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ABSTRACT: We briefly describe the algorithms behind some of the functions provided by the *Macaulay*2 package *MultiprojectiveVarieties*, a package for multi-projective varieties and rational maps between them.

INTRODUCTION. This paper is a natural sequel of [Staglianò 2018], where we presented some of the algorithms implemented in the *Macaulay*2 package [Cremona], related to computations with rational and birational maps between closed subvarieties of projective spaces.

Here we describe methods for working with rational and birational maps between multiprojective varieties, that is, closed subvarieties of products of projective spaces. For instance, we explain how to compute the degrees of such maps, their graphs, and the inverses when they exist. All these methods are implemented in the Macaulay2 package *MultiprojectiveVarieties*.

From a theoretical point of view, we know that every multiprojective variety is isomorphic, via the Segre embedding, to a projective variety embedded into a single projective space. Therefore, every rational map between multiprojective varieties can be regarded as a rational map between ordinary subvarieties of projective spaces. This, however, introduces a lot of new variables, making computation more difficult.

Moreover, basic constructions on rational maps naturally lead one to consider rational maps between multiprojective varieties. For instance, the graph of a rational map is a closed subvariety of the product of the source and of the target of the map. Using the package *Cremona*, it is generally easy to verify that the first projection from the graph is birational, but to calculate, for instance, its inverse we need the tools provided by the package presented here.

In Section 1, we give a concise overview of the theory of rational maps between multiprojective varieties, emphasizing the computational aspects and making clear how they can be represented in a computer. For more details on the theory see, e.g., [Harris 1992; Hartshorne 1977]. In Section 2, with the help of an example, we show how one can work with such maps using *Macaulay*2.

1. AN OVERVIEW OF RATIONAL MAPS BETWEEN MULTIPROJECTIVE VARIETIES.

1A. *Notation and terminology.* Throughout this paper, we keep the following notation. Let K denote an arbitrary field. Consider the polynomial ring

$$R = K[x_0^{(1)}, \dots, x_{n_1}^{(1)}; \dots; x_0^{(r)}, \dots, x_{n_r}^{(r)}]$$

in r groups of variables, equipped with the $\mathbb{Z}^r$-grading, where the degree of each variable is a standard

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basis vector. More precisely, we set $\deg(x_i^{(j)}) = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^r$, where 1 occurs at position j; we call this the *standard* $\mathbb{Z}^r$*-grading* on R. The polynomial ring R is the homogeneous coordinate ring of the product of r projective spaces

$$\boldsymbol{P}^{n_1,\dots,n_r} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}.$$

The closed subsets (of the Zariski topology) of $\boldsymbol{P}^{n_1,\dots,n_r}$ are of the form

$$V(\mathfrak{a}) = \{p \in \boldsymbol{P}^{n_1,\dots,n_r} : F(p) = 0 \text{ for all homogeneous } F \in \mathfrak{a}\},$$

where $\mathfrak{a}$ is a homogeneous ideal in R. For any homogeneous ideal $\mathfrak{a} \subseteq R$, the *multisaturation* of $\mathfrak{a}$ is the homogeneous ideal

$$\operatorname{sat}(\mathfrak{a}) = \Bigl(\cdots\Bigl(\Bigl(\mathfrak{a} : \bigl(x_0^{(1)}, \dots, x_{n_1}^{(1)}\bigr)^{\infty}\Bigr) : \bigl(x_0^{(2)}, \dots, x_{n_2}^{(2)}\bigr)^{\infty}\Bigr) : \cdots\Bigr) : \bigl(x_0^{(r)}, \dots, x_{n_r}^{(r)}\bigr)^{\infty}.$$

One says that $\mathfrak{a}$ is *multisaturated* if $\mathfrak{a} = \operatorname{sat}(\mathfrak{a})$. Two homogeneous ideals $\mathfrak{a}, \mathfrak{a}' \subseteq R$ define the same subscheme of $\boldsymbol{P}^{n_1,\dots,n_r}$ if and only if $\operatorname{sat}(\mathfrak{a}) = \operatorname{sat}(\mathfrak{a}')$, and they define the same subset if and only if $\sqrt{\operatorname{sat}(\mathfrak{a})} = \sqrt{\operatorname{sat}(\mathfrak{a}')}$.

We fix a homogeneous absolutely prime ideal $I \subset R$, and we may also assume that I is multisaturated. The graded domain R/I is the homogeneous coordinate ring of an absolutely irreducible multiprojective variety

$$X = V(I) \subseteq \boldsymbol{P}^{n_1,\dots,n_r} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}.$$

There is a similar correspondence between homogeneous ideals in R/I and closed subsets of X. The two most important invariants of X are: the dimension (as a topological space), which is the (Krull) dimension of the homogeneous coordinate ring R/I minus r, and the multidegree, an integral homogeneous polynomial of degree $\operatorname{codim} X = n_1 + \cdots + n_r - \dim X$ in r variables (see [Harris 1992, Lecture 19] and [Miller and Sturmfels 2005, p. 165]).

Similarly, let us take another polynomial ring in s groups of variables,

$$S = K[y_0^{(1)}, \dots, y_{m_1}^{(1)}; \dots; y_0^{(s)}, \dots, y_{m_s}^{(s)}],$$

equipped with the standard $\mathbb{Z}^s$-grading. Let $J \subset S$ be a multisaturated homogeneous absolutely prime ideal, and let

$$Y = V(J) \subseteq \boldsymbol{P}^{m_1,\dots,m_s} = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$$

be the absolutely irreducible multiprojective variety defined by J.

1B. ***Rational maps to an embedded projective variety.*** In this subsection we consider the particular case when $s = 1$, and we set $\mathbb{P}^m = \boldsymbol{P}^{m_1,\dots,m_s}$. Then $Y \subseteq \mathbb{P}^m$ is an embedded projective variety.

Definition of rational map. We call *multiform* (or simply *form*) a homogeneous element of R/I. To a vector $\boldsymbol{F} = (F_0, \dots, F_m)$ of $m+1$ forms in R/I of the same multidegree, which are not all zero, we associate a continuous map

$$\phi_{\boldsymbol{F}} : X \setminus V(\boldsymbol{F}) \longrightarrow \mathbb{P}^m, \qquad \text{defined by } p \in X \setminus V(\boldsymbol{F}) \xmapsto{\phi_{\boldsymbol{F}}} (F_0(p), \dots, F_m(p)) \in \mathbb{P}^m.$$

If $\boldsymbol{G} = (G_0, \dots, G_m)$ is another such vector of forms in R/I of the same multidegree, then we say that $\boldsymbol{F} \sim \boldsymbol{G}$ if $\phi_{\boldsymbol{F}}(p) = \phi_{\boldsymbol{G}}(p)$ for each $p \in X \setminus (V(\boldsymbol{F}) \cup V(\boldsymbol{G}))$. We have $\boldsymbol{F} \sim \boldsymbol{G}$ if and only if $\phi_{\boldsymbol{F}} = \phi_{\boldsymbol{G}}$ on some nonempty open subset U of $X \setminus (V(\boldsymbol{F}) \cup V(\boldsymbol{G}))$; in particular $\sim$ is an equivalence relation. A *rational map* $\Phi : X \dashrightarrow Y$ is defined as an equivalence class of nonzero vectors of $m+1$ forms $\boldsymbol{F} = (F_0, \dots, F_m)$ in R/I of the same multidegree, with respect to the relation $\sim$, such that for some (and hence every) representative $\boldsymbol{F}$ we have that the image of $\phi_{\boldsymbol{F}}$ is contained in Y. If $p \in X \setminus V(\boldsymbol{F})$ for some representative $\boldsymbol{F}$, we set $\Phi(p) = \phi_{\boldsymbol{F}}(p)$ and we say that Φ is *defined* at p. The *domain* of Φ, denoted by $\mathrm{Dom}(\Phi)$, is the set of points where Φ is defined, that is, it is the largest open subset of X such that the map $\phi_{\boldsymbol{F}}$ is defined for some representative $\boldsymbol{F}$. The complementary set in X of the domain of Φ is called *base locus*. A rational map $\Phi : X \dashrightarrow Y$ is called a *morphism* if it everywhere defined, that is, if its base locus is empty.

Establishing the equality of rational maps. Notice that if a vector $\boldsymbol{F} = (F_0, \dots, F_m)$ of forms in R/I represents a rational map $\Phi : X \dashrightarrow Y$, then also the vector $H \cdot \boldsymbol{F} = (HF_0, \dots, HF_m)$ represents Φ, for each nonzero form H in R/I. More generally, two vectors $\boldsymbol{F} = (F_0, \dots, F_m)$ and $\boldsymbol{G} = (G_0, \dots, G_m)$, as the ones considered above, represent the same rational map $\Phi : X \dashrightarrow Y$ if and only if

$$\mathrm{rk} \begin{pmatrix} F_0 & \cdots & F_m \\ G_0 & \cdots & G_m \end{pmatrix} < 2,$$

that is, if and only if $F_i G_j - F_j G_i$ vanishes identically on X, for every $i, j = 0, \dots, m$.

Determining the domain of a rational map. Let $\Phi : X \dashrightarrow Y$ be a rational map and let $\boldsymbol{F} = (F_0, \dots, F_m)$ be one of its representatives. A *syzygy* of $\boldsymbol{F}$ is a vector $\boldsymbol{H} = (H_0, \dots, H_m)$ of forms in R/I such that $\sum_{i=0}^m H_i F_i = 0$. Let $M_{\boldsymbol{F}}$ be a matrix whose columns form a set of generators for the module of syzygies of $\boldsymbol{F}$. The following result is proved in [Simis 2004, Proposition 1.1], although stated there only for $r = 1$.

Proposition 1.1. The representatives of the rational map Φ correspond bijectively to the homogeneous vectors in the rank one graded (R/I)-module

$$\ker(M_{\boldsymbol{F}}^t) \subset (R/I)^{m+1}.$$

Let $\boldsymbol{F}_1, \dots, \boldsymbol{F}_p$ be a set of minimal homogeneous generators of $\ker(M_{\boldsymbol{F}}^t)$. The base locus of Φ is the closed subset of X where all the entries of $\boldsymbol{F}_i$, for $i = 1, \dots, p$, vanish. The sequence of multidegrees $(\deg \boldsymbol{F}_1, \dots, \deg \boldsymbol{F}_p)$, defined up to ordering, is called the *degree sequence* of Φ.

Example 1.2. In the case when R/I is a unique factorization domain (e.g., $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$), then a rational map $\Phi : X \dashrightarrow Y$ is uniquely represented up to proportionality, that is, the degree sequence of Φ consists of a unique element.

Direct and inverse images via rational maps. Let $\Phi : X \dashrightarrow Y$ be a rational map, and let $\mathcal{M}$ be a set of generators for the (R/I)-module of representatives of Φ. For $\boldsymbol{F} = (F_0, \ldots, F_m) \in \mathcal{M}$, we consider the graded K-algebra homomorphism $\varphi_{\boldsymbol{F}} : S/J \to R/I$ defined by $\varphi_{\boldsymbol{F}}(y_i) = F_i \in R/I$.

For each homogeneous ideal $\mathfrak{a} \subseteq R/I$ (resp. $\mathfrak{b} \subseteq S/J$), we have a closed subset $V(\mathfrak{a}) \subseteq X$ (resp. $V(\mathfrak{b}) \subseteq Y$). The *direct image* of $V(\mathfrak{a})$ via Φ, denoted by $\overline{\Phi(V(\mathfrak{a}))}$, and the *inverse image* of $V(\mathfrak{b})$ via Φ, denoted by $\overline{\Phi^{-1}(V(\mathfrak{b}))}$, as sets, are given by the closure

$$\overline{\Phi(V(\mathfrak{a}))} = \overline{\{\Phi(p) : p \in \mathrm{Dom}(\Phi) \cap V(\mathfrak{a})\}}, \quad \overline{\Phi^{-1}(V(\mathfrak{b}))} = \overline{\{p \in \mathrm{Dom}(\Phi) : \Phi(p) \in V(\mathfrak{b})\}}.$$

The following result follows from elementary commutative algebra, and it tells us how to calculate direct and inverse images.

Proposition 1.3. The following formulas hold:

$$\overline{\Phi(V(\mathfrak{a}))} = \bigcup_{\boldsymbol{F} \in \mathcal{M}} V(\varphi_{\boldsymbol{F}}^{-1}(\mathfrak{a})) = V\Bigg(\bigcap_{\boldsymbol{F} \in \mathcal{M}} \varphi_{\boldsymbol{F}}^{-1}(\mathfrak{a})\Bigg);$$

$$\overline{\Phi^{-1}(V(\mathfrak{b}))} = \bigcup_{\boldsymbol{F} \in \mathcal{M}} V(\varphi_{\boldsymbol{F}}(\mathfrak{b}) : (\boldsymbol{F})^{\infty}) = V\Bigg(\bigcap_{\boldsymbol{F} \in \mathcal{M}} \varphi_{\boldsymbol{F}}(\mathfrak{b}) : (\boldsymbol{F})^{\infty}\Bigg).$$

As a consequence, we obtain that if $\boldsymbol{F}$ is any of the representatives of Φ, then

$$\overline{\Phi(X)} = V(\ker \varphi_{\boldsymbol{F}}).$$

The direct image $\overline{\Phi(X)}$ is called the (closure of the) *image* of Φ. We say that Φ is *dominant* if $\overline{\Phi(X)} = Y$.

1C. ***Rational maps to a multiprojective variety.*** We now consider the general case when $s \geq 1$, and hence $Y \subseteq \boldsymbol{P}^{m_1,\ldots,m_s} = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$ is a multiprojective variety. Let us denote by $\pi_i : \boldsymbol{P}^{m_1,\ldots,m_s} \to \mathbb{P}^{m_i}$ the i-th projection, and let $Y_i = \pi_i(Y)$.

Definition of multirational map. We define a *multirational map* (or simply rational map)

$$\Phi : X \dashrightarrow Y$$

as an s-tuple of rational maps $\Phi_i : X \dashrightarrow \mathbb{P}^{m_i}$ such that the image of Φ_i is contained in Y_i, for $i = 1, \ldots, s$. The domain of a multirational map Φ is the intersection

$$\mathrm{Dom}(\Phi) = \bigcap_{i=1}^{s} \mathrm{Dom}(\Phi_i).$$

In other words, Φ is defined at a point $p \in X$ if and only if Φ_i is defined at p for all $i = 1, \ldots, s$, and in that case we set $\Phi(p) = (\Phi_1(p), \ldots, \Phi_s(p)) \in \boldsymbol{P}^{m_1,\ldots,m_s}$. Analogously with the case $s = 1$, we call the *base locus* the complementary set in X of the domain of Φ, and we say that Φ is a *morphism* if

$X = \mathrm{Dom}(\Phi)$. We say that Φ is *dominant* if for some (and hence every) open subset U of the domain of Φ, the set $\{\Phi(p) : p \in U\}$ is dense in Y.

Composition of multirational maps. If $\Psi = (\Psi_1, \dots, \Psi_t) : Y \dashrightarrow Z$ is another multirational map, then Φ and Ψ can be composed if $\Phi(\mathrm{Dom}(\Phi)) \cap \mathrm{Dom}(\Psi) \neq \varnothing$; in particular, this happens when either Φ is dominant or Ψ is a morphism. If $\boldsymbol{F}^{(1)}, \dots, \boldsymbol{F}^{(s)}$ are, respectively, representatives of $\Phi_1, \dots, \Phi_s$, and if $\boldsymbol{G}^{(j)}$ is a representative of Ψ_j, then the vector $\boldsymbol{G}^{(j)}(\boldsymbol{F}^{(1)}, \dots, \boldsymbol{F}^{(s)})$ is a representative of $(\Psi \circ \Phi)_j = \Psi_j \circ \Phi$.

So we can consider the category of (multi)-projective varieties and dominant (multi)-rational maps. An "*isomorphism*" in this category is called a birational map, that is, $\Phi : X \dashrightarrow Y$ is a birational map if it admits an inverse, namely a multirational map $\Phi^{-1} : Y \dashrightarrow X$ such that $\Phi^{-1} \circ \Phi = \mathrm{id}_X$ and $\Phi \circ \Phi^{-1} = \mathrm{id}_Y$ as (multi)-rational maps. A birational morphism $\Phi : X \dashrightarrow Y$ is called an *isomorphism* if Φ^{-1} is a morphism. Also (multi)-projective varieties and morphisms form a category.

Example: *the Segre embedding.* Let $N = (n_1 + 1) \cdots (n_r + 1) - 1$, and let us consider $\mathbb{P}^N$ with the homogeneous coordinate ring $K[z_{(\iota_1,\dots,\iota_r)} : \iota_j = 0, \dots, n_j,\ j = 1, \dots, r]$, where the variables are the entries of the generic r-dimensional matrix of shape $(n_1 + 1) \times \cdots \times (n_r + 1)$. The *Segre embedding* of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ into $\mathbb{P}^N$ is the rational map

$$\mathfrak{S}_{n_1,\dots,n_r} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \dashrightarrow \mathbb{P}^N,$$

represented by the ring map

$$K[z_{(\iota_1,\dots,\iota_r)} : \iota_j = 0, \dots, n_j,\ j = 1, \dots, r] \to K[x_0^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_0^{(r)}, \dots, x_{n_r}^{(r)}],$$
$$z_{(\iota_1,\dots,\iota_r)} \mapsto x_{\iota_1}^{(1)} \cdots x_{\iota_r}^{(r)}.$$

This ring map (or better the forms defining it) represents uniquely up to proportionality the rational map $\mathfrak{S}_{n_1,\dots,n_r}$, and it is also clear that it is an injective morphism. The image of $\mathfrak{S}_{n_1,\dots,n_r}$ is the projective variety of all r-dimensional matrices of rank 1. If we consider $\mathfrak{S}_{n_1,\dots,n_r}$ as a rational map onto its image, then we have that $\mathfrak{S}_{n_1,\dots,n_r}$ is an isomorphism. Indeed, for $j = 1, \dots, r$, the module of representatives of the j-th component $\mathfrak{T}_j$ of the inverse $\mathfrak{T} = \mathfrak{S}_{n_1,\dots,n_r}^{-1}$ is generated by the $(n_1+1) \cdots (n_{j-1}+1)(n_{j+1}+1) \cdots (n_r+1)$ vectors $(z_{(\iota_1,\dots,\iota_r)} : \iota_j = 0, \dots, n_j)$, as $\iota_1, \dots, \iota_{j-1}, \iota_{j+1}, \dots, \iota_r$ vary. Note, in particular, that $\mathfrak{T}_j$ is not uniquely represented up to proportionality, provided that $n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_r$ are not all zero.

Multirational maps as ordinary rational maps. Let $\Phi = (\Phi_1, \dots, \Phi_s) : X \dashrightarrow Y$ be a multirational map. Then, by composing Φ with the restriction to Y of the Segre embedding $\mathfrak{S}_{m_1,\dots,m_s} : \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s} \to \mathbb{P}^M$, where $M = (m_1 + 1) \cdots (m_s + 1) - 1$, we get an ordinary rational map $\widetilde{\Phi} : X \dashrightarrow \mathfrak{S}_{m_1,\dots,m_s}(Y) \subseteq \mathbb{P}^M$. The rational map $\widetilde{\Phi}$ is the unique rational map that makes the following diagram commutative:

$$\begin{tikzcd}
X \arrow[rrr, dashed, "\widetilde{\Phi}"] \arrow[dr, dashed, "\Phi_1"'] \arrow[drr, dotted] \arrow[drrrr, dashed, "\Phi_s"] & & & \mathfrak{S}_{m_1,\dots,m_s}(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}) \arrow[dll] \arrow[dl, dotted] \arrow[dr] & \\
& \mathbb{P}^{m_1} & \cdots & & \mathbb{P}^{m_s}
\end{tikzcd}$$

Since $\mathfrak{S}_{m_1,\dots,m_s}$ is an isomorphism onto its image, we have that Φ is a morphism (resp. birational; resp. an isomorphism) if and only if $\widetilde{\Phi}$ is a morphism (resp. birational; resp. an isomorphism). Thus, from a theoretical point of view, it would be enough to consider only "*ordinary*" rational maps. In practice, however, this complicates things considerably since the ambient space of the target of $\mathfrak{S}_{m_1,\dots,m_s}$ is much larger with respect to the source, and moreover the homogeneous coordinate ring of the image of $\mathfrak{S}_{m_1,\dots,m_s}$ is no longer a unique factorization domain (ruling out trivial cases).

Graph of a (multi)-rational map. Let $\boldsymbol{F}^{(1)}, \dots, \boldsymbol{F}^{(s)}$ be, respectively, representatives of the components $\Phi_1, \dots, \Phi_s$ of a multirational map $\Phi : X \dashrightarrow Y$. Consider the $\mathbb{Z}^r \times \mathbb{Z}^s$-graded coordinate ring of

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}, \tag{1-1}$$

given by

$$T = K[\boldsymbol{x}_1; \dots; \boldsymbol{x}_r; \boldsymbol{y}_1; \dots; \boldsymbol{y}_s],$$

where $\boldsymbol{x}_j = (x_0^{(j)}, \dots, x_{n_j}^{(j)})$ and $\boldsymbol{y}_i = (y_0^{(i)}, \dots, y_{m_i}^{(i)})$, for $j = 1, \dots, r$ and $i = 1, \dots, s$. Moreover, let $t_1, \dots, t_s$ be new variables, and consider the extended polynomial ring

$$\overline{T} = K[t_1, \dots, t_s; \boldsymbol{x}_1; \dots; \boldsymbol{x}_r; \boldsymbol{y}_1; \dots; \boldsymbol{y}_s].$$

We define an ideal in $\overline{T}$ as the following sum of ideals (by abuse of notation we also denote by $\boldsymbol{F}^{(i)}$ chosen lifts of $\boldsymbol{F}^{(i)}$ to R):

$$\mathcal{I}_{(\boldsymbol{F}^{(1)},\dots,\boldsymbol{F}^{(s)})} := I + \sum_{i=1}^{s} (\boldsymbol{y}_i - t_i\, \boldsymbol{F}^{(i)}). \tag{1-2}$$

The *graph* $\Gamma(\Phi)$ of the multirational map Φ is the subvariety of (1-1) defined by the contraction ideal

$$\mathcal{I}_{(\boldsymbol{F}^{(1)},\dots,\boldsymbol{F}^{(s)})} \cap T, \tag{1-3}$$

which no longer depends on the choice of the representatives $\boldsymbol{F}^{(i)}$. Equivalently, we can consider the homogeneous ideal in T given by

$$\mathcal{J}_{(\boldsymbol{F}^{(1)},\dots,\boldsymbol{F}^{(s)})} := I + \left(2 \times 2 \text{ minors of } \begin{pmatrix} y_0^{(i)} & \cdots & y_{m_i}^{(i)} \\ F_0^{(i)} & \cdots & F_{m_i}^{(i)} \end{pmatrix},\ i = 1, \dots, s\right), \tag{1-4}$$

and then we can calculate the ideal of $\Gamma(\Phi)$ by the saturation:

$$(\cdots(\mathcal{J}_{(\boldsymbol{F}^{(1)},\dots,\boldsymbol{F}^{(s)})} : (\boldsymbol{F}^{(1)})^{\infty}) : \cdots) : (\boldsymbol{F}^{(s)})^{\infty}. \tag{1-5}$$

We point out that the homogeneous coordinate ring of $\Gamma(\Phi)$ is also known as "*Rees algebra*"; see [Eisenbud 2018]. We have two projections (which are morphisms) that fit in a commutative diagram

$$\begin{tikzcd} & \Gamma(\Phi) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "\Phi"] & & Y \end{tikzcd}$$

The first projection $\pi_1 : \Gamma(\Phi) \to X$ is also known as the *blowing up of X along B*, where $B = X \setminus \mathrm{Dom}(\Phi)$ is the base locus of Φ. It is a birational morphism, and it is an isomorphism if and only if Φ is a morphism. See, e.g., [Hartshorne 1977, Chapter II, Section 7] for more details. The second projection $\pi_2 : \Gamma(\Phi) \to Y$ is birational if and only if Φ is birational, and in that case the graph of Φ^{-1} is the same as that of Φ, by exchanging the two projections. Moreover, π_2 and Φ have always the same image in Y; in particular, we can calculate the homogeneous ideal of the image of Φ as the contraction of the ideal of $\Gamma(\Phi)$ to $S = K[\boldsymbol{y}_1; \dots ; \boldsymbol{y}_s]$.

Computing the inverse map of a birational map. Keep the notation as above, and assume moreover that $\Phi : X \dashrightarrow Y$ is birational. We want to find the components $\Psi_j : Y \dashrightarrow \mathbb{P}^{n_j}$, for $j = 1, \dots , r$, of the inverse multirational map $\Psi : Y \dashrightarrow X$ of Φ.

Fix a minimal set of multiforms generating the homogeneous ideal of the graph $\Gamma(\Phi)$ in the $\mathbb{Z}^r \times \mathbb{Z}^s$-graded coordinate ring of (1-1). For each $j = 1, \dots , r$, we select in this set those of multidegree $(0, \dots , 0, 1, 0, \dots , 0; d_1, \dots , d_s)$, where 1 occurs at position j, and $d_1, \dots , d_s$ are not subject to conditions. Let us denote these multiforms by $H_1(\boldsymbol{x}_j, \boldsymbol{y}_1, \dots , \boldsymbol{y}_s), \dots , H_q(\boldsymbol{x}_j, \boldsymbol{y}_1, \dots , \boldsymbol{y}_s)$. Thus, for $k = 1, \dots , q$, we can write

$$H_k(\boldsymbol{x}_j, \boldsymbol{y}_1, \dots , \boldsymbol{y}_s) = x_0^{(j)} G_0^{(j,k)}(\boldsymbol{y}_1, \dots , \boldsymbol{y}_s) + \cdots + x_{n_j}^{(j)} G_{n_j}^{(j,k)}(\boldsymbol{y}_1, \dots , \boldsymbol{y}_s),$$

for suitable uniquely determined forms $G_{\iota_j}^{(j,k)} \in S = K[\boldsymbol{y}_1, \dots , \boldsymbol{y}_s]$. We regard the $q \times (n_j{+}1)$-matrix

$$\mathfrak{J}^{(j)} = \bigl(G_{\iota_j}^{(j,k)}\bigr)_{k=1,\dots,q}^{\iota_j=0,\dots,n_j}$$

as a matrix over the homogeneous coordinate ring S/J of Y.

Proposition 1.4. The (S/J)-module of representatives of Ψ_j is given by $\ker(\mathfrak{J}^{(j)})$. More explicitly we have that the rank of $\mathfrak{J}^{(j)}$ is n_j, and Ψ_j is represented by the vector of signed $n_j \times n_j$-minors of any full rank $n_j \times (n_j{+}1)$-submatrix of $\mathfrak{J}^{(j)}$.

A proof of the previous result can be found in [Simis 2004, Theorem 2.4], in the particular case when $r = s = 1$ (see also [Doria et al. 2012] and [Busé et al. 2020, Theorem 4.4] for the case when $s = 1$ and the source is a product of projective varieties). The proof in the general case is not so different; its main ingredients are: the description of the equations of the graph $\Gamma(\Phi)$ given by (1-4) and (1-5), and the fact that $\Gamma(\Phi)$ can be identified with $\Gamma(\Psi)$. We leave the details to the reader.

Direct and inverse images via multirational maps. If $Z \subseteq X$ is an irreducible subvariety such that $Z \cap \mathrm{Dom}(\Phi) \neq \varnothing$, we can consider the restriction of Φ to Z, $\Phi|_Z : Z \dashrightarrow Y$, defined as usual by the composition of the inclusion $Z \hookrightarrow X$ with Φ. Note that the graph (and hence the image) of $\Phi|_Z$, can be calculated as above, just by replacing in (1-2) the ideal I with the multisaturated homogeneous ideal of Z, and by choosing the representatives $\boldsymbol{F}^{(i)}$ such that $Z \nsubseteq V(\boldsymbol{F}^{(i)})$. This gives us a way to calculate the direct image $\overline{\Phi(Z)} = \overline{\Phi|_Z(Z)}$.

If $W \subseteq Y$ is a subvariety, using Proposition 1.3, we can calculate the inverse image $\overline{\Phi^{-1}(W)} \subseteq X$ as $\overline{\Phi^{-1}(W)} = \widetilde{\Phi}^{-1}(\mathfrak{S}_{m_1,\dots,m_s}(W))$. Alternatively (and more efficiently), let $I_W \subseteq S$ be the defining ideal of W, and let $\varphi_{(\boldsymbol{F}^{(1)},\dots,\boldsymbol{F}^{(s)})} : S \to R/I$ be the map defined by $y^{(i)}_{\iota_i} \mapsto F^{(i)}_{\iota_i} \in R/I$, for $i = 1, \dots, s$ and $\iota_i = 0, \dots, m_i$. Then the saturation of the extended ideal $(\varphi_{(\boldsymbol{F}^{(1)},\dots,\boldsymbol{F}^{(s)})}(I_W)) \subseteq R/I$ with respect to all the ideals $(\boldsymbol{F}^{(i)})$, for $i = 1, \dots, s$, gives us the ideal of the closure of $\overline{\Phi^{-1}(W)} \setminus V(\boldsymbol{F}^{(1)}, \dots, \boldsymbol{F}^{(s)})$.

Multidegree of a multirational map. Let $\Phi : X \dashrightarrow Y$ be a rational map. The *projective degrees*

$$d_0(\Phi), d_1(\Phi), \dots, d_{\dim X}(\Phi)$$

of Φ are defined as the components of the multidegree of the graph, embedded as a subvariety of

$$\mathfrak{S}_{n_1,\dots,n_r}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}) \times \mathfrak{S}_{m_1,\dots,m_s}(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}) \subset \mathbb{P}^N \times \mathbb{P}^M,$$

where $N = \Pi^r_{j=1}(n_j + 1) - 1$ and $M = \Pi^s_{i=1}(m_i + 1) - 1$. It follows that the composition $\widetilde{\Phi} : X \dashrightarrow \mathbb{P}^M$ of Φ with the restriction to Y of the Segre embedding $\mathfrak{S}_{m_1,\dots,m_s}$ has the same projective degrees as Φ. If L denotes the intersection of Y with $\dim X - i$ general hypersurfaces of multidegree $(1, \dots, 1)$, then we have

$$d_i(\Phi) = \deg(\mathfrak{S}_{n_1,\dots,n_r}(\overline{\Phi^{-1}(L)})),$$

if $\dim(\overline{\Phi^{-1}(L)}) = i$ and $d_i(\Phi) = 0$ otherwise. See also [Harris 1992, Example 19.4, p. 240]. This gives us a probabilistic algorithm to compute the projective degrees, as already remarked in [Staglianò 2018]. A nonprobabilistic algorithm can be obtained by calculating the multidegree of the graph of Φ as a subvariety of $\boldsymbol{P}^{n_1,\dots,n_r} \times \boldsymbol{P}^{m_1,\dots,m_s}$ and then applying the following remark.

Remark 1.5. Let

$$P(a_1, \dots, a_r, b_1, \dots, b_s) \in \mathbb{Z}[a_1, \dots, a_r, b_1, \dots, b_s]$$

be the multidegree of a k-dimensional subvariety of $\boldsymbol{P}^{n_1,\dots,n_r} \times \boldsymbol{P}^{m_1,\dots,m_s}$. Then the multidegree of the same variety embedded as a subvariety of $\mathfrak{S}_{n_1,\dots,n_r}(\boldsymbol{P}^{n_1,\dots,n_r}) \times \mathfrak{S}_{m_1,\dots,m_s}(\boldsymbol{P}^{m_1,\dots,m_s}) \subset \mathbb{P}^N \times \mathbb{P}^M$, is given by

$$\sum_{i=\max(0,k-M)}^{\min(k,N)} d_i a^{N-i} b^{M-k+i} \in \mathbb{Z}[a, b],$$

where d_i denotes the coefficient of the monomial $a_1^{n_1} \cdots a_r^{n_r} b_1^{m_1} \cdots b_s^{m_s}$ in the polynomial

$$(a_1 + \cdots + a_r)^i (b_1 + \cdots + b_s)^{k-i} P(a_1, \dots, a_r, b_1, \dots, b_s).$$

In particular, when $m_1 = \cdots = m_s = 0$, we get the degree of the variety embedded in $\mathbb{P}^N$ from its multidegree as a subvariety of $\boldsymbol{P}^{n_1,\dots,n_r}$.

The last projective degree $d_{\dim X}(\Phi)$ is the degree of $\mathfrak{S}_{n_1,\dots,n_r}(X) \subseteq \mathbb{P}^N$. The first projective degree $d_0(\Phi)$ is the product of the degree of $\mathfrak{S}_{m_1,\dots,m_s}(\overline{\Phi(X)}) \subseteq \mathbb{P}^M$ with the *degree* of Φ. We have that Φ is birational onto its image if and only if its degree is 1, that is, if and only if $d_0(\Phi) = \deg(\mathfrak{S}_{m_1,\dots,m_s}(\overline{\Phi(X)}))$. Thus we can determine whether Φ is birational without computing its inverse.

2. Implementation in Macaulay2. The *Macaulay*2 package *MultiprojectiveVarieties* provides support for multiprojective varieties and multirational maps. It implements, among other things, the methods described in the previous section. As we previously said, a multirational map can be represented by a list of rational maps having as target a projective space. Partial support for this particular kind of rational maps is provided by the package Cremona, on which the first one depends.

Here we give just one simple example to illustrate how one can work with these packages. We refer to the online documentation of *Macaulay*2 for more examples and technical details.

It is classically well known that a smooth cubic hypersurface $X \subset \mathbb{P}^5$ containing two disjoint planes is birational to $\mathbb{P}^2 \times \mathbb{P}^2$, and that the inverse map $\mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow X$ is not defined along a K3 surface of degree 14. We now analyze this example using *Macaulay*2.

In the following lines of code, we first define the two projections $f : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ and $g : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ from two disjoint planes in $\mathbb{P}^5$, then we define the multirational map $(f, g) : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ and restrict it to a smooth cubic hypersurface X containing the two planes. So we get a multirational map $\Phi : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$.

```
M2 --no-preload
Macaulay2, version 1.18
i1 : needsPackage "MultiprojectiveVarieties"; -- version 2.2
i2 : K = QQ, K[t,u,v,x,y,z];
i3 : f = rationalMap {t,u,v};
o3 : RationalMap (linear rational map from PP^5 to PP^2)
i4 : g = rationalMap {x,y,z};
o4 : RationalMap (linear rational map from PP^5 to PP^2)
i5 : Phi = rationalMap {f,g};
o5 : MultirationalMap (rational map from PP^5 to PP^2 x PP^2)
i6 : X = projectiveVariety ideal(t*u*x-u^2*x+u*v*x-v^2*x+t*x^2-u*x^2+t^2*y-t*u*y-t*v*y-t*x*y
                                 -v*x*y-t*y^2+t*u*z+v^2*z-t*x*z-u*y*z-v*y*z-t*z^2+u*z^2);
o6 : ProjectiveVariety, hypersurface in PP^5
i7 : Phi = Phi|X;
o7 : MultirationalMap (rational map from X to PP^2 x PP^2)
```

Next, we verify that Φ is dominant and birational, compute the inverse map Φ^{-1}, and "describe" the base locus of Φ^{-1}.

```
i8 : image Phi == target Phi
o8 = true
i9 : degree Phi
o9 = 1
i10 : inverse Phi;
o10 : MultirationalMap (birational map from PP^2 x PP^2 to X)
i11 : describe baseLocus inverse Phi;
o11 = ambient:.............. PP^2 x PP^2
      dim:.................. 2
      codim:................ 2
      degree:............... 14
      multidegree:.......... 2 T_0^2 + 5 T_0 T_1 + 2 T_1^2
      generators:........... (2,1)^1 (1,2)^1
      purity:............... true
      dim sing. l.:......... -1
```

Now we take the graph of Φ with the two projections $p_1 : \Gamma(\Phi) \to X$ and $p_2 : \Gamma(\Phi) \to \mathbb{P}^2 \times \mathbb{P}^2$. We calculate the projective degrees of p_1 and p_2, the inverse of p_2, and verify that $p_1 \circ p_2^{-1} = \Phi^{-1}$ and that p_2 is a morphism but not an isomorphism.

```
i12 : (p1,p2) = graph Phi;

i13 : (multidegree p1, multidegree p2)
o13 = ({141, 63, 25, 9, 3}, {141, 78, 40, 18, 6})

i14 : inverse p2;
o14 : MultirationalMap (birational map from PP^2 x PP^2 to 4-dimensional
                        subvariety of PP^5 x PP^2 x PP^2)

i15 : (inverse p2) * p1 == inverse Phi, isMorphism p2, isIsomorphism p2
o15 = (true, true, false)
```

We now calculate the *exceptional locus* of the first projection p_1; this is the inverse image of the base locus of p_1^{-1}.

```
i16 : baseLocus Phi == baseLocus inverse p1
o16 = true

i17 : E = p1^* (baseLocus Phi);
o17 : ProjectiveVariety, threefold in PP^5 x PP^2 x PP^2

i18 : dim E, degree E
o18 = (3, 48)
```

Finally, we take the first projection $h : \Gamma(p_2) \to \Gamma(\Phi)$ from the graph of p_2. This multirational map, regarded as a rational map between embedded projective varieties, has as source a fourfold of degree 771 in $\mathbb{P}^{485}$ and as target a fourfold of degree 141 in $\mathbb{P}^{53}$.

```
i19 : h = first graph p2;
o19 : MultirationalMap (birational map from 4-dimensional subvariety of
                        PP^5 x PP^2 x PP^2 x PP^2 x PP^2 to 4-dimensional
                        subvariety of PP^5 x PP^2 x PP^2)

i20 : degree source h, degree target h
o20 = (771, 141)
```

By construction, we know (and *Macaulay*2 knows) that the map h is birational. We can also verify this experimentally, by reducing to prime characteristic and calculating the fiber of h at a random point p on its source.

```
i21 : h = h ** (ZZ/1000003),;

i22 : p = point source h;
o22 = ProjectiveVariety, a point in PP^5 x PP^2 x PP^2 x PP^2 x PP^2

i23 : p == h^* h p
o23 = true
```

On a standard laptop, the time to execute the 23 lines of code above is less than 5 seconds.

SUPPLEMENT. The online supplement contains version 2.3 of *MultiprojectiveVarieties*.

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