x) - \Theta_m(y)\| \leq C_\Theta \|x - y\|. \end{cases}$$

Moreover, let  $\|\Phi_m 0\| \leq C_{\Phi_0}$ , and  $\|\Theta_m 0\| \leq C_{\Theta_0}$ .

- (A4) The inequality

$$C_{\Phi_0} + bC_{\Theta_0} + \frac{C_v(b-a)^\kappa}{\Gamma(\kappa+1)} + (C_\Phi + C_\Theta b + pC_I + C_{\bar{I}}(b-a))r + C_v\varphi(r) \leq r,$$

has a solution  $r_0 > 0$ .

The function  $\varphi$  that is satisfying the condition in (A1) is equivalent to the corresponding condition in Theorem 2 [35]. It is obvious by definition of  $\Psi = (\Psi_1, \Psi_2, \Psi_3)$  that, the value of its norm on  $J_0$  is less than or equal to the corresponding norm value on  $J_k$ . Therefore, without loss of generality, we apply the norms on  $J_k$ , in the next result ( $k = 1, 2, \dots, p$ ).

**Theorem 4.** Let (A1)–(A4) be held. Then, the impulsive tripled system (1) involves at least one solution.

**Proof.** Let  $Y = X^3$ , and  $(x_n)$  be a sequence in  $Y$  such that  $x_n \rightarrow x$  in  $Y$ . Let  $\mathfrak{R} > 0$ , and choose  $\delta = \mathfrak{R} (C_\Phi + C_\Theta b + C_f + pC_I + C_{\bar{I}}(b-a))^{-1}$  such that  $\|x_n - x\| < \delta$ . Using assumptions (A1)–(A3), we deuce that

$$\begin{aligned} |\Psi_m x_n(t) - \Psi_m x(t)| &\leq \|\Phi_m x_n - \Phi_m x\| + t\|\Theta_m x_n - \Theta_m x\| \\ &+ \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \|f_m(s, x_n(s)) - f_m(s, x(s))\| ds \\ &+ \sum_{i=1}^k \|I_{m,i}(x_n(t_i)) - I_{m,i}(x(t_i))\| \\ &+ \sum_{i=1}^k (t-t_i) \|\bar{I}_{m,i}(x_n(t_i)) - \bar{I}_{m,i}(x(t_i))\| \end{aligned}$$

$$\leq \left( C_\Phi + C_\Theta t + I_a^\kappa v_f(t) + kC_I + C_I \sum_{i=1}^k (t - t_i) \right) \|x_n - x\|.$$

Hence

$$\|\Psi x_n - \Psi x\| \leq (C_\Phi + C_\Theta b + C_v + pC_I + C_I(b - a)) \|x_n - x\| < \mathfrak{R}.$$

This shows that  $\Psi$  is a uniformly continuous operator on  $Y$ .

Let  $B_{r_0} = \{\zeta \in X : \|\zeta\| \leq r_0\} \neq \emptyset$ , then  $B_{r_0}$  is a closed, and convex set in  $X$  with boundedness property. Further, the subset  $B_{r_0}^3 = B_{r_0} \times B_{r_0} \times B_{r_0}$  inherits the properties of  $B_{r_0}$  but in  $Y$ . We estimate

$$\begin{aligned} |\Psi_m x(t)| &\leq \|\Phi_m x\| + t\|\Theta_m x\| + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \|f_m(s, x(s))\| ds \\ &\quad + \sum_{i=1}^k \|I_{m,i}(x(t_i))\| + \sum_{i=1}^k (t-t_i) \|\bar{I}_{m,i}(x(t_i))\| \\ &\leq \left( C_\Phi + C_\Theta t + kC_I + C_I \sum_{i=1}^k (t - t_i) \right) \|x\| \\ &\quad + I_a^\kappa v_f(t) \varphi(\|x\|) + \|\Phi_m 0\| + t\|\Theta_m 0\| + \frac{C_v(t-a)^\kappa}{\Gamma(\kappa+1)}. \end{aligned}$$

Therefore, if  $x_m \in B_{r_0}$ ,  $m = 1, 2, 3$ , we deduce by (A4) that

$$\begin{aligned} \|\Psi x\| &\leq (C_\Phi + C_\Theta b + pC_I + C_I(b - a))r_0 + C_v \varphi(r_0) \\ &\quad + \frac{C_v(b-a)^\kappa}{\Gamma(\kappa+1)} + C_{\Phi_0} + bC_{\Theta_0} \\ &\leq r_0. \end{aligned}$$

This shows that  $\Psi$  maps  $B_{r_0}^3$  into  $B_{r_0}^3$ .

Let  $A_m$ ,  $m = 1, 2, 3$  be a nonempty subset in  $B_{r_0}$ ; then  $A_m$  inherits the boundedness and convexity properties from  $B_{r_0}$ . Let  $\zeta \in \Psi_m(A_1 \times A_2 \times A_3)$ , then there exists  $y = (x_1, x_2, x_3) \in A_1 \times A_2 \times A_3$  such that  $\Psi_m y = \zeta$ , and  $x_m \in A_m$ ,  $m = 1, 2, 3$ . The definition of  $\Psi_m$  implies that

$$\zeta(t) = \Psi_m y(t) = \begin{cases} \Phi_m y + t\Theta_m y + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s, y(s)) ds, t \in J_0, \\ \Phi_m y + t\Theta_m y + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s, y(s)) ds \\ \quad + \sum_{i=1}^k I_{m,i}(y(t_i)) + \sum_{i=1}^k (t-t_i) \bar{I}_{m,i}(y(t_i)), t \in J_k, k = 1, 2, \dots, p. \end{cases}$$

For any  $\mathfrak{R} > 0$ , let  $\tau_1, \tau_2 \in J$  be such that  $|\tau_2 - \tau_1| \leq \mathfrak{R}$ , then, we have

$$\begin{aligned} &|\zeta(\tau_2) - \zeta(\tau_1)| \\ &\leq |\tau_2 - \tau_1| \|\Theta_m y\| + \varphi(\|y\|) \int_a^{\tau_1} \left| \frac{(\tau_2 - s)^{\kappa_m-1}}{\Gamma(\kappa_m)} - \frac{(\tau_1 - s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \right| v_f(s) ds \\ &\quad + \int_a^{\tau_1} \left| \frac{(\tau_2 - s)^{\kappa_m-1}}{\Gamma(\kappa_m)} - \frac{(\tau_1 - s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \right| |f(s, 0)| ds \end{aligned}$$

$$\begin{aligned}
 & +\varphi(\|y\|) \int_{\tau_1}^{\tau_2} \left| \frac{(\tau_2 - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \right| v_f(s) ds + \int_{\tau_1}^{\tau_2} \left| \frac{(\tau_2 - s)^{\kappa_m - 1}}{\Gamma(\kappa_m)} \right| |f(s, 0)| ds \\
 & + |\tau_2 - \tau_1| \sum_{i=1}^k \|\bar{I}_{m,i}(x(t_i))\|, a.e.on J, \\
 \leq & |\tau_2 - \tau_1| \|\Theta_m y\| \\
 & + \varphi(\|y\|) v_f(\tau_1) \left( \left| \frac{(\tau_2 - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} - \frac{(\tau_1 - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right| + \frac{|\tau_2 - \tau_1|^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\
 & + C_f \left( \left| \frac{(\tau_2 - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} - \frac{(\tau_1 - a)^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right| + \frac{|\tau_2 - \tau_1|^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\
 & + \varphi(\|y\|) v_f(\tau_2) \frac{|\tau_2 - \tau_1|^{\kappa_m}}{\Gamma(\kappa_m + 1)} + C_f \frac{|\tau_2 - \tau_1|^{\kappa_m}}{\Gamma(\kappa_m + 1)} + p(\tau_2 - \tau_1) C_{\bar{I}}.
 \end{aligned}$$

Without loss of generality, we take  $\tau_1 > a$ , and  $\mathfrak{R} < \tau_1 - a$ . Then,

$$\begin{aligned}
 \omega(\zeta, \mathfrak{R}) & = \sup\{|\zeta(\tau_2) - \zeta(\tau_1)| : \tau_1, \tau_2 \in J, |\tau_2 - \tau_1| \leq \mathfrak{R}\} \\
 & \leq \mathfrak{R}(C_{\Theta}\|y\| + C_{\Theta_0}) + \varphi(\|y\|) v_f(\tau_1) \\
 & \quad \times \left( \frac{|\tau_1 - a|^{\kappa_m} \left| O\left(\frac{\mathfrak{R}}{\tau_1 - a}\right) \right|}{\Gamma(\kappa_m + 1)} + \frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\
 & + C_f \left( \frac{|\tau_1 - a|^{\kappa_m} \left| O\left(\frac{\mathfrak{R}}{\tau_1 - a}\right) \right|}{\Gamma(\kappa_m + 1)} + \frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\
 & + \varphi(\|y\|) v_f(\tau_2) \frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} + \frac{C_f \mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} + \mathfrak{R} p C_{\bar{I}},
 \end{aligned}$$

where  $O\left(\frac{\mathfrak{R}}{\tau_1 - a}\right)$  is the big O function [50] that converges to 0 as  $\mathfrak{R}$  converges to zero. It follows that

$$\begin{aligned}
 \omega(\Psi_m(A_1 \times A_2 \times A_3), \mathfrak{R}) & = \sup\{\omega(x, \zeta) : \zeta \in \Psi_m(A_1 \times A_2 \times A_3)\} \\
 & \leq \mathfrak{R}(C_{\Theta_0} + C_{\Theta} r_0) + \varphi(r_0) v_f(b) \\
 & \quad \times \left( \frac{|\tau_1 - a|^{\kappa_m} \left| O\left(\frac{\mathfrak{R}}{\tau_1 - a}\right) \right|}{\Gamma(\kappa_m + 1)} + \frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\
 & + C_f \left( \frac{|\tau_1 - a|^{\kappa_m} \left| O\left(\frac{\mathfrak{R}}{\tau_1 - a}\right) \right|}{\Gamma(\kappa_m + 1)} + \frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} \right) \\
 & + \varphi(r_0) v_f(b) \frac{\mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} + \frac{C_f \mathfrak{R}^{\kappa_m}}{\Gamma(\kappa_m + 1)} + \mathfrak{R} p C_{\bar{I}}.
 \end{aligned}$$

Therefore, passing the limit as  $\mathfrak{R}$  approaches to zero, we deduce then  $\omega_0(\Psi(A_1 \times A_2 \times A_3))$  approaches to zero for any  $A_1 \times A_2 \times A_3 \subseteq B_{r_0}^3$ . Applying Schauder fixed point Theorem 1, the impulsive tripled system (1) has a solution in  $B_{r_0}^3 \in \ker \omega_0$ .  $\square$

The equivalent relations between the regular measures of noncompactness permit to use Kuratowski  $\alpha$  and Hausdorff  $\chi$  measures of noncompactness in the previous theorem.

It is still possible to use a general measure  $\mu$  together with general conditions to obtain the existence result.

(A5) Assume that

$$\mu(f_m(t, A)) \leq \rho_m(t)\psi(\mu(A)),$$

for any bounded subset  $A \subseteq X$ , where  $\rho_m : J \rightarrow \mathbb{R}_+$  be so that

$$\max_{t \in J} \{I_a^k \rho_1(t), I_a^k \rho_2(t), I_a^k \rho_3(t)\} \leq L_\rho,$$

and  $\psi : [0, \infty) \rightarrow [0, \infty)$  is increasing with  $\lim_{n \rightarrow \infty} \psi^n(t) = 0, \forall t \in [0, \infty)$  and  $\psi(ks) = k\psi(s), k \geq 0$ .

(A6) There exist constants  $L_{\Phi_m}, L_{\Theta_m}$  such that

$$\begin{aligned} \mu(\Phi_m A) &\leq L_{\Phi_m} \psi(\mu(A)), \\ \mu(\Theta_m A) &\leq L_{\Theta_m} \psi(\mu(A)). \end{aligned}$$

Moreover, let  $L_\Phi = \max\{L_{\Phi_1}, L_{\Phi_2}, L_{\Phi_3}\}$ , and  $L_\Theta = \max\{L_{\Theta_1}, L_{\Theta_2}, L_{\Theta_3}\}$ .

(A7) There exist constants  $L_{I_{m,i}}, L_{\bar{I}_{m,i}}$  such that

$$\mu\{I_{m,i}(x(t_i)), x \in A\} \leq L_{I_{m,i}} \psi(\mu(A)), \quad \mu\{\bar{I}_{m,i}(x(t_i)), x \in A\} \leq L_{\bar{I}_{m,i}} \psi(\mu(A)).$$

Furthermore, let  $L_{I_i} = \max\{L_{I_{1,i}}, L_{I_{2,i}}, L_{I_{3,i}}\}$ , and  $L_{\bar{I}_i} = \max\{L_{\bar{I}_{1,i}}, L_{\bar{I}_{2,i}}, L_{\bar{I}_{3,i}}\}$ .

(A8) Assume that

$$L_\Phi + bL_\Theta + \sum_{i=1}^p L_{I_i} + (b - a) \sum_{i=1}^p L_{\bar{I}_i} + L_\rho < 1.$$

The further lemma is needed for our goal.

**Lemma 3** ([33]). *Regard  $f : J \times X \rightarrow \mathbb{R}$  satisfying (A5), and let  $K : J \times J \rightarrow \mathbb{R}$  be a bounded continuous mapping. If  $A$  is an equicontinuous set of functions, then*

$$\mu\left(\left\{\int_I K(t, s)f(s, x(s))ds : x \in A\right\}\right) \leq \int_I |K(t, s)|\rho(s)\psi(\mu(\{x(s) : s \in A\}))ds,$$

for any subset  $I$  of  $J$  and any  $t \in I$ .

**Theorem 5.** *Let  $\mu$  be any measure of noncompactness. If (A1)–(A8) are fulfilled, then the impulsive tripled system (1) involves a solution.*

**Proof.** Let  $Y = X^3$ . The conditions (A1)–(A4) imply that the operator  $\Psi : B_{r_0}^3 \rightarrow B_{r_0}^3$  is continuous operator on the closed convex bounded subset  $B_{r_0}^3$  of  $Y$ . Moreover,  $\Psi B_{r_0}^3$  is compact subset in  $B_{r_0}^3$ . If  $\mu(B_{r_0}^3) = 0$ , then by (M2), we have  $\mu(A) = 0$  for any  $A \subseteq B_{r_0}^3$ , hence we have the result as in Theorem 4. Otherwise, let  $\mu(A) > 0$ , then, by (M4) and (M5), we have

$$\begin{aligned} \mu((\Psi_m A)(t)) &\leq \mu\left(\left\{\Phi_m x + t\Theta_m x + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s, x(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^k I_{m,i}(x(t_i)) + \sum_{i=1}^k (t-t_i)\bar{I}_{m,i}(x(t_i)) : \right. \right. \\ &\quad \left. \left. x \in A, t \in J_k, k = 0, 1, 2, \dots, p\right\}\right) \end{aligned}$$

$$\begin{aligned} &\leq \mu(\{\Phi_m x : x \in A\}) + t\mu(\{\Theta_m x : x \in A\}) + \\ &\quad + \sum_{i=1}^k \mu(\{I_{m,i}(x(t_i)) : x \in A\}) \\ &\quad + \sum_{i=1}^k (t - t_i)\mu(\{\bar{I}_{m,i}(x(t_i)) : x \in A\}) \\ &\quad + \mu\left(\left\{\int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} f_m(s, x(s)) ds : x \in A\right\}\right). \end{aligned}$$

Using Lemma 3,

$$\begin{aligned} \mu((\Psi_m A)(t)) &\leq \mu(\{\Phi_m x : x \in A\}) + t\mu(\{\Theta_m x : x \in A\}) + \\ &\quad + \sum_{i=1}^k \mu(\{I_{m,i} A : x \in A\}) \\ &\quad + \sum_{i=1}^k (t - t_i)\mu(\{\bar{I}_{m,i}(x(t_i)) : x \in A\}) \\ &\quad + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \mu(\{f_m(s, x(s)) : x \in A\}) ds. \end{aligned}$$

In view of assumptions (A5)–(A7), we deduce

$$\begin{aligned} \mu((\Psi_m A)(t)) &\leq L_{\Phi_m} \psi(\mu(A)) + tL_{\Theta_m} \psi(\mu(A)) + \\ &\quad + \psi(\mu(A)) \sum_{i=1}^k L_{I_{m,i}} + \psi(\mu(A)) \sum_{i=1}^k L_{\bar{I}_{m,i}} (t - t_i) \\ &\quad + \int_a^t \frac{(t-s)^{\kappa_m-1}}{\Gamma(\kappa_m)} \mu(\{f_m(s, x(s)) : x \in A\}) ds \\ &\leq \left( L_{\Phi_m} + tL_{\Theta_m} + \sum_{i=1}^k L_{I_{m,i}} + \sum_{i=1}^k L_{\bar{I}_{m,i}} (t - t_i) + I_a^\kappa \rho_m(t) \right) \psi(\mu(A)). \end{aligned}$$

Taking the maximum over  $m = 1, 2, 3$ , we have

$$\begin{aligned} \mu((\Psi A)(t)) &= \sup\{\mu((\Psi_1 A)(t)), \mu((\Psi_2 A)(t)), \mu((\Psi_3 A)(t))\} \\ &\leq \left( L_{\Phi} + tL_{\Theta} + \sum_{i=1}^k L_{I_i} + \sum_{i=1}^k L_{\bar{I}_i} (t - t_i) + L_{\rho} \right) \psi(\mu(A)). \end{aligned}$$

Taking the supremum over  $J$ , we have

$$\begin{aligned} \mu((\Psi A)) &\leq \left( L_{\Phi} + bL_{\Theta} + \sum_{i=1}^p L_{I_i} + (b - a) \sum_{i=1}^p L_{\bar{I}_i} + L_{\rho} \right) \psi(\mu(A)) \\ &= \phi(\gamma(A)), \end{aligned} \tag{10}$$

where  $\phi(s) = \left( L_{\Phi} + bL_{\Theta} + \sum_{i=1}^p L_{I_i} + (b - a) \sum_{i=1}^p L_{\bar{I}_i} + I_a^{\kappa} \rho(b) \right) \psi(s)$  which satisfies  $\phi^n(s) = \left( L_{\Phi} + bL_{\Theta} + \sum_{i=1}^p L_{I_i} + (b - a) \sum_{i=1}^p L_{\bar{I}_i} + I_a^{\kappa} \rho(b) \right)^n \psi^n(s)$  by using (A5) and (A8). In accordance with (A8), we get the required results by applying Theorem 2. This finishes the proof.  $\square$

#### 4. Application

We give a general example to examine the obtained results.

**Example 1.** Consider the following impulsive tripled system:

$$\begin{cases} {}^c D_0^{\frac{6}{5}} x_1(t) = \frac{1}{\sqrt{100+t^2}} \left( \frac{|x_1(t)|}{1+|x_1(t)|} + \frac{|x_2(t)|}{1+|x_2(t)|} + \frac{|x_3(t)|}{1+|x_3(t)|} \right), \\ {}^c D_0^{\frac{3}{2}} x_2(t) = \frac{t|x_1(t)|}{10} + \frac{t|x_2(t)|}{10(t^3+1)} + \frac{t|x_3(t)|}{10(1+t)}, \\ {}^c D_0^{\frac{9}{8}} x_3(t) = \frac{|x_1(t)|}{\sqrt{100+t^2}} + \frac{|x_2(t)|}{10+t} + \frac{|x_3(t)|}{10\sqrt{1+t^2}}, \end{cases} \tag{11}$$

where  $t \in [0, 1]$ , subject to boundary conditions

$$\begin{cases} 10m x_m(0) = \int_0^1 |\sin(x_m(s))| ds, 10m x'_m(0) = \frac{|x_m(1)|}{1+(x'_0(0)+x'_1(0)+x'_3(0))^2}, \\ m\Delta x_m|_{t=0.5} = \frac{0.1|x_m(0.5)|}{1+|x_m(0.5)|}, m\Delta x'_m|_{t=0.5} = \frac{0.1|x_m(0.5)|}{1+2|x'_m(0.5)|}. \end{cases}$$

In view of the above data, we have ( $m = 1, 2, 3$ )

$$\Phi_m(x) = \frac{1}{10m} \int_0^1 |\sin(x_m(s))| ds,$$

$$\Theta_m(x) = \frac{1}{10m} \frac{|x_m(1)|}{1+(x'_0(0)+x'_1(0)+x'_3(0))^2},$$

$$I_{m,1}(x) = \frac{1}{m} \frac{0.1|x_m(0.5)|}{1+|x_m(0.5)|},$$

and

$$\bar{I}_{m,1}(x) = \frac{1}{m} \frac{0.1|x_m(0.5)|}{1+2|x'_m(0.5)|},$$

where  $\kappa_1 = \frac{6}{5}, \kappa_2 = \frac{3}{2}, \kappa_3 = \frac{9}{8}, t_0 = 0, t_1 = 0.5$ , and  $t_2 = 1$ . Furthermore, we notice that  $\psi_{f_1}(t) = \frac{1}{\sqrt{100+t^2}}, \psi_{f_2}(t) = \frac{t}{10}$ , and  $\psi_{f_3}(t) = \frac{1}{10}$ . Therefore, by the assumptions (H1)–(H3), we can find that  $L_{\Phi} = L_{\Theta} = L_{\psi} = L_{I_i} = L_{\bar{I}_i} \leq \frac{1}{10}$ . Hence

$$L_{\Phi} + L_{\Theta} b + L_{\psi} + \sum_{i=1}^p L_{I_i} + \sum_{i=1}^p (t_{i+1} - t_i) L_{\bar{I}_i} = 0.45 < 1.$$

Thus, all hypotheses of Theorem 3 are satisfied, then there exists a unique solution to the impulsive tripled system (11).

On other hand, by referring to hypotheses (A1)–(A4), similar calculations can be performed and so we find  $C_{\Phi_0} = C_{\Theta_0} = 0, a = 0, p = 1 = b, \kappa = 1.5$  and  $C_{\Phi} = C_{\Theta} = C_I = C_{\bar{I}} = 0.1$ ,

$C_v \leq 0.1$ . Therefore, we solve the inequality given in (A4) and we find that  $r_0 \approx 0.23$  satisfies this inequality. Therefore using Theorem 4, there exists a solution to the supposed impulsive tripled system (11).

## 5. Conclusions

In this paper, we designed a tripled system consisting of impulsive fractional equations involving the generalized boundary conditions with some given operators. By introducing two types of measure of noncompactness (Kuratowski and Hausdorff), we investigated necessary hypotheses and conditions implying the existence of solutions with the help of the tripled fixed point and modulus of continuity. Furthermore, the Banach principle was applied to confirm the uniqueness property. In this research, we showed that our results are valid for both Kuratowski and Hausdorff measures of noncompactness. To confirm this correctness, we designed an example of the control problem. For future works, we can generalize our results to such an impulsive problem with nonsingular operators.

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