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Intrinsic square functions and commutators on Morrey-Herz spaces with variable exponents

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In this article, we will study the boundedness of intrinsic square functions on the Morrey-Herz spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$. The boundedness of commutators generated by *BMO* functions and intrinsic square functions is also discussed on the aforementioned Morrey-Herz spaces.

KEYWORDS

BMO space, commutators, intrinsic square functions, Morrey-Herz spaces, variable exponent

MSC CLASSIFICATION

42B20; 42B25

1 | INTRODUCTION

For $0 < \gamma \leq 1$, let C_γ be the family of all functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that φ has support contained in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \varphi(x)dx = 0$, and such that for any $x_1, x_2 \in \mathbb{R}$ the following inequality holds:

$$|\varphi(x_1) - \varphi(x_2)| \leq |x_1 - x_2|^\gamma. \tag{1}$$

For $(y,t) \in \mathbb{R}^{n+1}_+$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let us set

$$A_\gamma(f)(y,t) = \sup_{\varphi \in C_\gamma} |f * \varphi_t(y)| = \sup_{\varphi \in C_\gamma} \left| \int_{\mathbb{R}^n} \varphi_t(y-z) f(z) \mathrm{d}z \right|. \tag{2}$$

The intrinsic square function of f of order γ is defined by

$$S_\gamma(f)(x) = \left(\iint_{\Gamma(x)} \left(A_\gamma(f)(y,t)\right)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}, \tag{3}$$

where $\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)$ and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

Andrea Scapellato dedicates this paper to the memory of his beloved aunt Anna Porto.

The definition of intrinsic square function S_γ was first introduced by Wilson.[1,2] Wilson[2] proved the weighed L^p boundedness of intrinsic square functions. Lerner[3] proved sharp $L^p(w)$ norm inequalities for the intrinsic square function in terms of the A_p characteristic of w for all $1<p<\infty$. The boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz Morrey and Campanato spaces was considered in Liang *et al.*[4]

Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $b \in BMO(\mathbb{R}^n)$. The commutator generated by b and the intrinsic square function $S_\gamma(f)(x)$ is defined by

$$[b,S_\gamma](f)(x)=\left(\int\int_{\Gamma(x)}\sup_{\varphi\in C_\gamma}\left|\int_{\mathbb{R}^n}(b(x)-b(z))\varphi_t(y-z)f(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{4}$$

Wang[5] established the commutators of intrinsic square functions $[b,S_\gamma]$ on weighted L^p space. Guliyev *et al.*[6] proved the boundedness of intrinsic square function and their commutators on weighted Orlicz-Morrey space.

Moreover, Izuki[7] defined the Herz-Morrey spaces with one variable exponent $p(\cdot)$ and investigated the boundedness of fractional integrals on those space. Lu and Zhu[8] considered the Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with two variable exponent $\alpha(\cdot)$ and $p(\cdot)$ and obtained some boundedness results for certain sublinear operators and their commutators in these spaces. Wang[9] proved the boundedness of the commutator of the intrinsic square function in variable exponent spaces.

We also mention that Deringoz *et al.*[10] studied the boundedness of intrinsic square functions and their commutators on vanishing generalized Orlicz-Morrey spaces. Deringoz *et al.*[10] obtain some conditions for the boundedness are given in terms of Zygmund-type integral inequalities without assuming any monotonicity property.

Finally, it is interesting to point out that the boundedness of several singular integral operator on Herz-type spaces was used in the study of the regularity properties of solutions of second-order elliptic equations with discontinuous coefficients. We mention the work of Ragusa[11] in the context of homogeneous Herz spaces and the works of Scapellato[12,13] in which the authors extended the results contained in Ragusa[11] to Herz spaces with variable exponents. Furthermore, we refer to Ragusa,[14] who studied Herz spaces endowed with a parabolic metric and proved regularity results for weak solutions to divergence form parabolic equations with discontinuous coefficients, using some boundedness results for integral operators and commutators.

The aim of this paper is to discuss boundedness properties of intrinsic square functions and their commutators on the non-homogeneous Morrey-Herz spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ with three variable exponents.

2 | MATHEMATICAL BACKGROUND

Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E|>0$. Let us denote by χ_E the characteristic function of E. We mention that, throughout the paper, C denotes a positive constant, not necessarily the same at each occurrence.

We recall some definitions.

Definition 2.1 (Cruz-Uribe & Fiorenza,[15] Chapter 2, p. 18). Let $p(\cdot):E\to[1,\infty)$ be a measurable function. The *variable exponent Lebesgue space* is defined by

$$L^{p(\cdot)}(E)=\left\{f\ \text{ is measurable }:\ \int_E\left(\frac{|f(x)|}{\eta}\right)^{p(x)}\mathrm{d}x<\infty\ \text{ for some constant }\ \eta>0\right\}.$$

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E)=\{f\ \text{ is measurable }:\ f\in L^{p(\cdot)}(K)\ \text{ for any compact set }\ K\subset E\}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)}=\inf\left\{\eta>0:\ \int_E\left(\frac{|f(x)|}{\eta}\right)^{p(x)}\mathrm{d}x\le 1\right\}.$$

We set $p_-=\text{ess inf}\{p(x):x\in E\}$, $p_+=\text{ess sup}\{p(x):x\in E\}$. $\mathcal{P}(E)$ is the set of all measurable functions $p(\cdot)$ satisfying $p_->1$ and $p_+<+\infty$ and $\mathcal{P}^0(E)$ is the set of all measurable functions $p(\cdot)$ satisfying $p_->0$ and $p_+<+\infty$. For any $f\in$

$L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B\subseteq\mathbb{R}^n} \frac{1}{|B|}\int_B |f(y)|dy,$$

being B a sphere in $\mathbb{R}^n$. The set $\mathcal{B}(\mathbb{R}^n)$ consists of all $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfies the following inequalities,

$$|p(x)-p(y)| \le \frac{-C}{\log(|x-y|)}, \text{ if } |x-y|\le 1/2,$$
$$|p(x)-p(y)| \le \frac{C}{\log(e+|x|)}, \text{ if } |y|\ge|x|,$$

then we have $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.

Let us now recall the definition of space $BMO(\mathbb{R}^n)$. This space consists of all locally integrable functions f such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \|f\|_* = \sup_Q |Q|^{-1}\int_Q |f(x)-f_Q|dx < \infty,$$

where $f_Q = |Q|^{-1}\int_Q f(y)dy$, the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of Q.

Now, we give the definition of Morrey-Herz space with variable exponents $q(\cdot), p(\cdot), \alpha(\cdot)$.

Let $B_k = \{x\in\mathbb{R}^n : |x|\le 2^k\}$, $C_k = B_k\setminus B_{k-1}$, $\chi_k = \chi_{C_k}$, $k\in\mathbb{Z}$.

Definition 2.2 (Wang & Tao[16]). Let $q(\cdot), p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $0\le\lambda<\infty$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha\in L^\infty(\mathbb{R}^n)$. The nonhomogeneous Morrey-Herz space with variable exponents $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ is defined by

$$MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \{f\in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\beta>0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\beta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}\le 1\right\}.$$

The homogeneous Morrey-Herz space with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \{f\in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\beta>0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\sum_{k=-\infty}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\beta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}\le 1\right\}.$$

Remark 1. If $q(\cdot)$ is a constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. If both $\alpha(\cdot)$, $q(\cdot)$ are constants, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. If the variable exponents $\alpha(\cdot)$, $p(\cdot)$, and $q(\cdot)$ are constants, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p}(\mathbb{R}^n)$. Moreover, if $\lambda=\alpha(\cdot)\equiv 0$ and $p(\cdot)=q(\cdot)$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$.

Next, we need some lemmas that will be used in the proofs of our main results.

Lemma 2.3 (Cruz-Uribe & Fiorenza[15]). *(Generalized Hölder's inequality) If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, then there exists a constant C such that, for all $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, the following inequality holds:*

$$\int_{\mathbb{R}^n}|f(x)g(x)|\mathrm{d}x \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

Lemma 2.4 (Izuki[17]). *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then, there exists a constant* $C > 0$ *such that for any ball* $B \subset \mathbb{R}^n$, *the following inequality holds:*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.5 (Izuki[17]). *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *For* $h = 1, 2$, *there exist constants* $\delta_{h1}, \delta_{h2}, C > 0$ *such that for all balls* $B \subset \mathbb{R}^n$ *and all measurable* $S \subset B$ *the following inequalities hold:*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{h1}}, \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{h2}}.$$

Lemma 2.6 (Wang & Tao[18]). *Let* $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ *and* $f \in L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)$. *Then,*

$$\min(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}) \leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}} \leq \max(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}).$$

Lemma 2.7 (Izuki[17]). *Let us assume that* $b \in BMO(\mathbb{R}^n)$ *and that* n *is a positive integer. Then, there exists a constant* $C > 0$, *such that for any* $k, j \in \mathbb{Z}$ *with* $k > j$, *the following inequalities hold:*

(1) $C^{-1}\|b\|_* \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*$,
(2) $\|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k - j)\|b\|_*\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

3 | BOUNDEDNESS OF THE INTRINSIC SQUARE FUNCTIONS

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$ and let w be a weight (i.e., a nonnegative locally integrable function on $\mathbb{R}^n$). We say that $w \in A_p$ if there exists $C > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\mathrm{d}x\right)^{p-1} \leq C < \infty.$$

Wilson[1] proved the following weighted ($L^p - L^p$) boundedness of the intrinsic square functions.

Lemma 3.1 (Wilson[1]). *Let* $1 < p < \infty$, $0 < \gamma \leq 1$ *and* $w \in A_p$. *Then, there exists a constant* $C > 0$ *such that*

$$\|S_\gamma(f)\|_{L^p_w} \leq C\|f\|_{L^p_w}.$$

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Lemma 3.2. (Cruz-Uribe *et al.*[19]). *Given a family of functions* $\mathcal{F}$, *assume that for* p_0, $1 < p_0 < \infty$, $p_0 \leq p_-$, *and* $\left(\frac{p(\cdot)}{p_0}\right)' \in \mathcal{B}(E)$ *and every* $w_0 \in A_{p_0}$,

$$\int_{\mathbb{R}^n} f(x)^{p_0}w_0(x)\mathrm{d}x \leq \int_{\mathbb{R}^n} g(x)^{p_0}w_0(x)\mathrm{d}x, \qquad (f, g) \in \mathcal{F}.$$

If $p(\cdot) \in \mathcal{P}(E)$, *then for all* $(f, g) \in \mathcal{F}$ *and* $f \in L^{p(\cdot)}(E)$, *we have*

$$\|f\|_{L^{p(\cdot)}(E)} \leq C\|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p/s'} \subset A_\infty$, by applying Lemmas 3.1 and 3.2, it is easy to get the boundedness of the intrinsic square functions S_γ on $L^{p(\cdot)}$.

Theorem 3.3. *Let us assume that* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *with* $(q_2)_- \geq (q_1)_+$ *and* $0 < \gamma \leq 1$. *If* $(\lambda_1)(q_2)_+ = (\lambda_2)(q_1)_-$ *and* $-n\delta_{12} < \alpha_+ < n\delta_{11} + (\lambda_1)/(q_1)_-$, *where* δ_{11} *and* δ_{12} *are the constants in Lemma 2.5; then, the operator* S_γ *is bounded from* $MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$ *to* $MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)$.

Before starting the proof of Theorem 3.3, we state a simple inequality that will be used in the proof.

Remark. Let $h \in \mathbb{N}$, $a_h \geq 0$, $1 \leq p_h < \infty$. We have

$$\sum_{h=0}^{\infty} a_h^{p_h} \leq \left(\sum_{h=0}^{\infty} a_h \right)^{p_*},$$

here

$$p_* = \begin{cases} \min\limits_{h\in\mathbb{N}} p_h & \text{if } \sum\limits_{h=0}^{\infty} a_h \leq 1, \\ \max\limits_{h\in\mathbb{N}} p_h & \text{if } \sum\limits_{h=0}^{\infty} a_h > 1. \end{cases}$$

Proof. Let $f \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$. We decompose f as follows:

$$f(x) = \sum_{j=0}^{\infty} f(x)\chi_j(x) = \sum_{j=0}^{\infty} f_j(x).$$

By the definition of the norm in $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$, we have

$$\|S_\gamma(f)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{ \beta > 0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)}|S_\gamma(f)\chi_k|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1 \right\}.$$

For any $k_0 \in \mathbb{Z}$, we get

$$\begin{aligned} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)}|S_\gamma(f)\chi_k|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=0}^{\infty} S_\gamma(f_j)\chi_k \right|}{\beta_{11}+\beta_{12}+\beta_{13}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ &\leq C2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=0}^{k-2} S_\gamma(f_j)\chi_k \right|}{\beta_{11}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} + C2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right|}{\beta_{12}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ &\quad + C2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=k+2}^{\infty} S_\gamma(f_j)\chi_k \right|}{\beta_{13}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}, \end{aligned}$$

where

$$\beta_{11} = \left\| \sum_{j=0}^{k-2} S_\gamma(f_j)\chi_k \right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}$$

$$= \inf\left\{ \beta > 0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=0}^{k-2} S_\gamma(f_j)\chi_k \right|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1 \right\},$$

$$\beta_{12} = \left\| \sum_{k-1}^{k+1} S_\gamma(f_j)\chi_k \right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}$$

$$= \inf\left\{ \beta > 0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1 \right\},$$

$$\beta_{13} = \left\| \sum_{j=k+2}^{\infty} S_\gamma(f_j)\chi_k \right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}$$

$$= \inf\left\{ \beta > 0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=k+2}^{\infty} S_\gamma(f_j)\chi_k \right|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1 \right\}.$$

If $\beta = \beta_{11} + \beta_{12} + \beta_{13}$, thus,

$$2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |S_\gamma(f_j)\chi_k|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C.$$

Then,

$$\|S_\gamma(f)\chi_k\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} \leq C\beta \leq C[\beta_{11} + \beta_{12} + \beta_{13}].$$

Hence, if we prove that

$$\beta_{11} \leq C\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}, \qquad \beta_{12} \leq C\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}, \qquad \beta_{13} \leq C\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)},$$

we are done. Let us set $\beta_1 = \|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}$.

We consider β_{12} first. From Lemma 2.6 and the boundedness of S_γ on $L^{p(\cdot)}(\mathbb{R}^n)$, it follows that

$$\begin{aligned}
2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right\|_{L^{p(\cdot)}}^{(q_2^1)_k} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{(k-j)\alpha_+}2^{j\alpha_+}|S_\gamma(f_j)\chi_k|}{\beta_1}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|f_j|}{\beta_1}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k},
\end{aligned}$$

where

$$(q_2^1)_k = \begin{cases} (q_2)_- & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}} \leq 1, \\ (q_2)_+ & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

Since $f \in MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$, then we have

$$2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \leq 1.$$

From this, and again applying Lemma 2.6, if $(q_1)_+ \leq (q_2)_-$ and $\lambda_1(q_2)_+ = \lambda_2(q_1)_-$, we obtain that

$$\begin{aligned}
2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|f_j|}{\beta_1}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\left\|\frac{2^{k\alpha_+}|f\chi_k|}{\beta_1}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1^1)_k}}\\
&\leq C\sum_{k=0}^{k_0}\left\{2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{\frac{(q_2^1)_k}{(q_1^1)_k}}\\
&\leq C,
\end{aligned}$$

where

$$(q_1^1)_k = \begin{cases}(q_1)_+ & \text{if } \left\|\frac{2^{k\alpha_+}|f_{\chi_k}|}{\beta_1}\right\|_{L^{p(\cdot)}} \leq 1,\\ (q_1)_- & \text{if } \left\|\frac{2^{k\alpha_+}|f_{\chi_k}|}{\beta_1}\right\|_{L^{p(\cdot)}} > 1.\end{cases}$$

The previous calculations imply that

$$\beta_{12} \leq C\beta_1 \leq C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Next, we estimate β_{11}. If $x \in C_k$, $(y,t) \in \Gamma(x)$, $z \in C_j \cap \{z: |y-z| \leq t\}$, $j \leq k-2$, then

$$t \geq \frac{1}{2}(|x-y| + |y-z|) \geq \frac{1}{2}|x-z| \geq \frac{1}{4}|x|.$$

Thus, we have

$$\begin{aligned}
|A_\gamma(f_j)(x)| &= \left(\int\int_{\Gamma(x)}\sup_{\varphi\in C_\gamma}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f_j(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\leq C\left(\int_{\frac{|x|}{4}}^\infty\int_{|x-y|<t}\left|t^{-n}\int_{C_j\bigcap\{z:|y-z|\leq t\}}\varphi_t(y-z)f_j(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\leq C\left(\int_{C_j}|f_j(z)|dz\right)\left(\int_{\frac{|x|}{4}}^\infty\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}\\
&\leq C2^{-kn}\int_{C_j}|f_j(z)|dz\\
&= C2^{-kn}\|f_j\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Then, by using Lemma 2.6, we deduce that

$$
\begin{aligned}
2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}2^{-kn}\|f_j\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\beta_1}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k(\alpha_+-n)}\sum_{j=0}^{k-2}\left\|\frac{f_j}{\beta_1}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}}\right)^{(q_2^2)_k},
\end{aligned}
$$

where

$$
(q_2^2)_k=\begin{cases}(q_2)_-, & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}}\le 1,\\[2ex] (q_2)_+, & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}}> 1.\end{cases}
$$

From Hölder's inequality and applying Lemmas 2.4-2.6, it follows that

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[\sum_{j=0}^{k-2}2^{k(\alpha_+-n)}\left\|\frac{f_j}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\right]^{(q_2^2)_k}\\
&\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[\sum_{j=0}^{k-2}2^{k(\alpha_+-n)}\left\|\frac{f_j}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p'(\cdot)}}^{-1}|B_k|\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=-\infty}^{\infty}\left[\sum_{j=0}^{k-2}2^{k\alpha_+}\left\|\frac{f_j}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}}\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\frac{2^{j\alpha_+}(f_j)\chi_k}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)_k}.
\end{aligned}
$$

Notice that $f \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+ = (\lambda_1)/(q_1)_-$, and $\alpha_+ < n\delta_{11} + (\lambda_1)/(q_1)_-$. Then, we have

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\beta_1}\right)^{q_1(\cdot)}\right\|^{\frac{(q_2^1)_k}{(q_1^1)_k}}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}_n)}\right]^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^2)j}}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\\
&\quad\times\left[\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n}\right)^{\frac{1}{(q_1^2)j}}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left[\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\right]^{(q_2^2)_k}\\
&\leq C,
\end{aligned}
$$

where

$$
(q_1^2)_j = \begin{cases}(q_1)_- & \text{if } \left\|\frac{2^{j\alpha_+}|f_{\chi_j}|}{\beta_1}\right\|_{L^{p(\cdot)}} \leq 1,\\ (q_1)_+ & \text{if } \left\|\frac{2^{j\alpha_+}|f_{\chi_j}|}{\beta_1}\right\|_{L^{p(\cdot)}} > 1.\end{cases}
$$

Then, from the above calculations, it follows that

$$
\beta_{11} \leq C\beta_1 \leq C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.
$$

Finally, we estimate β_{13}. If $x \in C_k$, $(y,t) \in \Gamma(x)$, $z \in C_j \cap \{z: |y-z| \leq t\}$, $j \geq k+2$, then

$$
t \geq \frac{1}{2}(|x-y|+|y-z|) \geq \frac{1}{2}|x-z| \geq \frac{1}{2}(|z|-|x|) \geq \frac{1}{4}|z|.
$$

Thus, we have

$$
\begin{aligned}
|A_\gamma(f_j)(x)| &= \left(\int \int_{\Gamma(x)} \sup_{\varphi \in C_\gamma} \left| \int_{\mathbb{R}^n} \varphi_t(y-z) f_j(z) \mathrm{d}z \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\frac{|x|}{4}}^{\infty} \int_{|x-y|<t} \left| t^{-n} \int_{C_j \bigcap \{z: |y-z| \leq t\}} \varphi_t(y-z) f_j(z) \mathrm{d}z \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{C_j} |f_j(z)| dz \right) \left(\int_{\frac{|x|}{4}}^{\infty} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} \\
&\leq C 2^{-jn} \int_{C_j} |f_j(z)| dz \\
&= C 2^{-jn} \|f_j\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

By using Lemma 2.6 and applying Hölder's inequality, it follows that

$$
\begin{aligned}
& 2^{-k_0 \lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{k+2}^{\infty} S_\gamma(f_j) \chi_k \right|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\leq 2^{-k_0 \lambda_2} \sum_{k=0}^{k_0} \left\| \frac{2^{k\alpha(\cdot)} \left| \sum_{j=k+2}^{\infty} S_\gamma(f_j) \chi_k \right|}{\beta_1} \right\|_{L^{p(\cdot)}}^{(q_2^3)_k} \\
&\leq C 2^{-k_0 \lambda_2} \sum_{k=0}^{k_0} \left\| \frac{2^{k\alpha(\cdot)} \left| \sum_{j=k+2}^{\infty} 2^{-jn} \|f_j\|_{L^1(\mathbb{R}^n)} \chi_k \right|}{\beta_1} \right\|_{L^{p(\cdot)}}^{(q_2^3)_k} \\
&\leq C 2^{-k_0 \lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=k+2}^{\infty} 2^{-jn} \left\| \frac{f_j}{\beta_1} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right]^{(q_2^3)_k},
\end{aligned}
$$

where

$$
(q_2^3)_k = \begin{cases} (q_2)_- & \text{if } \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=k+2}^{\infty} S_\gamma(f_j) \chi_k \right|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{p(\cdot)}} \leq 1, \\ (q_2)_+ & \text{if } \left\| \left(\frac{2^{k\alpha(\cdot)} \left| \sum_{j=k+2}^{\infty} T_{\Omega,\mu}^A(f_j) \chi_k \right|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{p(\cdot)}} > 1. \end{cases}
$$

Therefore, applying Lemmas 2.4 and 2.5, we get

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{k+2}^{\infty}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{f_j}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}}|B_j|\right]^{(q_2^3)_k}$$
$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_++n\delta_{12})}\left\|\frac{2^{j\alpha_+}f\chi_j}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^3)_k}$$
$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_++n\delta_{12})}\left\|\left(\frac{2^{j\alpha_+}f\chi_j}{\beta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1^3)_j}}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right]^{(q_2^3)_k},$$

where

$$(q_1^3)_j=\begin{cases}(q_1)_- & \text{if } \left\|\frac{2^{j\alpha_+}f_{\chi_j}}{\beta_1}\right\|_{L^{p_1(\cdot)}}\leq 1,\\ (q_1)_+ & \text{if } \left\|\frac{2^{j\alpha_+}f_{\chi_j}}{\beta_1}\right\|_{L^{p_1(\cdot)}}>1.\end{cases}$$

Notice that $f\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+=(\lambda_1)/(q_1)_-$, and $\alpha_+>-n\delta_{12}+(\lambda_1)/(q_1)_-$. Then, we have

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{k+2}^{\infty}S_\gamma(f_j)\chi_k\right|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\leq C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_++n\delta_{12}-(\lambda_1)/(q_1)_-)}\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^3)_j}}\right]^{(q_2^3)_k}$$
$$\leq C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_++n\delta_{12}-(\lambda_1)/(q_1)_-)}\right]^{(q_2^3)_k}\leq C.$$

The above calculations imply that

$$\beta_{13}\leq C\beta_1=C\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.3. □

4 | BMO ESTIMATE FOR THE COMMUTATOR OF INTRINSIC SQUARE FUNCTIONS

Let $b\in BMO(\mathbb{R}^n)$. Wang[5] obtained some boundedness results for the commutator $[b,S_\gamma]$ in the framework of weighted Morrey spaces.

Lemma 4.1. *Let $1<p<\infty, 0<\beta\leq 1$, and $w\in A_p$. Suppose that $b \in BMO(\mathbb{R}^n)$, then there exists a constant $C>0$, independent of f, such that*

$$\|[b,S_\gamma](f)\|_{L^p_w} \leq C\|f\|_{L^p_w}.$$

We can apply Lemmas 4.1 and 3.2 to get the boundedness of the commutator $[b,S_\gamma]$ in $L^{p(\cdot)}$.

Theorem 4.2. *Let $b \in BMO(\mathbb{R}^n)$. Suppose that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$ and $0<\gamma\leq 1$. If $(\lambda_1)(q_2)_+ = (\lambda_2)(q_1)_-$ and $-n\delta_{12}<\alpha_+<n\delta_{11}+(\lambda_1)/(q_1)_-$, where δ_{11} and δ_{12} are the constants in Lemma 2.5, then the operator $[b,S_\gamma]$ is bounded from $MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$ to $MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $b \in BMO(\mathbb{R}^n)$, $f \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$. Let us decompose f as follows:

$$f(x) = \sum_{j=0}^{\infty} f(x)\chi_j(x) = \sum_{j=0}^{\infty} f_j(x).$$

By the definition of the norm in $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$, we have

$$\|[b,S_\gamma](f)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\beta>0 : \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|[b,S_\gamma](f)\chi_k|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1\right\}.$$

For any $k_0 \in \mathbb{Z}$, we see that

$$\begin{aligned}
2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|[b,S_\gamma](f)\chi_k|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{\infty}[b,S_\gamma](f_j)\chi_k\right|}{\beta_{21}+\beta_{22}+\beta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} + C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}[b,S_\gamma](f_j)\chi_k\right|}{\beta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k\right|}{\beta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}.
\end{aligned}$$

Let

$$\beta_{21}=\left\|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}$$
$$=\inf\left\{\beta>0:\ \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le 1\right\},$$

$$\beta_{22}=\left\|\sum_{k-1}^{k+1}[b,S_\gamma](f_j)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}$$
$$=\inf\left\{\beta>0:\ \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}[b,S_\gamma](f_j)\chi_k\right|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le 1\right\},$$

$$\beta_{23}=\left\|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}$$
$$=\inf\left\{\beta>0:\ \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,\ge S_\gamma](f_j)\chi_k\right|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le 1\right\},$$

and

$$\beta=\beta_{21}+\beta_{22}+\beta_{13}.$$

That is,

$$\|[b,S_\gamma](f)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}\le C\beta\le C[\beta_{21}+\beta_{22}+\beta_{23}].$$

Hence, once we prove that

$$\beta_{21}\le C\|b\|_*\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)},\ \beta_{22}\le C\|b\|_*\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)},\ \beta_{23}\le C\|b\|_*\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)},$$

we are done. Let us set $\beta_1=\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}$.

For β_{22}, by the boundedness of $[b,S_\gamma]$ on $L^{p(\cdot)}$, and using an argument similar to that in the estimate for β_{12}, it follows that

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le C,$$

which implies that

$$\beta_{22}\le C\beta\|b\|_*\le C\|b\|_*\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Now let us deal with the estimate for β_{21}. Let $x \in C_k, j \le k-2$. By the estimate of $S_\gamma(f_j)(x)$ in the proof of Theorem 3.3, we have

$$S_\gamma(f_j)(x) \le C2^{-kn}\|f_j\|_{L^1(\mathbb{R}^n)}.$$

From this inequality, we obtain that

$$[b, S_\gamma](f_j)(x) = |S_\gamma[(b(x)-b)f_j](x)| \le C2^{-kn}\|(b(\cdot)-b)f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, using Lemma 2.6, we have

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right\|_{L^{p(\cdot)}}^{(q_2^1)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}|x|^{-n}\|(b(\cdot)-b)f_j\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\beta_1\|b\|_*}\right\|_{L^{p(\cdot)}}^{(q_2^1)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k(\alpha_+-n)}\sum_{j=0}^{k-2}\left\|\frac{|(b-b_j)f_j|}{\beta_1\|b\|_*}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}\\
&\quad + C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k(\alpha_+-n)}\sum_{j=0}^{k-2}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^1(\mathbb{R}^n)}\frac{1}{\|b\|_*}\|(b-b_j)\chi_k\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k},
\end{aligned}$$

where

$$(q_2^1)_k = \begin{cases} (q_2)_- & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}} \le 1,\\ (q_2)_+ & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

Applying the generalized Hölder's inequality (Lemma 2.3) and Lemmas 2.4,2.5, and 2.7, we get that

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k(\alpha_+-n)}\sum_{j=0}^{k-2}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\frac{|(b-b_j)\chi_{B_j}|}{\|b\|_*}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}\\
&\quad+C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k(\alpha_+-n)}\sum_{j=0}^{k-2}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}(k-j)\|\chi_{B_k}\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k(\alpha_+-n)}\sum_{j=0}^{k-2}(k-j+1)\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}}|B_k|\right)^{(q_2^1)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\frac{2^{j\alpha_+}|f\chi_j|}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^1)_k}.
\end{aligned}
$$

Notice that $f\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+=(\lambda_1)/(q_1)_-$, and $\alpha_+<n\delta_{11}+(\lambda_1)/(q_1)_-$. Then, we have

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\left(\frac{2^{j\alpha_+}(f_j)\chi_k}{\beta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1^1)_j}}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right]^{(q_2^1)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11})}\left(2^{j\lambda}2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^1)_j}}\right]^{(q_2^1)_k}\\
&\leq \sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left[\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^1)_j}}\right]^{(q_2^1)_k}\\
&\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left[\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\right]^{(q_2^1)_k}\leq C,
\end{aligned}
$$

where

$$
(q_1^1)_j=\begin{cases}(q_1)_- & \text{if } \left\|\frac{2^{j\alpha_+}|f_{\chi_j}|}{\beta_1}\right\|_{L^{p(\cdot)}}\leq 1,\\ (q_1)_+ & \text{if } \left\|\frac{2^{j\alpha_+}|f_{\chi_j}|}{\beta_1}\right\|_{L^{p(\cdot)}}>1.\end{cases}
$$

This implies that

$$
\beta_{21}\leq C\beta_1\|b\|_*\leq C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.
$$

Finally, we estimate β_{23}. Let $x \in C_k, j \geq k+2$. By the estimation of $S_\gamma(f_j)(x)$ in the proof of Theorem 3.3, we have

$$S_\gamma(f_j)(x) \leq C2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)}.$$

From the above inequality, we obtain that

$$[b, S_\gamma](f_j)(x) = |S_\gamma[(b(x)-b)f_j](x)| \leq C2^{-jn}\|(b(\cdot)-b)f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, when $\alpha_+ > -n\delta_{12} + \lambda_1/(q_1)_-$, proceeding as in the estimate of β_{21}, we get that

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}2^{-jn}\|(b(\cdot)-b)f_j\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\beta_1\|b\|_*}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{|(b-b_k)f_j|}{\beta_1\|b\|_*}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}}\right)^{(q_2^2)_k}\\
&\quad + C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^1(\mathbb{R}^n)}\frac{1}{\|b\|_*}\|(b-b_k)\chi_k\|_{L^{p(\cdot)}}\right)^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{-jn}(j-k)|B_j|\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)(\alpha_++n\delta_{12})}\left\|\frac{2^{j\alpha_+}|f\chi_j|}{\beta_1}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k} \leq C,
\end{aligned}$$

where

$$(q_2^2)_k = \begin{cases} (q_2)_-, & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}} \leq 1,\\ (q_2)_+, & \text{if } \left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k\right|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

The above calculations imply that

$$\beta_{23} \leq C\beta_1\|b\|_* \leq C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 4.2. □

AUTHOR CONTRIBUTIONS

All authors contributed equally to this work. All authors read and approved the final manuscript.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article.

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