




Performance measurement with expectiles

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Abstract

Financial performance evaluation is intimately linked to risk measurement methodologies. There exists a well-developed literature on axiomatic and operational characterization of measures of performance. Hinged on the duality between coherent risk measures and reward associated with investment strategies, we investigate representation of acceptability indices of performance using expectile-based risk measures that recently attracted a lot of attention inside the financial and actuarial community. We propose two purely expectile-based performance ratios other than the classical gain-loss ratio and the Omega ratio. We complement our analysis with elicibility of expectile-based acceptability indices and their conditional version accounting for new information flow.

Keywords Acceptability indices · Expectile-based coherent risk measures · Elicibility · Conditional performance measure

JEL Classification C02 · C21 · C65 · G11 · G17 · G20

1 Introduction

Mostly used in practice to quantify market risk, value-at-risk at probability level $\alpha \in (0, 1)$ is the α -quantile of the profit/loss (P&L henceafter) distribution corresponding to a single asset or a portfolio, see for example the Basel Committee on Banking Supervision (2006). But it is a well-documented fact that value-at-risk is not subadditive in general and hence it is not a coherent risk measure. Moreover it does not account for the size of losses beyond the threshold α , because quantiles hardly depend upon the frequency of tail losses and not on their values. On one hand, coherence is a good theoretical property satisfied instead by expected shortfall, see Artzner et al. (1999), that also accounts for extreme losses in the tail of the P&L distribution, as

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recognized by the recent market risk framework of the Basel Committee on Banking Supervision (2019). On the other hand from the regulatory and statistical point of view, methodologies for testing the accuracy of a risk model are essential then value-at-risk is preferred. Expected shortfall requires a huge amount of data for accurate estimation, in fact it is not *elicitable* as pointed out for example by Gneiting (2011). A standard risk measure that is coherent and appropriate for backtesting is the mean, but it cannot capture tail risk. Expectiles have been receiving increasing attention as risk measures in mathematical finance and actuarial science since the contribution of Kuan et al. (2009). In fact expectiles are both coherent for a threshold level range and elicitable risk measures, see Bellini et al. (2014) and the references therein. From a statistical point of view, expectiles are the least square analogue of quantiles, both being M -functionals¹ with asymmetric convex loss function, but the latter are not coherent as risk measures. There are other advantages of using expectiles as coherent risk measures: they rely on tail expectations rather than tail probabilities, thus they are sensitive to extreme losses (in contrast to quantiles) and may lead to more prudent and reactive risk management: altering the shape of extreme losses does impact the expectiles in contrast to value-at-risk. In addition, expectiles avoid the use of regularity conditions on the underlying distribution of losses. For a detailed account of all these aspects in connection with the problem of estimation of expectiles see for example Daouia et al. (2018) and Daouia et al. (2021). Moreover, numerical evidence found in Bellini and Di Bernardino (2017) is in favor of expectiles as a reasonable alternative to value-at-risk or expected shortfall, especially when using them in portfolio management based on the gain-loss ratio.

Evaluating the performance of a financial trade is a crucial point for making informed choices among alternative investments. A popular guideline suggests that rational analysts and investors should select their portfolio according to a reward/risk criterion acting as a *performance measure*. Performance measurement is also a device to judge the quality of the value-added service provided by fund managers, when processing information not reflected by market prices. Needless to say, the classical Sharpe ratio is the prototypical of performance measures. Other indices have been proposed over the past four decades based on criticisms about the Sharpe ratio, concerning distributional assumptions needed for its compatibility with the mean-variance approach to portfolio selection, or the lack of consistency with arbitrage principles. Thus Bernardo and Ledoit (2000) proposed the alternative *gain-loss ratio*, beside Cherny and Madan (2009) elaborates on this and provides an axiomatic approach to frame performance indices meant to measure the largest nonnegative level at which the risk of loss is still acceptable. Following their approach, for a risky position X modelled by a bounded random variable, its *index of acceptability* is characterized as a performance measure satisfying four axioms: (increasing) monotonicity, quasi-concavity, scale invariance and Fatou property (see Sect. 3 below). Moreover, Cherny and Madan (2009) pointed out that such an index corresponds to a continuum of degrees of acceptability based on a system of acceptable bounded trades X , and Cherny and Madan (2009, Theorem 1)

¹ Recall that given a loss function $g(x, s)$ which is Borel on $\mathbb{R} \times \mathbb{R}$ an M -functional $T(F_X)$, related to a cumulative distribution function F_X of a random variable X , is the solution of $\int g(x, T(F_X))dF_X(x) = \min_{s \in \Theta} \int g(x, s)dF_X(x)$. Here Θ is an open subset of \mathbb{R} representing the parameter set. This is a generalization of maximum likelihood estimates.

provides its representation through an associated system of scenarios that support the dual representation of corresponding coherent risks in such a way their expectations are positive accordingly. This is because a system of acceptable trades can be equivalently defined in term of an increasing family of coherent risk measures for which acceptability is typically given by their nonnegativity.

The contribution of this article is structured as follows. First, we propose *expectile-based* performance measures as special cases of acceptability indices introduced by Cherny and Madan (2009) but which generalize the gain-loss ratio and the Omega ratio used in the finance industry, see Bellini et al. (2018) and the references therein. We characterize acceptability indices of performance using the dual representation of expectiles as coherent risk measures as studied in Bellini et al. (2014). To this end we additionally give a representation of performance measures for position modelled as random variables with finite expectation. Our expectile-based performance measure leads to a more general risk-adjusted return on capital (RAROC) other than the gain-loss ratio or the Omega ratio, where the coherent risk measure is the expectile itself or the expectile-based analogue to the expected shortfall, as proposed by Daouia et al. (2020) and Daouia et al. (2021). Second, given the law-invariance of our expectile-based performance measure we also provide the corresponding Kusuoka representation. Moreover, we analyze the elicibility of the proposed performance index and highlight some practical aspects of choosing competing point forecasts. Third, we provide a conditional characterization of expectile-based performance indices whenever the coherent risk measure used is the conditional expectile as introduced by Bellini et al. (2018) as a generalization of the conditional mean.

The outline of this article is as follows. Section 2 anticipates the definition of expectile-based performance ratio and highlights the advantage of using it from an economic point of view. Section 3 sets up the definition and properties of acceptability indices for financial positions with finite expectation, then delivers the representation of coherent acceptability indices. Section 4 presents a quick review of expectiles together with their use as coherent risk measure, and provides the representation of expectile-based acceptability indices. Section 5 provides two main examples of expectile-based performance ratios. Moreover, the connection of expectile-based acceptability indices with the Omega ratio as well as the gain-loss ratio is highlighted. Section 6 is on elicibility of the expectile-based acceptability index. Section 7 develops a conditional version of expectile-based acceptability indices. Section 8 contains some concluding remarks.

2 Motivation of the paper

The framework we introduce later in this article is based on the well-understood concept of *acceptability index of performance* due to Cherny and Madan (2009). Originally developed for positions² $X \in L^\infty$, given an indexed family $(\rho_x)_{x \in \mathbb{R}_+}$ of coherent risk measures increasing in x we say that a position is acceptable at the level

² Loosely speaking, the space L^∞ contains bounded random variables.

$x \geq 0$ if and only its financial performance can be measured as

$$\sup\{x \in \mathbb{R}_+ \mid \rho_x(X) \leq 0\}. \quad (1)$$

This establishes a natural duality between coherent risk measures and performance measures. As showed by Cherny and Madan (2009), using the dual representation of each $\rho_x(X)$ one can obtain different acceptability indices of performance. In fact for every $x \geq 0$ the acceptability condition $\rho_x(X) \leq 0$ is equivalently characterized by a system of acceptable trades $\mathcal{A}_x := \{X \in L^\infty \mid \alpha(X) \geq x\}$ and the supremum in (1) can be taken over $\{x \in \mathbb{R}_+ \mid X \in \mathcal{A}_x\}$. Observe that $\rho_x(X) = \inf_{\varphi \in \mathcal{S}_x} \mathbb{E}(\varphi X)$ and for each $x \geq 0$ we have \mathcal{S}_x as a set of scenarios, i.e. the class of all Radon–Nikodym derivatives $\varphi = \frac{d\mathbb{Q}}{d\mathbb{P}}$ of probability measures \mathbb{Q} equivalent to the physical probability measure \mathbb{P} . Choosing ρ_x as the negative of expectiles (see Sect. 4 for a formal definition) with the appropriate probability level and applying the definition of acceptability systems of X in term of the best gain-loss ratio we get

$$\sup \left\{ x \in \mathbb{R}_+ \mid \frac{\mathbb{E}(X^+)}{\mathbb{E}(X^-)} \geq x \right\} = \frac{\mathbb{E}(X^+)}{\mathbb{E}(X^-)}.$$

Therefore, for bounded X this yields the trivial representation of expectile-based performance measure as the gain-loss ratio itself: this is not surprising since the connection between expectiles and gain-loss ratios is already known in the literature. Our first goal is to define an expectile-based performance index with domain L^1 to account for potential extreme losses possibly given by heavy-tailed distributions. We keep the axiomatic approach of Cherny and Madan (2009) with appropriate modifications to account for the new domain, but as a novel approach we will make use of the dual representation of expectiles as analyzed in Bellini et al. (2014). This gives us the advantage to deduce acceptability indices other than gain-loss or Omega ratios.

Remark 1 For the mapping $\alpha : L^\infty \rightarrow \mathbb{R}_+ \cup \{+\infty\} =: [0, +\infty]$ given in Cherny and Madan (2009, Theorem 1) we adopt in our article a different name (see Sect. 3) to avoid confusion with the standard symbol used to identify the probability level for value-at-risk.

Anticipating a little our main examples of expectile-based performance ratio³ that will be given in Sect. 5, we propose the following main alternative to the gain-loss ratio:

$$\frac{\mathbb{E}(X)}{-e_\alpha(X)},$$

where $-e_\alpha(X)$ is the expectile risk measure at probability level α , for which we usually choose the interval $(0, \frac{1}{2}]$, see Sect. 4 for further details. To appreciate the advantage of using this performance ratio instead of the gain-loss ratio from an economic point of view, we recall the needed asset pricing framework. Jaschke and Küchler (2001)

³ There is also a slight abuse of notation since we report $\alpha \in (0, 1)$ as the probability level that will be denoted by $\alpha(x)$, see Sects. 4 and 5.

establish a one-to-one correspondence among the partial ordering defined on the set of traded positions X (interpreted from the decision theory perspective as a *preference relation*), sets of acceptable positions, valuation bounds (actually *good-deal bounds*), coherent risk measures and price systems. Indeed, Bernardo and Ledoit (2000) introduced the gain-loss ratio as an alternative to the Sharpe ratio whose exacerbated high values yield a portfolio regarded as a quasi-arbitrage, namely a good-deal as studied also in Cochrane and Saa-Requejo (2001). This given raise to asset pricing theory based on no-good deals that postulates to restrict the Sharpe ratio and the corresponding pricing kernels. Therefore, denoting \mathcal{K} the set of (non-trivial) portfolio gains in a two-period economy and for a general probability space, the *best* gain-loss ratio is always attained

$$\sup_{X \in \mathcal{K}, X \neq 0} \frac{E(X^+)}{E(X^-)} = \max_{X \in \mathcal{K}, X \neq 0} \frac{E(X^+)}{E(X^-)} = \min_{Z \in [c, C]} \frac{\text{ess sup } Z}{\text{ess inf } Z} < +\infty,$$

for a bounded and bounded away from zero state-price density Z with constants $c, C > 0$, see Bernardo and Ledoit (2000). This result is correct only for finite sample spaces Ω , in fact Biagini and Pinar (2013) showed the best gain-loss ratio is a poor performance measure in a quite general continuous-time model, after providing a refined dual representation on gain-loss free markets based on utility maximization of terminal wealth. Specifically, they assumed a filtered probability space satisfying the usual conditions and over which a semimartingale $S = (S_t^1, \dots, S_t^n)_{t \in [0, T]}$ representing the underlying price processes only satisfies $\sup_{t \in [0, T]} |S_t^i| \in L^1$, for any $i = 1, \dots, n$: in this way \mathcal{K} becomes the set of financial positions K replicable at zero cost through an admissible (predictable and bounded) self-financing trading strategy $\pi = (\pi_t^1, \dots, \pi_t^n)_{t \in [0, T]}$, i.e. K equals the stochastic integral $(\pi \cdot S)_T := \sum_{i=1}^n \pi_0^i S_0^i + \sum_{i=1}^n \int_0^T \pi_t^i dS_t^i$, a terminal gain over the time horizon $[0, T]$. The main result Biagini and Pinar (2013, Theorem 2.4) yields the *finiteness* of the best gain-loss ratio $\sup_{X \in \mathcal{K}, X \neq 0} \frac{E(X^+)}{E(X^-)} < +\infty$ if and only if the set of equivalent martingale measures is nonempty and the corresponding pricing kernels Z are Radon–Nikodým derivatives bounded and bounded away from zero, provided that the above market is gain-loss free.⁴ Based on this characterization, the best gain-loss ratio is likely to be infinite⁵ and then the underlying market is not gain-loss free even if it should be arbitrage free, see Biagini and Pinar (2013, Example 2.7) where a Black–Scholes market is assumed with typical unbounded (unique) price kernel $Z = e^{-\frac{\mu-r}{\sigma} W_T - (\frac{\mu-r}{\sigma})^2 \frac{T}{2}}$, where r is the risk-free rate and $\frac{\mu-r}{\sigma}$ is the market price of risk with a standard Brownian motion $W = (W_t)_{t \in [0, T]}$ defined over the P-augmentation of its natural filtration. In the setting of the same example, let $K_\epsilon := \mathbf{I}_{A_\epsilon} - c_\epsilon$ where $\mathbf{I}_{\{\bullet\}}$ is the indicator of the event $A_\epsilon = \{Z < \epsilon\}$ of having a very small state-price density Z for $0 < \epsilon < 1$. Assuming a zero risk-free rate $r = 0$, one considers \mathbf{I}_{A_ϵ} as a cash-or-nothing digital call option on the stock price $S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$ with very large strike price provided that $\epsilon \rightarrow 0$. By the market completeness, K_ϵ is a trading gain where $c_\epsilon = E(Z \mathbf{I}_{A_\epsilon})$ is the

⁴ The precise concept is *λ gain-loss free*, see Biagini and Pinar (2013, Definition 2.2).

⁵ Moreover, it could be finite but with no supremum, see Biagini and Pinar (2013, Example 2.8).

replicating cost of the option that satisfies $c_\epsilon < \epsilon p_\epsilon < 1$ and p_ϵ being the probability $P(A_\epsilon)$, since $c_\epsilon < E(\epsilon \mathbf{I}_{A_\epsilon})$. The gain-loss ratio is

$$\frac{E(K_\epsilon^+)}{E(K_\epsilon^-)} = \frac{(1 - c_\epsilon)p_\epsilon}{c_\epsilon(1 - p_\epsilon)} > \frac{1}{\epsilon} - p_\epsilon,$$

which goes to $+\infty$ as $\epsilon \rightarrow 0$. On the other hand, our proposed ratio

$$\frac{E(K_\epsilon)}{-e_\alpha(K_\epsilon)} = \frac{p_\epsilon - c_\epsilon}{-e_\alpha(K_\epsilon)}$$

is finite because clearly $0 < p_\epsilon - c_\epsilon < +\infty$ which implies $K_\epsilon \in L^1$ and by Bellini et al. (2014, Proposition 8) it also holds $-e_\alpha(K_\epsilon) < +\infty$ for every $\alpha \in (0, \frac{1}{2}]$. The same conclusion is valid (with minor modifications) in the case of a cash-or-nothing digital put option \mathbf{I}_{B_ϵ} where $B_\epsilon = \{Z > \epsilon\}$ and very small strike price when $\epsilon \rightarrow 0$: one takes $b_\epsilon = E(Z\mathbf{I}_{B_\epsilon})$ with $\frac{q_\epsilon}{\epsilon} < b_\epsilon < 1$ and $P(B_\epsilon) = q_\epsilon$; the gain is now $\mathbf{I}_{B_\epsilon} - b_\epsilon$ as well as its opposite K_ϵ , and the gain-loss ratio is bounded below by $\frac{1-q_\epsilon}{\epsilon}$ so that it goes again to $+\infty$ as $\epsilon \rightarrow 0$. In a complete arbitrage-free market with unbounded price kernels, our expectile-based performance measure assigns a finite value to gains such as K_ϵ while the gain-loss ratio requires a state-pricing density bounded above and bounded away from zero, thus a further advantage of our proposal is that it involves a more flexible performance measurement approach compatible with no-arbitrage asset pricing theory.

Remark 2 Dybvig and Ingersoll (1982) showed that in the CAPM framework arbitrage does exist also with bounded Sharpe ratios. On the other hand, only a bounded gain-loss ratio implies absence of arbitrage and one must impose both narrowed no-arbitrage and no-good deal bounds also in incomplete markets. While the gain-loss ratio is attractive from the perspective of determining price bounds for a specified asset pricing model or to measure funds' performance with respect to benchmark state-price densities, it suffers from the *curse of infinity* in many standard models.

3 Acceptability indices on L^1

We review the definition of a coherent risk measure. Let $\mathcal{X} \subset L^0$ be a linear space of financial positions containing the constants where $L^0 := L^0(\Omega, \mathcal{F}, P)$ is the equivalence class of all random variables over a common atomless probability space (Ω, \mathcal{F}, P) . Throughout this article we work with random variables $X \in L^1$, where $L^1 := L^1(\Omega, \mathcal{F}, P)$ is the equivalence class of random variables with finite first moment modelling financial positions with respect to a fixed final date. The reason why we choose the space L^1 is mainly due to the findings in Filipović and Svindland (2012). Moreover, if X is a portfolio, then the restriction to L^1 is not a problem from the management perspective, as empirical evidence shows losses have finite first moment. From now on all equalities, inequalities and convergence concepts concerning random variables are understood in the P-a.s. sense, i.e. with probability one. Among different

sign conventions for X we assume it represents P&L with losses being in the left-tail of the underlying distribution. Then, a mapping $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is a *monetary risk measure* if:

- It is *increasing monotone*, for any $X, Y \in \mathcal{X}$ such that $X \leq Y$ implies $\rho(X) \geq \rho(Y)$;
- It is *cash additive*, any choice $m \in \mathbb{R}$ implies $\rho(X + m) = \rho(X) - m$.

If in addition ρ is

- *Positive homogeneous*, $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \geq 0$ and,
- *Subadditive*, for any $X, Y \in \mathcal{X}$ it holds $\rho(X + Y) \leq \rho(X) + \rho(Y)$,

then it is a coherent risk measure and a convex mapping too. The financial meaning of the above conditions is well understood. Working with positions $X \in L^1$ requires to restate the essential properties of acceptability indices of performance. We consider acceptable P&L's at a level $x \in \mathbb{R}_+$ forming a convex superlevel set

$$\mathcal{A}_x := \left\{ X \in L^1 \mid \text{PERF}(X) \geq x \right\}, \tag{2}$$

which is a natural requirement for *any* performance measure and in particular for $\text{PERF} \equiv \text{at}$ given in the next definition.

Definition 1 A mapping $\text{at} : L^1 \rightarrow [0, +\infty]$ is an **acceptability index** if it satisfies the following four properties.

- **Quasi-concavity** for any pair $X, Y \in L^1$ and for every $\lambda \in [0, 1]$ such that $\text{at}(X) \geq x$ and $\text{at}(Y) \geq x$ one has

$$\text{at}(\lambda X + (1 - \lambda)Y) \geq x. \tag{3}$$

- **Monotonicity** for any $X, Y \in L^1$

$$X \leq Y \implies \text{at}(X) \leq \text{at}(Y). \tag{4}$$

- **Scale invariance** for every $\lambda > 0$ and $X \in L^1$

$$\text{at}(\lambda X) = \text{at}(X). \tag{5}$$

- **Upper semi-continuity** given a sequence $(X_n)_{n \in \mathbb{N}} \subset L^1$ converging to $X \in L^1$ in the L^1 -norm, $\|X_n - X\|_1 \rightarrow 0$, we have

$$\limsup_{n \rightarrow \infty} \text{at}(X_n) \leq \text{at}(X), \tag{6}$$

which implies $\text{at}(X) \geq x$ provided that $\text{at}(X_n) \geq x$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$.

Quasi-concavity is equivalent to the convexity of any \mathcal{A}_x , yielding a diversified position performs as well as its components. Acceptable positions are valued monotonically: al is an increasing mapping and Y is at least as acceptable as X provided that the latter is dominated by the former. By scale invariance any acceptance set \mathcal{A}_x is a convex cone: the level of acceptance remains the same whenever we scale the financial positions. Eventually, the acceptability functional is required to be $\|\cdot\|_1$ -upper semi-continuous and as byproduct the acceptance set \mathcal{A}_x is norm-closed in L^1 for a fixed $x \in \mathbb{R}_+$.

Remark 3 In Definition 1 there are two differences with respect to the original definition of acceptability indices given by Cherny and Madan (2009), namely the domain of al is now L^1 and the Fatou property is replaced by the upper semi-continuity of al in the norm topology (fourth axiom above).

Now we come to the basic representation of acceptability indices on L^1 . To this end we provide two lemmas which we will use in the proof of the sufficiency part of Proposition 1 below. Let us define

$$\rho_x(X) := \inf \{m \in \mathbb{R} \mid \text{al}(X + m) \geq x\}, \quad \text{for every } x \in \mathbb{R}_+, X \in L^1, \quad (7)$$

and take the infimum over $m \in \mathbb{R}$ of both sets $\{m \in \mathbb{R} \mid \text{al}(X + m) \geq y\} \subset \{m \in \mathbb{R} \mid \text{al}(X + m) \geq x\}$, for any $0 < x \leq y$. Thus, $x \mapsto \rho_x(X)$ increases for fixed $X \in L^1$. Any mapping in this increasing family can be represented by acceptability indices as is established in Lemma 1 below. We refer to coherent risk measures defined on L^1 that are $\|\cdot\|_1$ -lower semicontinuous, i.e.

$$\liminf_{n \rightarrow +\infty} \rho_x(X_n) \geq \rho_x(X), \quad \text{for any } (X_n)_{n \in \mathbb{N}} \subset L^1 \text{ s.t. } \|X_n - X\|_1 \rightarrow 0.$$

Lemma 1 Let $\rho_x(X)$ be defined as in (7), for any $x \in \mathbb{R}_+$ and $X \in L^1$, by an acceptability index al . Then $\rho_x(X)$ is a coherent risk measure on L^1 .

Proof For $m \in \mathbb{R}$ and $x \in \mathbb{R}_+$ condition $\text{al}(X + m) \geq x$ is equivalent to $X + m \in \mathcal{A}_x \subset L^1$ and we have that $X \leq Y$ together with $X \in \mathcal{A}_x$ implies $Y \in \mathcal{A}_x$. We check monotonicity. Take $x \in \mathbb{R}_+$ and select $X, Y \in L^1$ such that $X \geq Y$. By monotonicity of al we have

$$\text{al}(Y + m) \leq \text{al}(X + m), \quad \text{for every } m \in \mathbb{R}.$$

Thus we deduce $\{m \in \mathbb{R} \mid \text{al}(X + m) \geq x\} \supset \{m \in \mathbb{R} \mid \text{al}(Y + m) \geq x\}$, and taking the infimum of both sets we get

$$\rho_x(X) := \inf \{m \in \mathbb{R} \mid \text{al}(X + m) \geq x\} \leq \inf \{m \in \mathbb{R} \mid \text{al}(Y + m) \geq x\} := \rho_x(Y).$$

To show positive homogeneity, it suffices to call for the scale invariance of al . Next, we show subadditivity. Take $m_1, m_2 \in \mathbb{R}$ such that $\text{al}(X + m_1) \geq x$ and $\text{al}(Y + m_2) \geq x$,

for every $X, Y \in L^1$ and $x \in \mathbb{R}_+$. By quasi-concavity of at , for any $\lambda \in [0, 1]$ we have

$$\text{at}(\lambda X + \lambda m_1 + (1 - \lambda)Y + (1 - \lambda)m_2) \geq x,$$

at the same acceptability level x . Choosing $\lambda = \frac{1}{2}$ and using again scale invariance of the acceptability index entails

$$\text{at}(X + Y + (m_1 + m_2)) \geq x.$$

Therefore, the scalar $m_1 + m_2$ belongs to the set $\{m \in \mathbb{R} \mid \text{at}(X + Y + m) \geq x\}$ and it is greater than or equal to the infimum over the same set, which in turn is just $\rho_x(X + Y)$. This inequality holds true for all m_1 and all m_2 belonging to $\{m \in \mathbb{R} \mid \text{at}(X + m) \geq x\}$ and to $\{m \in \mathbb{R} \mid \text{at}(Y + m) \geq x\}$, respectively. As a consequence, taking the infimum with respect to m_1 and then with respect to m_2 we get

$$\rho_x(X) + \rho_x(Y) \geq \rho_x(X + Y).$$

To show cash invariance, we exhibit the following for every $x \in \mathbb{R}_+$ and $X \in L^1$:

$$\begin{aligned} \rho_x(X + c) &:= \inf\{m \in \mathbb{R} \mid \text{at}(X + c + m) \geq x\} \\ &= \inf\{m \in \mathbb{R} \mid \text{at}(X + (c + m)) \geq x\} \\ &= \inf\{c + m \in \mathbb{R} \mid \text{at}(X + (c + m)) \geq x\} - c \\ &= \inf\{r \in \mathbb{R} \mid \text{at}(X + r) \geq x\} - c \\ &=: \rho_x(X) - c. \end{aligned}$$

To check the lower semicontinuity of ρ_x in the $\|\cdot\|_1$ -norm it suffices to take $\epsilon > \liminf_{n \rightarrow +\infty} \rho_x(X_n)$ and (if necessary) passing to a subsequence to get $X_n + \epsilon \in \mathcal{A}_x$ for all $n \in \mathbb{N}$. Thus, $\text{at}(X_n + \epsilon) \geq x$ and using the upper semi-continuity of the acceptability index we also have $\text{at}(X + \epsilon) \geq x$, i.e. $\rho_x(X_n) \leq \epsilon$, and by the arbitrariness of ϵ it holds $\liminf_{n \rightarrow +\infty} \rho_x(X_n) \geq \rho_x(X)$. A byproduct, ρ_x is a coherent risk measure on L^1 . □

Remark 4 The finiteness of $\|\cdot\|_1$ -lower semicontinuous coherent risk measures on L^1 is equivalent to the $\|\cdot\|_1$ -continuity, see for example Rüschemdorf (2013, Section 7.2.2).

We can represent acceptability indices in terms of an increasing family of coherent risk measures on L^1 .

Lemma 2 *Let $(\rho_x)_{x \in \mathbb{R}_+}$ be a family of coherent risk measures on L^1 increasing in x . Then, the mapping $\text{at} : L^1 \rightarrow [0, +\infty]$ defined as*

$$\text{at}(X) := \sup \{x \in \mathbb{R}_+ \mid \rho_x(X) \leq 0\} \tag{8}$$

is an acceptability index of performance (we assume $\sup \emptyset = 0$).

Proof Let $x \geq 0$, then by monotonicity of the risk measures ρ_x in x we have

$$\rho_x(X) \leq \rho_x(Y), \quad \text{for all } X, Y \in L^1 \text{ such that } X \geq Y.$$

For any $x_0 \in \{x \in \mathbb{R}_+ \mid \rho_x(X) \leq 0\}$ we also have $\rho_{x_0}(X) \leq 0$, which together with the monotonicity entails $\rho_{x_0}(X) \leq \rho_{x_0}(Y) \leq 0$, for all $X \geq Y$. As a consequence the set inclusion

$$\{x \in \mathbb{R}_+ \mid \rho_x(X) \leq 0\} \supset \{x \in \mathbb{R}_+ \mid \rho_x(Y) \leq 0\}$$

holds, and taking the supremum of both sides monotonicity of ai is proved. To check quasi-concavity of ai , we first pick $X, Y \in L^1$ such that $\text{ai}(X) \geq x_0$ and $\text{ai}(Y) \geq x_0$ whenever $x_0 \in (0, +\infty)$. By definition (8) together with monotonicity of ρ_x we have $\rho_x(X) \leq \rho_{x_0}(X) \leq 0$ and $\rho_x(Y) \leq \rho_{x_0}(Y) \leq 0$, for all $x < x_0$. This combined with the positive homogeneity of ρ_x yields

$$\rho_x(\lambda X) = \lambda \rho_x(X) \leq 0, \quad \rho_x((1 - \lambda)Y) = (1 - \lambda)\rho_x(Y) \leq 0,$$

for every $\lambda \in [0, 1]$. Moreover, by subadditivity of ρ_x and again for every $x < x_0$ we additionally have

$$\rho_x(\lambda X + (1 - \lambda)Y) \leq 0$$

which entails $\sup \{x \in \mathbb{R}_+ \mid \rho_x(\lambda X + (1 - \lambda)Y) \leq 0\} \geq x_0$. Eventually, this combined with definition (8) yields $\text{ai}(\lambda X + (1 - \lambda)Y) \geq x_0$ and quasi-concavity follows. Scale invariance of ai follows immediately from the positive homogeneity of ρ_x . Finally take $(X_n)_{n \in \mathbb{N}} \subset L^1$ such that $\|X_n - X\|_1 \rightarrow 0$ and $\text{ai}(X_n) \geq x$ for every $n \in \mathbb{N}$, $x \in \mathbb{R}_+$. Now, since coherent risk measures on L^1 are continuous and then lower semicontinuous (this follows from, for instance, Ruzscazyński and Shapiro (2006, Proposition 3.1), as byproduct we have $\|\rho_x(X_n) - \rho_x(X)\|_1 \rightarrow 0$ and then $\rho_x(X) \leq 0$ because $\rho_x(X_n) \leq 0$ which implies $\text{ai}(X) \geq x$. \square

Proposition 1 A mapping $\text{ai} : L^1 \rightarrow [0, +\infty]$ is an acceptability index if and only if there exists a family of sets of scenarios $(\mathcal{S}_x)_{x \in \mathbb{R}_+}$ increasing in x such that the representation

$$\text{ai}(X) = \sup \left\{ x \in \mathbb{R}_+ \mid \inf_{\varphi \in \mathcal{S}_x} E(\varphi X) \geq 0 \right\} \quad (9)$$

holds, where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Proof (Necessity) Let ai be given by (9). Firstly, we check property 1 of Definition 1. Pick $X, Y \in L^1$ with $\text{ai}(X) \geq z$ and $\text{ai}(Y) \geq z$ for $z \in \mathbb{R}_+$. For any $y < z$ we can find a representation set $\mathcal{S}_y \subset \mathcal{S}_z$ such that the two inequalities together with the supremum in (9) entail $E(\varphi X) \geq 0$ and $E(\varphi Y) \geq 0$ for the corresponding scenario $\varphi \in \mathcal{S}_y$. Taking the convex combination $Z = \lambda X + (1 - \lambda)Y$, for any $\lambda \in [0, 1]$,

and considering $\min\{X, Y\} \leq Z \leq \max\{X, Y\}$ we have $E(\varphi Z) \geq 0$ too, and the quasi-concavity follows by enlarging the representation set, taking the infimum of the expectations and finally taking the supremum with respect to $x \in \mathbb{R}_+$. Secondly, consider the sets

$$A = \left\{ x \in \mathbb{R}_+ \mid \inf_{\varphi \in \mathcal{S}_x} E(\varphi X) \geq 0 \right\} \quad \text{and} \quad B = \left\{ x \in \mathbb{R}_+ \mid \inf_{\varphi \in \mathcal{S}_x} E(\varphi Y) \geq 0 \right\}.$$

Now for $X, Y \in L^1$ such that $X \leq Y$, pick $x_0 \in A$ and observe that $0 \leq E(\varphi X) \leq E(\varphi Y)$ whenever $\varphi \in \mathcal{S}_{x_0}$ which implies $x_0 \in B$. On the other hand, for any $y_0 \in B$ and using the same inequality between expectations for $\varphi \in \mathcal{S}_{y_0}$ we can consider $Y = 0$ which implies $E(\varphi X) < 0$ yielding $y_0 \notin B$. We conclude that $A \subset B$ and passing to the supremum property 2 is proved. Property 3 is an easy consequence of linearity of $E(\cdot)$. To check that at is upper-semi continuous in the L^1 -norm, pick a sequence $(X_n)_{n \in \mathbb{N}} \subset L^1$ with $X_n \in \mathcal{A}_x$ each $n \in \mathbb{N}$ for a fixed $x \in \mathbb{R}_+$. Clearly $\text{at}(X_n) \geq x$. Assume further $\|X_n - X\|_1 \rightarrow 0$. Then whatever the choice of $y < x$ is, for any $\varphi \in \mathcal{S}_y$ and any $n \in \mathbb{N}$ we have $E(\varphi X_n) \geq 0$ and by the L^1 -convergence $E(\varphi X) \geq 0$ too. As a consequence, $\text{at}(X) \geq x$, then after passing to the supremum we have that \mathcal{A}_x is $\|\cdot\|_1$ -closed and the mapping at is then upper-semi continuous. (Sufficiency). Let's assume at is an acceptability index. Then let

$$\rho_x(X) := \inf \{ m \in \mathbb{R} \mid X + m \in \mathcal{A}_x \}$$

with $\mathcal{A}_x = \{ X \in L^1 \mid \text{at}(X) \geq x \}$ for $x \in \mathbb{R}_+$. Since $X + m \in \mathcal{A}_x$ is equivalent to $\text{at}(X) \geq x$, by Lemma 1 each $\rho_x(X)$ is a coherent risk measure of X and moreover $x \mapsto \rho_x(X)$ is increasing in x . Since $\mathcal{A}_x \subset L^1_+$ is a $\|\cdot\|_1$ -closed convex cone, its polar $(\mathcal{A}_x)^\circ = \{ X \in L^\infty \mid E(XY) \geq 0, \forall Y \in \mathcal{A}_x \} \subset L^\infty_+$ is a $\sigma(L^\infty, L^1)$ -closed convex cone. A basis for this cone is \mathcal{S}_x , i.e. $(\mathcal{A}_x)^\circ = \cup_{\lambda \geq 0} \lambda \mathcal{S}_x$. By the Bipolar theorem (see for example Delbaen (2012)) we get

$$((\mathcal{A}_x)^\circ)^\circ = \mathcal{A}_x = \left\{ X \in L^1 \mid E(XY) \geq 0, \text{ for every } Y \in \mathcal{S}_x \right\}, \quad \text{with } Y = \varphi.$$

Observe that $\mathcal{A}_y \subset \mathcal{A}_x$ is equivalent to $(\mathcal{A}_x)^\circ \subset (\mathcal{A}_y)^\circ$, for $0 < x < y$. Thus, X lies in some \mathcal{A}_{x_0} so that $\rho_{x_0}(X) \leq 0$. This is in turn equivalent to $\text{at}(X) \geq x$, which by Lemma 2 implies the desired representation (9) of the acceptability index by passing to $\inf_{\varphi \in \mathcal{S}_x} E(\varphi X)$. □

4 Representation of expectile-based acceptability indices

Given a financial position $X \in L^1$, the definition of expectile at a probability level $\alpha \in (0, 1)$ originally due to Newey and Powell (1987) is

$$e_\alpha(X) = \operatorname{argmin}_{m \in \mathbb{R}} \{ E(\ell(X - m) - \ell(X)) \}, \tag{10}$$

also called the α -expectile of X . Here ℓ is the asymmetric quadratic loss function defined as

$$\ell(x) := |\alpha - \mathbf{I}_{\{x \leq 0\}}|x^2,$$

thus one could see that expectiles are asymmetric generalization of the mean and also quadratic versions of the usual quantiles. The corresponding first order condition

$$\alpha \mathbf{E}(X - e_\alpha(X))^+ - (1 - \alpha)\mathbf{E}(X - e_\alpha(X))^- = 0 \quad (11)$$

entails $e_\alpha(X)$ as the unique minimizer, where as usual $X^+ = \max\{0, X\}$ is the positive part and $X^- = \max\{0, -X\}$ is the negative part of a random variable.⁶ Indeed, the function

$$f_X(m) := \alpha(x - m)\mathbf{I}_{\{x \geq 0\}} + (1 - \alpha)(x - m)\mathbf{I}_{\{x < 0\}} \quad (12)$$

is such that (after taking expectation) for $X \in L^1$ the first order condition for the unique minimizer in Eq. (11) is given with derivative $m \mapsto -2 f_X(m)$, which implies $f_X(m) = 0$. Now, by Krättschmer and Zähle (2017, Lemma A.1) the mapping $m \mapsto f_X(m)$ is real-valued, continuous and strictly decreasing, thus the α -expectile of a position X with finite expectation can be characterized by its inverse function as follows:

$$m = f_X^{-1}(f_X(m)) = f_X^{-1}(0), \quad \text{with } m = e_\alpha(X).$$

The above defines a mapping $e_\alpha : L^1 \rightarrow \mathbb{R}$ whose negative $-e_\alpha(X)$ is by Krättschmer and Zähle (2017, Proposition A.2) together with Bellini et al. (2014, Proposition 7(c)) a coherent risk measure for $\alpha \in (0, \frac{1}{2}]$ ensures subadditivity, and which is continuous in the L^1 -norm $\|\cdot\|_1$ (this is especially due to Cheridito and Li (2009, Theorem 4.1)). The dual representation we use in this article is given by

$$e_{1-\alpha}(-X) = \max_{\varphi \in \mathcal{S}_\alpha} (-\mathbf{E}(\varphi X)), \quad \text{if } \alpha \in (0, \frac{1}{2}], \quad (13)$$

where the underlying set of *scenarios* is

$$\mathcal{S}_\alpha = \left\{ \varphi \in L^\infty \mid \varphi > 0, \quad \mathbf{E}(\varphi) = 1, \quad \frac{\text{ess sup } \varphi}{\text{ess inf } \varphi} \leq \frac{1-\alpha}{\alpha} \right\}, \quad (14)$$

see Bellini et al. (2014, Proposition 8). Now, by Bellini et al. (2014, Proposition 7(c)) it holds $-e_\alpha(X) = e_{1-\alpha}(-X)$ thus we may reformulate representation (13) for a position $X \in L^1$ being *acceptable*:

⁶ Recall that L^1 is a Banach lattice with respect to the cone L^1_+ , i.e. for all $X, Y \in L^1$ we have $\max\{X, Y\} \in L^1$ and $|X| \leq |Y|$ implies $\|X\|_1 \leq \|Y\|_1$.

$$\begin{aligned}
 -e_\alpha(X) &= \max_{\varphi \in \mathcal{S}_\alpha} (-E(\varphi X)) \leq 0 \iff -\min_{\varphi \in \mathcal{S}_\alpha} E(\varphi X) \leq 0 \\
 &\iff \\
 e_\alpha(X) &= \min_{\varphi \in \mathcal{S}_\alpha} E(\varphi X) \geq 0.
 \end{aligned}
 \tag{15}$$

Remark 5 For every $\alpha \in (0, \frac{1}{2}]$ the set of scenarios \mathcal{S}_α is $\sigma(L^\infty, L^1)$ -compact and $\sigma(L^\infty, L^1)$ -closed-convex in L^∞ . As usual we write $\{X \in L^1 \mid \rho(X) \leq 0\}$ for the corresponding acceptance set which turns out to be a $\|\cdot\|_1$ -closed convex cone in L^1 . In Sect. 4 we use the duality pair (L^1, L^∞) with L^1 endowed with the $\|\cdot\|_1$ -topology and L^∞ endowed with the $\sigma(L^\infty, L^1)$ -topology.

We call $-e_\alpha(X)$ the α -expectile risk measure of the position X whose relevant properties are summarized in the following lemma for ease of reference, see Newey and Powell (1987) and Bellini et al. (2014).

Lemma 3 *Let $X \in L^1$ be the terminal value of a traded position. Then:*

- (a) $-e_\alpha(X + h) = -e_\alpha(X) - h$, for each $h \in \mathbb{R}$,
- (b) $-e_\alpha(\lambda X) = -\lambda e_\alpha(X)$, for every $\lambda > 0$,
- (c) $X \leq Y$ implies $-e_\alpha(X) \geq -e_\alpha(Y)$, for any $Y \in L^1$,
- (d) $-e_\alpha(X + Y) \leq -e_\alpha(X) - e_\alpha(Y)$, for $\alpha \in (0, \frac{1}{2}]$ and every $Y \in L^1$,
- (e) $-e_\alpha(X)$ is strictly decreasing and continuous with respect to $\alpha \in (0, 1)$.

Remark 6 (Financial Interpretation) Since for a position $X \in L^1$ we can write

$$\alpha = \frac{E((X - e_\alpha(X))^-)}{E(|X - e_\alpha(X)|)}, \quad \text{for all } \alpha \in (0, 1),$$

the financial interpretation of expectiles, that especially holds for the α -expectile risk measure of Lemma 3,⁷ is nowadays more clear: the *index of prudence* α is the ratio of expected margin shortfall to the total cost of capital requirement given by e_α ; the greater the expectile, the smaller the expected loss resulting in a smaller α . Indeed, lower values of the probability level represent more risk aversion. This interpretation is originally due to Kuan et al. (2009) and relies on $e_\alpha(X)$ being a location parameter for the distribution of X such that the average distance of X below the expectile equals the fraction α of the total distance. Therefore, while quantiles are only sensitive to the ordering of the X 's values,⁸ expectiles depend on the whole distribution, then α associated to an expectile changes with the distribution.

Given this definition of expectile risk measure, we prove the analogous of Cherny and Madan (2009, Theorem 1) characterizing acceptability indices $a_l(X)$. The main idea of the original representation is that any acceptability index is linked to an increasing one-parameter family of sets of scenario $(\mathcal{S}_x)_{x \in \mathbb{R}_+}$ supporting the corresponding coherent risk measure ρ_x , thus the value $a_l(X)$ yields the largest level of acceptability x such that X is valued positively under each scenario from the associated level x .

⁷ Some authors call it *expectile value-at-risk* (EVaR), see for example Bellini and Di Bernardino (2017).

⁸ Namely, the ordered statistics of a sample from the X 's distribution.

Corollary 1 *A mapping $\text{at} : L^1 \rightarrow [0, +\infty]$ is an expectile-based acceptability index if and only if there exists a family of sets of scenarios $(\mathcal{S}_{\alpha(x)})_{x \in \mathbb{R}_+}$ increasing in x such that the representation*

$$\text{at}(X) = \sup \left\{ x \in \mathbb{R}_+ \mid \min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\varphi X) \geq 0 \right\} \quad (16)$$

holds, where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Proof We apply Proposition 1 identifying coherent risk measures $\rho_x = -e_{\alpha(x)}$ with $\alpha(x) = \frac{1}{1+x}$. Indeed for $\alpha(x) \in (0, \frac{1}{2}]$ we have $\alpha(x) \rightarrow 0$ provided $x \rightarrow \infty$ so that $-e_{\alpha(x)}$ increases in x . The sets of scenarios are identified with those given by (14) with the appropriate re-labeling $\alpha \equiv \alpha(x)$. Under these conditions each $\alpha(x)$ -expectile risk measure can be represented as the unique minimizer $\min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\varphi X)$. Moreover, expectiles are proved to be Lipschitz continuous with respect to the Wasserstein metric

$$d_p(F_X, F_Y) = \inf \{ E(|X - Y|^p) \mid X \sim F_X \text{ and } Y \sim F_Y \}, \quad \text{for } p \geq 1$$

where $F_X(x) = P(X \leq x)$ and similarly for Y . Thus lower semicontinuity of $\alpha(x)$ -expectile risk measures is guaranteed and we are done. \square

Observe that Lipschitz continuity of expectiles with respect to the $\|\cdot\|_1$ -norm is proved in Krättschmer and Zähle (2017, Proposition A.2), but it can also be deduced by Bellini et al. (2018, Theorem 2.3(e)) using the trivial sigma-algebra $\{\emptyset, \Omega\}$, see Sect. 7.

Remark 7 In the Proof of Corollary 1 the used re-labeling $\alpha = \frac{1}{1+x}$ yields $\min_{\varphi \in \mathcal{S}_{\frac{1}{1+x}}} E(\varphi X) \geq \min_{\varphi \in \mathcal{S}_{\frac{1}{1+y}}} E(\varphi X)$ for $x < y$ since for the corresponding expectile-based coherent risk measures we have $-e_{\frac{1}{1+x}}(X) \leq -e_{\frac{1}{1+y}}(X)$. Indeed α -expectiles are strictly increasing in α . All this is equivalent to have a nested family of sets of scenarios $\mathcal{S}_{\frac{1}{1+x}} \subset \mathcal{S}_{\frac{1}{1+y}}$.

5 Main examples of expectile-based performance measures

With Corollary 1 in mind we are able to characterize those expectile-based performance indices other than the gain-loss ratio, for positions $X \in L^1$. In particular, we are interested in finding the appropriate scenarios used in representation (16) such that the corresponding acceptability index is a new performance ratio. By Corollary 1 acceptability at level $x \in \mathbb{R}_+$ of a position in term of risk and reward does correspond to a positive expectation $E(\varphi X)$ under each scenario $\varphi \in \mathcal{S}_{\alpha(x)}$. Now, define

$$\text{RAROC}(X) := \begin{cases} \frac{E(X)}{-e_{\alpha(x)}(X)}, & \text{if } E(X) > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

By convention we let $\text{RAROC}(X) = +\infty$ if the coherent $\alpha(x)$ -expectile risk measure is ≤ 0 . We have the following chain of equivalences:

$$\begin{aligned} \text{RAROC}(X) \geq x &\iff \frac{E(X)}{-e_{\alpha(x)}(X)} \geq x \\ &\iff E(X) \geq -x \min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\varphi X) \\ &\iff \frac{1}{1+x} E(X) + \frac{x}{1+x} \min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\varphi X) \geq 0 \\ &\iff \min_{\varphi \in \mathcal{S}_{\alpha(x)}} \left\{ \frac{1}{1+x} E(X) + \frac{x}{1+x} E(\varphi X) \right\} \geq 0 \\ &\iff \min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\tilde{\varphi} X) \geq 0. \end{aligned}$$

To see why the fifth equivalence holds let us use the Change of Variable formula to rewrite $\frac{1}{1+x} E(X) + \frac{x}{1+x} E(\varphi X)$ as

$$\int_{\Omega} \frac{1}{1+x} X dP + \int_{\Omega} \frac{x}{1+x} \varphi X dP,$$

where the Radon–Nikodým Theorem provides each scenario $\varphi = \frac{dQ}{dP}$ as the density of some probability measure Q equivalent to P . As byproduct the above becomes

$$\int_{\Omega} X d\left(\frac{1}{1+x} P + \frac{x}{1+x} Q\right) = \int_{\Omega} X d\tilde{Q} = \int_{\Omega} X \tilde{\varphi} dP,$$

with density $\tilde{\varphi} = \frac{d\tilde{Q}}{dP}$ belonging to $\mathcal{S}_{\alpha(x)}$. In fact any set of scenarios is a $\sigma(L^\infty, L^1)$ -closed convex cone, hence any of its element can be given as a Radon–Nikodým derivative corresponding to some convex combination $\frac{1}{1+x} P + \frac{x}{1+x} Q$. The second equivalence is obvious when $\min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\varphi X) < 0$. If $\min_{\varphi \in \mathcal{S}_{\alpha(x)}} E(\varphi X) \geq 0$ then by our convention $\text{RAROC}(X) = +\infty$ so the inequality on the left-hand side is again satisfied with $E(\varphi X) \geq 0$ thanks to $\varphi \in \mathcal{S}_{\alpha(x)}$, and this implies the inequality on the right-hand side is satisfied too. As a consequence, $(\mathcal{S}_{\alpha(x)})_{x \in \mathbb{R}_+}$ supports the representation of $\text{RAROC}(X) = a_l(X)$ as required by Corollary 1, provided that for every acceptability level $x \in \mathbb{R}_+$ we let $\mathcal{S}_{\alpha(x)} = \cap_{y>x} \mathcal{S}_{\alpha(y)}$, see Cherny and Madan (2009, Lemma 1).

Another expectile-based coherent risk measure recently proposed by Daouia et al. (2020, 2021) is the analogue of expected shortfall, namely the *average expectile coherent risk measure* defined as

$$-\frac{1}{\alpha} \int_0^\alpha e_t(X) dt, \quad \alpha \in (0, \frac{1}{2}]. \tag{18}$$

Since in the current article we treat positions $X \in L^1$ modelling P&L's and what matter for risk measurement is the left-tail of the X 's distribution, Eq. (18) resembles

the original one $\frac{1}{1-\alpha} \int_{\alpha}^1 e_t(X)dt$ with $\alpha \in [\frac{1}{2}, 1)$ where X is typically a random variable whose values are the negative of P&L's with *extreme* losses correspond to a level α close to one. Coherence of the average expectile coherent risk measure is proved in Daouia et al. (2020) and easily transfers to our resembled version (18). This enable us to propose a second example of expectile-based acceptability index of performance as follows. We claim that the ratio of expectation to (18) is:

$$\begin{aligned} \text{RAROC}(X) \geq x &\iff \frac{E(X)}{-\frac{1}{\alpha(x)} \int_0^{\alpha(x)} e_t(X)dt} \geq x \\ &\iff \frac{1}{\alpha(x)} \int_0^{\alpha(x)} E(X)dt \geq -x \frac{1}{\alpha(x)} \int_0^{\alpha(x)} e_t(X)dt \\ &\iff \frac{1}{\alpha(x)} \int_0^{\alpha(x)} [E(X) + x e_t(X)]dt \geq 0 \\ &\iff E(X) + x e_t(X) \geq 0 \\ &\iff \frac{1}{1+x} E(X) + \frac{x}{1+x} e_t(X) \geq 0. \end{aligned}$$

From the last equivalence and the dual representation of $e_t(X)$, the same reasoning used in building $\text{RAROC}(X) = \frac{E(X)}{-e_{\alpha(x)}(X)}$ as an expectile-based acceptability index now confirm our claim. Observe that $t \in \text{range}(\alpha(x))$ but we do not need integration by substitution. Indeed by Corollary 1 the second expectile-based coherent risk measure must be indexed by acceptability levels $x \in \mathbb{R}_+$.

5.1 Point estimators of expectile-based performance measures

To make statistical inference of our expectile-based RAROC, we consider the following procedure of non-parametric point estimation. First, from a random sample X_1, \dots, X_n let $X_{(1)}, \dots, X_{(n)}$ be the corresponding n th order statistics. The empirical counterpart of the probability level $\alpha \in (0, \frac{1}{2}]$ is

$$\tilde{\alpha}_i = \frac{i X_{(i)} - \sum_{k=1}^i X_{(k)}}{\sum_{k=1}^i |X_{(k)} - X_{(i)}|}, \quad i = 1, \dots, n. \tag{19}$$

Second, the empirical $\tilde{\alpha}_i$ -expectile is $\tilde{e}_{\alpha,n} = X_{(i)}$ if and only if $\alpha = \tilde{\alpha}_i$ for $i = 1, \dots, n$ with $\tilde{\alpha}_0 = 0$ and $\tilde{\alpha}_n = 1$. In fact, its expression is

$$\tilde{e}_{\alpha,n} = \frac{(1-\alpha) \sum_{k=1}^i X_{(k)} + \alpha \sum_{k=i+1}^n X_{(k)}}{(1-\alpha)i + \alpha(n-i)}, \tag{20}$$

and $\tilde{e}_{\alpha,n} \in [X_{(i)}, X_{(i+1)})$. Since the empirical expectile is nondecreasing in α and $\tilde{\alpha}_i \leq \tilde{\alpha}_{i+1}$ one get

$$\tilde{e}_{\alpha,n} \in [X_{(i)}, X_{(i+1)}) \iff \alpha \in [\tilde{\alpha}_i, \tilde{\alpha}_{i+1}),$$

see Holzmann and Klar (2016). To emphasize the dependence on rank i we rewrite $\tilde{e}_{\alpha,n(i)}$. As byproduct, to compute the nonparametric estimator

$$\widetilde{\text{RAROC}}_n := \frac{n^{-1} \sum_{i=1}^n X_i}{-\tilde{e}_{\alpha,n(i)}} \tag{21}$$

we suggest to:

- (1) Compute recursively each empirical level $\tilde{\alpha}_i$ for $i = 1, \dots, n$;
- (2) Identify the interval $[\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$;
- (3) Go back to the corresponding $[X_{(i)}, X_{(i+1)})$ and find the right $-\tilde{e}_{\alpha,n(i)}$.

The last step can be done by interpolation. When the population RAROC is defined in term of the average expectile coherent risk measure (18) we need to compute all $\tilde{e}_{\alpha,n(i)}$ in the left-tail of the empirical distribution. Therefore, we suggest to consider all the levels $\tilde{\alpha}_k \leq \alpha$ for $k = 1, \dots, i$ yet provided by step (1) above where i is related to the interval found in step (2). Then, one repeat step (3) to get $[X_{(k)}, X_{(k+1)})$ and then the corresponding empirical expectiles $\tilde{e}_{\alpha,n(k)}$. Eventually one compute the average $-i^{-1} \alpha \sum_{k=1}^{\lfloor i\alpha \rfloor} \tilde{e}_{\alpha,n(k)}$ which now replaces the denominator in $\widetilde{\text{RAROC}}_n$. Here $\lfloor \cdot \rfloor$ is the floor function.

By our main assumption $X \in L^1$ and the well-known stylized facts about the behavior of financial return series, is reasonably to consider the class of heavy-tailed distributions corresponding to the maximum domain of attraction of the Fréchet family. We recall that Extreme Value Theory is developed around limiting distribution functions of properly normalized maxima of i.i.d. X_1, \dots, X_n . Distributional properties of $\max_{1 \leq i \leq n} \{X_i\} = X_{(n)}$ are easily transferred to $\min_{1 \leq i \leq n} \{X_i\} = X_{(1)}$, the latter pertaining the current context where we work with P&L's (especially when they represent returns). Hence, we can refer to some asymptotic results of high expectiles without worry of our chosen sign convention about X . In particular, Bellini et al. (2014) showed that assuming X with heavy-tailed distribution in the Fréchet maximum domain of attraction (and then having a Paretian right-tail) it holds $e_\alpha(X) < q_\alpha(X)$, where $q_\alpha(X) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq \alpha\}$ is the α -quantile of the distribution function $F_X(x) = P(X \leq x)$, provided the tail index of the Fréchet distribution is greater than 2 and $\alpha \in (\bar{\alpha}, 1)$ with $\bar{\alpha} < 1$. Passing to the negative we have

$$\text{VaR}_\alpha(X) = -q_\alpha(X) < -e_\alpha(X), \quad \alpha \in (0, \bar{\alpha}) \tag{22}$$

and the expectile-based coherent risk measure is less conservative than value-at-risk. As a result, the performance ratio $\frac{E(X)}{\text{VaR}_\alpha(X)}$ is greater than our RAROC-type $\frac{E(X)}{-e_\alpha(X)}$, meaning more prudent performance evaluation. The same remains true when comparing the RAROC-types $\frac{E(X)}{\frac{1}{\alpha} \int_0^\alpha \text{VaR}_t(X) dt}$ and $\frac{E(X)}{-\frac{1}{\alpha} \int_0^\alpha e_t(X) dt}$, by simply taking the integrated version of (22). Recall that $\frac{1}{\alpha} \int_0^\alpha \text{VaR}_t(X) dt$ is the classical expected shortfall.

5.2 The omega ratio connection

As pointed out in Delbaen (2013) and Bellini and Di Bernardino (2017), the acceptance set for the α -expectile of a position $X \in L^1$ as a coherent risk measure is

$$\mathcal{A}_{-e_\alpha} = \left\{ X \mid \frac{\mathbb{E}(X^+)}{\mathbb{E}(X^-)} \geq \frac{1-\alpha}{\alpha} \right\}, \quad \text{for every } \alpha \in (0, \frac{1}{2}], \quad (23)$$

namely the acceptability of a position is given for a sufficiently high gain-loss ratio. This entails a *natural* link between risk and performance measures. Recall that the Omega ratio of $X \in L^1$ with respect to a benchmark $t \in \mathbb{R}$ is defined as

$$\Omega_X(t) := \frac{\mathbb{E}((X-t)^+)}{\mathbb{E}((X-t)^-)}, \quad (24)$$

see also Remillard (2013, Section 4.4.4) for a reference at a textbook level. Since the first order condition defining expectiles can be written as $\Omega_X(e_\alpha(X)) = \frac{1-\alpha}{\alpha}$, see Bellini et al. (2018) and the references therein, there is a one-to-one relation between expectiles and Omega ratios:

$$e_\alpha(X) = \Omega_X^{-1}\left(\frac{1-\alpha}{\alpha}\right), \quad \text{and} \quad \Omega_X(t) = \frac{1 - e_t^{-1}(X)}{e_t^{-1}(X)}.$$

The Omega ratio is a widespread performance measure and the gain-loss ratio is a special case for a benchmark $t = 0$. The advantage of its relation with expectiles is that besides the typical usage in performance measurement and especially in ranking investment funds, ordering all expectiles (in the sense of Bellini et al. (2018, Definition 6)) is equivalent to ordering all the Omega ratios (see Bellini et al. (2018, Theorem 8)) for all possible values of t . This can obviously help investors in making their risky decisions independently of the chosen benchmark in order to get the best reward.

Using the appropriate labeling $\alpha \equiv \alpha(x)$ as in Sect. 4, Corollary 1, the acceptance set (23) can be equivalently given using the nonnegativity condition concerning expectations and expectiles. Since $\alpha(x) = \frac{1}{1+x}$ if and only if $x = \frac{1-\alpha}{\alpha}$, we write for $\alpha(x) \in (0, \frac{1}{2}]$

$$X - e_{\alpha(x)}(X) = X + \min_{\varphi \in \mathcal{S}_{\alpha(x)}} \mathbb{E}(\varphi X).$$

Now, observe that $\alpha(x) \mapsto -e_{\alpha(x)}(X)$ is continuous and strictly increasing in $x \in \mathbb{R}_+$ provided $\alpha(x) \rightarrow 0$ and $x \rightarrow \infty$. Together with the increasing family of sets of scenarios $(\mathcal{S}_{\alpha(x)})_{x \in \mathbb{R}_+}$ the acceptability $X \in \mathcal{A}_{-e_{\alpha(x)}}$ associated to a suitable level of performance is given by

$$\frac{\mathbb{E}\left(\left(X + \min_{\varphi \in \mathcal{S}_{\alpha(x)}} \mathbb{E}(\varphi X)\right)^+\right)}{\mathbb{E}\left(\left(X + \min_{\varphi \in \mathcal{S}_{\alpha(x)}} \mathbb{E}(\varphi X)\right)^-\right)} \geq x. \quad (25)$$

Hence we have the following:

Corollary 2 *A mapping $a_t : L^1 \rightarrow [0, +\infty]$ is an expectile-based acceptability index if and only if there exists a family $(\mathcal{S}_{\alpha(x)})_{x \in \mathbb{R}_+}$ increasing in x such that the equivalent representation*

$$a_t(X) = \sup \left\{ x \in \mathbb{R}_+ \mid \text{condition (25) is fulfilled} \right\} \tag{26}$$

holds, where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Clearly, Eq. (25) can be rewritten as

$$\frac{\mathbb{E} \left(\left(X - e_{\frac{1}{1+x}}(X) \right)^+ \right)}{\mathbb{E} \left(\left(X - e_{\frac{1}{1+x}}(X) \right)^- \right)} \geq x \iff e_{\frac{1}{1+x}}(X) \geq 0. \tag{27}$$

To go back to the gain-loss ratio we find the right choice of scenarios, see Cherny and Madan (2009) where $\mathbb{E}(X)$ is now replaced by $\mathbb{E}(X^+)$. First, the special case $\Omega_X(0)$ gives the following acceptability index of performance

$$\text{GLR}(X) := \begin{cases} \frac{\mathbb{E}(X^+)}{\mathbb{E}(X^-)}, & \text{if } \mathbb{E}(X^+) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Each set of scenarios $\mathcal{S}_{\alpha(x)}$ is for every $x \in \mathbb{R}_+$ given by those φ satisfying conditions stated in Eq. (14) and in addition $\mathbb{E}(\varphi X) \geq 0$ for any $X \in L^1$ with $a_t(X) \geq x$. A modification of Cherny and Madan (2009, Lemma 1) yields each set of scenarios in the family $(\mathcal{S}_{\alpha(x)})_{x \in \mathbb{R}_+}$ as $\mathcal{S}_{\alpha(x)} = \cap_{y>x} \mathcal{S}_{\alpha(y)}$. Therefore, there exists a constant $\kappa \in \mathbb{R}_+$ such that any scenario can be represented as

$$\varphi = \kappa(\mathbf{I}_{\{X>0\}} + Y) \in \mathcal{S}_{\alpha(x)} \quad \text{for } 0 \leq Y \leq x.$$

The constant κ guarantees that $\mathbb{E}(\kappa(\mathbf{I}_{\{X>0\}} + Y)) = 1$, i.e. $\kappa = \frac{1}{\mathbb{P}(X>0) + \mathbb{E}(Y)}$. With all this in mind and calling for acceptability sets (1), it is easy to verify that $\min_{\varphi \in \mathcal{S}_{\alpha(x)}} \mathbb{E}(\varphi X) \geq 0$ is equivalent to $\min_{0 \leq Y \leq x} \mathbb{E}((\mathbf{I}_{\{X>0\}} + Y)X) \geq 0$ and the minimum is attained at $Y^* = x\mathbf{I}_{\{X \leq 0\}}$. Thus we have the following chain of equivalences

$$\begin{aligned} \min_{\varphi \in \mathcal{S}_{\alpha(x)}} \mathbb{E}(\varphi X) \geq 0 &\iff \mathbb{E}(\mathbf{I}_{\{X>0\}} + x\mathbf{I}_{\{X \leq 0\}}) \geq 0 \\ &\iff \mathbb{E}(X^+) \geq x\mathbb{E}(X^-) \\ &\iff \text{GLR}(X) \geq x. \end{aligned}$$

The last equivalence entails $a_t(X) = \text{GLR}(X)$, given the chosen set of scenarios $\mathcal{S}_{\alpha(x)}$. The optimum in the representation (26) is far from the too tolerant risk attitude given by $e_{\frac{1}{2}}(X) = \mathbb{E}(X)$, for $\alpha = \frac{1}{2}$ or $x = 1$. Anyway, by Corollary 2 and inequality (27)

the acceptability of the trade X at the level x corresponds to the greatest nonnegative α -expectile with more *risk aversion* produced by a greater $\mathcal{S}_{\alpha(x)}$, since by Delbaen (2012, Example 5) any expectile can be regarded as the von Neumann-Morgenstern utility function

$$u(x) = ax\mathbf{I}_{\{x \geq 0\}} + bx\mathbf{I}_{\{x < 0\}}$$

defined on \mathbb{R} with $0 < a \leq b$ for the concavity. Also, taking the supremum over $x \in \mathbb{R}_+$ of the left-hand expression in (27) implies that $\mathfrak{a}_t(X)$ is dominated by the Omega ratio $\Omega_X(e_{\frac{1}{1+x}}(X)) < +\infty$, or it is just equal to the Omega ratio for $t = e_{\frac{1}{1+x}}(X)$ and $x < +\infty$.

Remark 8 For $X, Y \in L^1$, let us assume $E(\psi(X)) \leq E(\psi(Y))$ meaning the usual stochastic order for all increasing functions ψ . This is equivalent to the *expectile order* $X \leq_e Y$ defined as $e_{\alpha}(X) \leq e_{\alpha}(Y)$ for all $\alpha \in (0, 1)$, see Bellini et al. (2018, Definition 6). Let x_1 be the optimum in the representation of $\mathfrak{a}_t(X)$ given by Corollary 2 and take $t_1 = e_{\frac{1}{1+x_1}}(X)$. Thus, $\Omega_X(t_1) \leq \Omega_Y(t_1)$ since the expectile order is equivalent to the pointwise ordering of Omega ratios, see Bellini et al. (2018, Theorem 8). Writing x_2 for the optimum in representation of $\mathfrak{a}_t(Y)$ given by Corollary 2, we have $t_2 = e_{\frac{1}{1+x_2}}(Y) < e_{\frac{1}{1+x_1}}(Y)$ since the expectile is strictly decreasing in $x \in \mathbb{R}_+$. Now we have two possibilities:

- (i) $0 \leq t_1 \leq t_2$ and consequently $\Omega_Y(t_1) \geq \Omega_Y(t_2)$ by the monotonicity of Omega ratios with respect to t , see Bellini et al. (2018, Theorem 2);
- (ii) $t_1 \geq t_2 \geq 0$ and $\Omega_X(t_1) \leq \Omega_Y(t_1) \leq \Omega_Y(t_2)$.

Only in the case (ii) the expectile order would imply $\mathfrak{a}_t(X) \leq \mathfrak{a}_t(Y)$, and taking the negative of the corresponding expectiles $-t_1 \leq -t_2 \leq 0$ this corresponds to the situation when a riskier trade Y in term of the expectile risk measure need more compensation for its reward, or equivalently a trade Y preferred to a trade X in the usual stochastic order have a higher performance.

6 Further properties of expectile-based indices of performance

Most of the risk measures proposed by academics and used in practice are law-invariant, that is the numerical value of the risk measure is only affected by the probability distribution of the underlying financial position. They are indeed special cases of statistical functionals $T(F)$ defined on a proper set of probability distributions \mathcal{M} over \mathbb{R} with cumulative distribution functions F . To fit in the current context, we assume F belongs to the class of all cumulative distribution functions \mathcal{D}^1 on \mathbb{R} with finite first moment $\int |x|dF(x) < +\infty$. Throughout this section we let $e_{\alpha}(F)$ for $X \sim F$. We give a direct representation of the proposed expectile-based acceptability index in term of the probability distribution of the financial position to assess. Following Delbaen (2012) and in particular Delbaen (2013, Theorem 7) we reformulate Corollary 1 by means of the Kusuoka representation of the α -expectile as a coherent,

law-invariant risk measure:

$$-e_\alpha(F) = \inf_{\nu \in \mathcal{M}_\alpha} \int_{[0,1]} -U_c(F) \nu(\mathrm{d}c), \tag{28}$$

where \mathcal{M}_α is a weak*-closed convex set of probability measures on $(0, 1]$ such that

$$\int_{(0,1]} \frac{1}{u} \nu(\mathrm{d}u) \leq \frac{1-\alpha}{\alpha} \nu(\{1\}), \quad \text{for } \alpha \in (0, \frac{1}{2}].$$

The integrand in the above representation is nothing but the negative of the expected shortfall,

$$U_c(F) = \frac{1}{c} \int_0^c q_c(F) \mathrm{d}c, \quad \text{for } c \in (0, 1],$$

where $q_c(F) = \inf \{m \in \mathbb{R} \mid F(m) \geq c\}$ is the c -quantile of F . Therefore representation (16) becomes:

$$a_l(F) = \sup \left\{ x \in \mathbb{R}_+ \mid \inf_{\nu \in \mathcal{M}_x} \int_{[0,1]} U_c(F) \nu(\mathrm{d}c) \geq 0 \right\}, \tag{29}$$

for an acceptable position $X \sim F$. Now $(\mathcal{M}_x)_{x \in \mathbb{R}_+}$ is an x -increasing family of sets of scenarios based on probability measures on $[0, 1]$, supporting the representation of each $-e_{\frac{1}{1+x}}(F)$, where as in Sect. 5.2 we let $\alpha = \frac{1}{1+x}$. The law-invariance of any such coherent expectile-based risk measure transfers to the acceptability index $a_l(F)$. There is also a Kusuoka representation using densities. By Delbaen (2013, Theorem 2), a direct application of Fubini’s theorem entails expectiles associated to $x \in \mathbb{R}_+$ given by

$$e_{\frac{1}{1+x}}(F) = \inf \left\{ \int_0^1 q_c(F) f'(1-c) \mathrm{d}c \mid f \in \mathbf{F}_x \right\}, \tag{30}$$

where \mathbf{F}_x is a convex set of convex (distortion) functions $f : [0, 1] \rightarrow [0, 1]$ such that $f(0) = 0, f(1) = 1$ and additionally $\frac{f'(1)}{f'(0)} \leq x$. Notice that the density $y \mapsto f'(y) = \int_{1-y}^1 \frac{1}{\alpha} \nu(\mathrm{d}\alpha)$ is nondecreasing, therefore $1 \leq \frac{f'(1)}{f'(0)} \leq x$. Taking the negative of expectiles given in (30) for a position $X \sim F$ yields acceptability at the level $x \in \mathbb{R}_+$ provided that the infimum in the above representation is nonnegative. In fact we have the following:

Corollary 3 *A mapping $a_l : L^1 \rightarrow [0, +\infty]$ is an expectile-based acceptability index if and only if there exists a family $(\mathbf{F}_x)_{x \in \mathbb{R}_+} \subset C[0, 1]$ increasing in x such that*

$$a_l(F) = \sup \left\{ x \in \mathbb{R}_+ \mid e_{\frac{1}{1+x}}(F) \geq 0 \right\} \tag{31}$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Proof It suffices to note that \mathbf{F}_x is a (convex) compact (in the topology of uniform convergence) set and the infimum in (30) is attained at $\bar{f} \in \mathbf{F}_x$ with $\int_0^1 q_c(F) \bar{f}'(1-c) dc \geq 0$. Then for $0 < x < y$ we have by construction $\mathbf{F}_x \subset \mathbf{F}_y$ which yields the desired result. \square

6.1 Elicitability of expectile-based performance indices

Elicitability of statistical functionals, and then of law-invariant risk measures, allows the assessment and comparison of competing point forecasts by means of an error measure called *scoring function* which is the analogue of the loss function in statistical decision theory. This implies for risk measures the possibility to perform backtesting through the *average score*

$$\bar{S} = \frac{1}{n} \sum_{i=1}^n S(T_i, X_i), \quad (32)$$

where T_i are point estimates of the statistical functional for given realizations X_i of a random sample with population $X \sim F$. We assume scoring functions $S : \mathbb{R} \times \mathbb{R} \rightarrow (0, +\infty)$ as defined in Bellini and Bignozzi (2015, Definition 3.1). In fact, whenever a statistical functional $T(F)$ is elicitable then, modulo some technical conditions, it can be represented as a minimizer

$$T(F) = \operatorname{argmin}_{z \in \mathbb{R}} \int S(z, x) dF(x), \quad \text{for every } F$$

where the cumulative distribution functions belong to a class \mathcal{D} such that for $\mu \in \mathcal{M}$ we have $F(x) = \mu(-\infty, x]$. The above integral is nothing but the expected score $E(S(z, X))$ based on the original probability measure \mathbb{P} , for example we can consider $\mu = \mathbb{P} \circ X^{-1}$. A statistical functional is not elicitable if its level sets $\{T = t\}$ are not convex for all $t \in \mathbb{R}$, see for example Delbaen et al. (2016). In our setting, the elicibility is relative to the class \mathcal{M} or equivalently \mathcal{D} . Since the negative of expectiles are law-invariant and elicitable by Bellini and Bignozzi (2015, Theorem 4.4(b)), our aim is to show that also $\mathfrak{a}_t(F)$ is elicitable⁹ too with respect to the class \mathcal{D}^1 .

Corollary 4 *The law-invariant acceptability index $\mathfrak{a}_t(F)$ with representation (31) is elicitable relative to the class \mathcal{D}^1 .*

Proof First, observe that $x \mapsto e_{\frac{1}{1+x}}(F)$ for fixed $F \in \mathcal{D}^1$ is continuous in $x \in \mathbb{R}_+$ and thus its range is the interval $[b, +\infty)$ for some $b \in \mathbb{R}_+$. Consider the correspondence $\Psi : [b, +\infty) \rightarrow 2^{\mathbb{R}_+ \times \mathbb{R}}$ defined by

$$\Psi(w) = [0, x] \times \left\{ e_{\frac{1}{1+s}}(F) \mid 0 \leq s \leq x \right\},$$

⁹ In what follows we do not need to assume this class corresponds to the set of all probability measures on \mathbb{R} with compact support.

for $w = e_{\frac{1}{1+x}}(F)$. We take its restriction to the range of Ψ , namely

$$\tilde{\Psi} = \Psi|_{\Psi([b, +\infty))}, \quad \text{where } \Psi([0, +\infty)) := \cup_{w \in [b, +\infty)} \Psi(w).$$

We further consider a second map $g : \Psi([b, +\infty)) \rightarrow [0, +\infty]$ defined by

$$g(\tilde{\Psi}(w)) = \sup \left\{ s \in [0, x] \mid e_{\frac{1}{1+s}}(F) \geq 0 \right\}.$$

Because the expectile is strictly decreasing in $x \in \mathbb{R}_+$, we have that

$$w \neq w' \iff e_{\frac{1}{1+x}}(F) \neq e_{\frac{1}{1+x'}}(F) \implies \tilde{\Psi}(w) \neq \tilde{\Psi}(w')$$

and the composition $h = g \circ \tilde{\Psi}$ is a 1-to-1 mapping between the range $[0, +\infty)$ of the expectile and the half line $[0, +\infty]$. By Gneiting (2011, Theorem 2.6) as a version of Osband’s *revelation principle* Osband (1985, p. 9) we additionally have that $h(e_{\frac{1}{1+x}}(F))$ is elicitable for every $x \in \mathbb{R}_+$ as the expectile is yet elicitable. Finally, taking

$$\lim_{x \rightarrow +\infty} h \left(e_{\frac{1}{1+x}}(F) \right) = \text{al}(F)$$

and applying again the above results yields the acceptability index as an elicitable statistical functional for $F \in \mathcal{D}^1$. □

When the performance index is just the Omega ratio with benchmark $t = e_{\frac{1}{1+x}}(F)$ (see Sect. 5.2, Eq. (27)) we can recover the scoring function strictly consistent with $\text{al}(F)$ in the case $E((X - t)^-) \neq 0$ by Gneiting (2011, Theorem 3.2(b),(c)) which is of the form

$$S(z, x) = s(x) (\phi(x) - \phi(z)) - \phi'(z) (r(x) - z s(x)) + \phi'(x) (r(x) - x s(x)),$$

where ϕ is a convex function with first derivative ϕ' and

$$r(x) = \max\{0, x - t\}, \quad s(x) = \max\{0, t - x\}$$

are such that $r, s : \mathbb{R} \rightarrow (0, +\infty)$. This only requires that $E((X - t)^+)$, $E((X - t)^-)$ and moreover $E((X - t)^+ \phi'(X - t))$, $E((X - t)^- \phi(X - t))$, $E(X (X - t)^- \phi'(X - t))$ are all finite which is the case since $X - t \in L^1$.

Corollary 4 above can be rephrased as follows: the expectile-based acceptability indices $\text{al}(X)$ for a traded position with terminal value $X \in L^1$ are elicitable. Consider for example the expectile-based performance ratio $\text{al}(X) = \frac{E(X)}{-e_\alpha(X)}$ introduced in Sect. 5. Clearly, it is a statistical functional¹⁰ and its estimation depends on past observations of asset prices and/or returns. As a *possible* point estimator we considered

¹⁰ Formally it would be correctly set as $\text{al}(F) = \frac{E(F)}{-e_\alpha(F)}$.

the nonparametric statistics $\hat{\theta}_n := \widetilde{\text{RAROC}}_n$ given in Eq. (21), in fact other estimators $\hat{\theta}_n$ such as¹¹ maximum likelihood should be considered and this opens the question of how to choose among them, given a sample X_1, \dots, X_n drawn from $X \sim F$. The discrepancy between $\text{at}(X)$ and $\hat{\theta}_n$ qualifies as estimation error and is intimately linked with the problem of *forecasting*, well known in financial econometrics. Now, theoretical elicibility of $\text{at}(X)$ means that we can assign a (strictly consistent) scoring function¹² such that

$$\frac{E(X)}{-e_\alpha(X)} = \underset{z \in \mathbb{R}}{\text{argmin}} E(S_{\text{at}}(z, X)), \quad (33)$$

where $S_{\text{at}}(z, X)$ is the score of our acceptability index. Applying the one-to-one division mapping $g(u, v) = \frac{u}{v}$ to the range of the bi-variate statistical functional¹³ $\mathbf{T}(X) := (E(X), -e_\alpha(X))$, by Gneiting (2011, Theorem 4) or Fissler and Ziegel (2016, Proposition 4.2(ii)) we get

$$S_{\text{at}}(z, X) = S(g^{-1}(z), X),$$

i.e. the scoring function of our performance ratio is that of $\mathbf{T}(X)$ before $g(u, v)$ acts. Then it can be written as the sum of the two separate scores of $E(X)$ and $-e_\alpha(X)$ respectively,

$$\begin{aligned} S_{\text{at}}(z, X) &:= \underbrace{S_1(u, X) + S_2(v, X)}_{=S(g^{-1}(z), X)} \\ &= \underbrace{(X - u)^2}_{\text{expectation}} + \underbrace{\alpha((X - v)^+)^2 + (1 - \alpha)((X - v)^-)^2}_{\text{expectile}}, \end{aligned} \quad (34)$$

because $\mathbf{T}(X)$ is a 2-elicitable functional, in the sense of Fissler and Ziegel (2016, Definition 2.1), with elicitable components.¹⁴ The sample counterpart of the expected score in (33), the so called *empirical score*, is

$$\bar{S}_n = \frac{1}{n} \sum_{t=1}^n S_{\text{at}}(z_t, x_t), \quad (35)$$

namely the estimated average score where the *ex-ante* forecast of the acceptability index¹⁵ is $z_t = \frac{u_t}{v_t}$, the ratio of the ex-ante forecast u_t of the mean to that of the expectile-based risk measure v_t , while x_t are ex-post realizations of a time series

¹¹ Observe that $\hat{\theta}_n$ is equivalent to the estimator $\text{at}(X_1, \dots, X_n)$.

¹² This is similar to the *risk function* of statistical decision theory, i.e. $L(\text{at}(X), \hat{\theta}_n)$, for example the L^p loss $L(x, y) = |x - y|^p$.

¹³ We have $u = E(X)$ and $v = -e_\alpha(X)$, for $-e_\alpha(X) \neq 0$.

¹⁴ It is well known that also the expectation is elicitable.

¹⁵ In defining the scoring function, we, without loss of generality, assume the observation domain for x_t and the action domain for z_t coincide and both are subsets of \mathbb{R} .

$(X_t)_{t \in \mathbb{N}}$ of periodic returns. More extensively, we have

$$S_{\text{at}}(z_t, x_t) = \begin{cases} (x_t - E_t)^2 + \alpha(x_t + e_{\alpha,t})^2, & \text{if } x_t > -e_{\alpha,t} \\ (x_t - E_t)^2 + (1 - \alpha)(x_t + e_{\alpha,t})^2, & \text{if } x_t \leq -e_{\alpha,t}, \end{cases} \quad (36)$$

by identifying $u_t = E_t$ with an estimate of the expected return and $v_t = -e_{\alpha,t}$ with an estimate of the corresponding expectile-based risk measure at a level $\alpha \in (0, \frac{1}{2}]$. Thus for competing estimation procedures we can assess the elicitation of $\text{at}(X)$ by ranking the corresponding empirical score (35), employing Eq. (36), and then choosing that with the lowest value. For example, we can refer to the forecast selection examples in Bellini and Di Bernardino (2017) where the competing estimation procedures are the standard Normal, the historical, and two alternative GARCH(1,1) models with Normal and with Student- t innovations, for a dataset of $n = 3818$ daily log-returns of the S&P500 Index; the author set $\alpha = 0.00145$ in order to get $-e_\alpha(X) = \text{VaR}_{0.01}(X)$. This approach can be easily adapted to the current framework by applying the aforementioned estimation procedures also to E_t other than $-e_{\alpha,t}$. Indeed, one can consider additionally estimators for $-e_\alpha(X)$ such as the weighted version of the empirical expectile in Daouia et al. (2021, equation (11)) which is in turn based on their definition of *expectHill* estimator of α , see Daouia et al. (2021, equation (8)). The entire procedure above can be also applied to the performance ratio $\frac{E(X)}{-\frac{1}{\alpha} \int_0^\alpha e_s(X) ds}$, for which further estimation procedures can be selected out of those employed in Bellini and Di Bernardino (2017), for example one can take the extrapolated estimator of the expectile-based risk measure $-\frac{1}{\alpha} \int_0^\alpha e_s(X) ds$ given in Daouia et al. (2021, equation (16)).

7 Conditional expectile-based performance index

In this section we develop a conditional representation of expectile-based acceptability indices. We write $L^p(\mathcal{F})$ for the space of all positions X with finite p th-moment which are measurable with respect to the initial *information set* \mathcal{F} . To account for new information and update the performance measurement we introduce a sub- σ -algebra $\mathcal{G} \subset \mathcal{F}$ and also consider positions in the smaller space $L^1(\mathcal{G})$. Recall that we in the rest of this section all inequalities, equalities, convergence concepts and statements about solution of equations applied to random variables are meant to hold P-a.s. if not stated otherwise. A *conditional* acceptability index is a mapping $\text{at}(\cdot | \mathcal{G}) : L^1(\mathcal{F}) \rightarrow L^1(\mathcal{G})$ satisfying the following conditions, where $X, Y \in L^1(\mathcal{F})$ and $W \in L^1(\mathcal{G})$ is a nonnegative random variable:

1. *Conditional quasi-concavity*, $\text{at}(\Lambda X + (1 - \Lambda)Y | \mathcal{G}) \geq W$ for any $\Lambda \in L^\infty(\mathcal{G})$ such that $\Lambda \geq 0$ and every W ;
2. *Monotonicity*, $X \leq Y \implies \text{at}(X | \mathcal{G}) \leq \text{at}(Y | \mathcal{G})$;
3. *Conditional scale invariance*, $\text{at}(\Lambda X | \mathcal{G}) = \text{at}(X | \mathcal{G})$ for any $\Lambda \in L^\infty(\mathcal{G})$ such that $\Lambda \geq 0$;
4. *Continuity property*, for any sequence $(X_n)_{n \in \mathbb{N}} \subset L^1(\mathcal{F})$ such that $X_n \uparrow X \in L^1(\mathcal{F})$, implies $\text{at}(X | \mathcal{G}) \geq W$ provided that $\text{at}(X_n | \mathcal{G}) \geq W$ for every $n \in \mathbb{N}$.

To establish a representation of conditional expectile-based acceptability indices we need a modified version of *conditional expectiles* originally defined by Bellini et al. (2018, Definition 2). For the sake of completeness we list below this definition together with the essential properties.

Definition 2 Given $X \in L^1(\mathcal{F})$ and $\alpha \in (0, 1)$, its *conditional α -expectile* is the unique solution $Z \in L^1(\mathcal{G})$ of the equation

$$\mathbb{E}(\alpha(X - Z)^+ - (1 - \alpha)(X - Z)^- | \mathcal{G}) = 0.$$

Existence and uniqueness of $Z = e_\alpha(X | \mathcal{G})$ is guaranteed by Bellini et al. (2018, Theorem 2.2). Conditional expectiles as coherent risk measures $\rho(X | \mathcal{G}) = -e_\alpha(X | \mathcal{G})$ for $\alpha \leq \frac{1}{2}$ satisfy:¹⁶

1. *Conditional cash invariance*, $\rho(X + H | \mathcal{G}) = \rho(X | \mathcal{G}) - H$, for any $H \in L^1(\mathcal{G})$;
2. *Monotonicity*, $X \leq Y$ implies $\rho(X | \mathcal{G}) \geq \rho(Y | \mathcal{G})$;
3. *Conditional positive homogeneity*, $\rho(\Lambda X | \mathcal{G}) = \Lambda \rho(X | \mathcal{G})$ for $\Lambda \in L^1(\mathcal{G})$;
4. *Subadditivity*, $\rho(X + Y | \mathcal{G}) \leq \rho(X | \mathcal{G}) + \rho(Y | \mathcal{G})$.

To provide the analogous of Lemma 2 in Sect. 3, we introduce a slightly different definition of conditional expectiles.

Definition 3 Given $X \in L^1(\mathcal{F})$ and a random level $A \in L^1(\mathcal{G})$ such that they form a random vector $(A, X) : \Omega \rightarrow \mathbb{R} \times (0, 1)$, the *randomized conditional A -expectile* is the solution $Z \in L^1(\mathcal{G})$ of the equation

$$\mathbb{E}(A(X - Z)^+ - (1 - A)(X - Z)^- | \mathcal{G}) = 0. \quad (37)$$

Essentially, the α -parameter of conditional expectile as given in Definition 2 is now a positive random variable in the conditional expected loss of Eq. (37). This can be interpreted as a risk measurement approach leading to capital requirement for a position X to assess that *minimizes* both risk overestimation and underestimation, conditioned on new information \mathcal{G} which also implies updating of the randomized probability level of occurrence A , see Moresco et al. (2019). We provide a short proof that the above version of conditional expectile exists as the unique solution of Eq. (37).

Lemma 4 Let $X \in L^1(\mathcal{F})$ and $A \in L^1(\mathcal{G})$ such that $A(\omega) \in (0, 1)$ for every $\omega \in \Omega$. There exists a unique solution $Z^* \in L^1(\mathcal{G})$ of Eq. (37).

Proof Let $F(a, x, \omega)$ be a version of $\mathbb{P}(A \leq a, X \leq x | \mathcal{G})$. Rewrite the conditional expectation in (37) as

$$\int [a(x - z)^+ (1 - a)(x - z)^-] dF(a, x, \omega) = 0,$$

¹⁶ This condition as usual ensures subadditivity, as after a change in sign conditional expectiles belongs to the family of conditional shortfall with concave loss function, see Weber (2006), and are conditional risk measures as introduced by Detlefsen and Scandolo (2005).

which has a unique solution given by $Z^*(\omega)$. The \mathcal{G} -measurability of Z^* is showed as in the proof of Bellini et al. (2018, Theorem 2.2) but using the regular conditional distribution function $F(a, x, \omega)$. To show that Z^* has finite first moment as a \mathcal{G} -measurable random variable, we proceed as in the proof of Bellini et al. (2018, Theorem 2.2) simply by considering

$$E(Z^* | \mathcal{G}) = Z^* \leq E(X | \mathcal{G}),$$

with $E\left(\mathbf{I}_{\{A \leq \frac{1}{2}\}} A(X - Z^*) \mid \mathcal{G}\right) = E\left(\mathbf{I}_{\{A \leq \frac{1}{2}\}} (1 - A)(X - Z^*) \mid \mathcal{G}\right)$, and analogously on $\{A \geq \frac{1}{2}\}$. □

Now, our randomized version of the conditional expectile is

$$Z^* = e_A(X | \mathcal{G}), \quad \text{with } A(\omega) \in (0, 1) \text{ for all } \omega \in \Omega, \tag{38}$$

satisfying properties 1 to 4 of the original conditional expectile in Definition 2. This can be obtained by observing that

$$e_A(X | \mathcal{G}) = \text{ess inf} \left\{ Z \in L^1(\mathcal{G}) \mid \text{expectation in (37)} \leq 0 \right\}, \tag{39}$$

which can be easily deduced by equation (9) in Bellini et al. (2018, Theorem 2.2.) and its proof. The aforementioned properties are then transferred from those of the conditional shortfall risk measures introduced by Weber (2006), if in addition one considers $\{A \leq \frac{1}{2}\}$ for the subadditivity. Nevertheless, we need two further properties of $e_A(X | \mathcal{G})$:

- 5. $A_1 \leq A_2 \implies e_{A_1}(X | \mathcal{G}) \leq e_{A_2}(X | \mathcal{G})$, with the reverse inequality holding by a change in sign;
- 6. *Continuity from below*, $X_n \uparrow X$ for a sequence $(X_n)_{n \in \mathbb{N}} \subset L^1(\mathcal{F})$, implies $e_A(X_n | \mathcal{G}) \uparrow e_A(X | \mathcal{G})$, which becomes continuity from above taking the negatives.

Property 5 is an easy generalization of the analogous monotonicity property of conditional shortfall risk measures, with respect to the parameter α . Property 6 can be easily deduced by Bellini et al. (2018, Theorem 2.3(d)). We call

$$\rho_W(X | \mathcal{G}) := -e_{\frac{1}{1+W}}(X | \mathcal{G}), \quad \text{on } \{A \leq \frac{1}{2}\}, \tag{40}$$

the *randomized conditional expectile risk measure* of the position X , where for the sake of consistency with the unconditional representation in Proposition 1 we have

$$W(\omega) = \frac{1 - A(\omega)}{A(\omega)} \geq 0, \quad \text{for every } \omega \in \Omega. \tag{41}$$

Note that

$$\left\{ \omega \in \Omega \mid W(\omega) \text{ is nondecreasing} \right\} = \left\{ \omega \in \Omega \mid A(\omega) \text{ is nonincreasing} \right\}. \tag{42}$$

Lemma 5 Let $(\rho_W(X|\mathcal{G}))_{W \in L^1(\mathcal{G})}$ be an increasing family of randomized conditional expectile risk measure for $X \in L^1$. Then, the mapping $\mathfrak{a}_t(\cdot|\mathcal{G}) : L^1(\mathcal{F}) \rightarrow L^1(\mathcal{G})$ defined by

$$\begin{aligned} \mathfrak{a}_t(X|\mathcal{G}) &:= \text{ess sup} \{W \in B \mid \rho_W(X|\mathcal{G}) \leq 0\} \\ \text{where } B &= \{A \in L^1(\mathcal{G}) \mid A \leq \tfrac{1}{2}\} \end{aligned} \quad (43)$$

is a conditional acceptability index of performance (we assume $\text{ess sup} \emptyset = 0$).

Although the obvious similarity between this lemma and Lemma 2, we present the proof to account for the randomization effect. Recall the all equalities and inequalities are valid P-a.s.

Proof Let $B = \{A \in L^1(\mathcal{G}) \mid A \leq \frac{1}{2}\}$ and recall Eqs. (40), (41). By the monotonicity of randomized conditional expectile risk measures we have for every $X, Y \in L^1(\mathcal{F})$ with $X \geq Y$ that $\rho_W(X|\mathcal{G}) \leq \rho_W(Y|\mathcal{G})$. Thus

$$W_0 \in \left\{ W \in L^1(\mathcal{G}) \mid \rho_W(X|\mathcal{G}) \leq 0 \right\}$$

implies $\rho_{W_0}(X|\mathcal{G}) \leq 0$ which together with monotonicity also entails $\rho_{W_0}(X|\mathcal{G}) \leq \rho_{W_0}(Y|\mathcal{G}) \leq 0$. Using (42) we consequently have

$$\{W \in B \mid \rho_W(X|\mathcal{G}) \leq 0\} \supset \{W \in B \mid \rho_W(Y|\mathcal{G}) \leq 0\},$$

then taking the essential supremum of both sets yields $\mathfrak{a}_t(X|\mathcal{G}) \geq \mathfrak{a}_t(Y|\mathcal{G})$. For the quasi-concavity of $\mathfrak{a}_t(\cdot|\mathcal{G})$ we consider two random trades $X, Y \in L^1$ such that $\mathfrak{a}_t(X|\mathcal{G})$ and $\mathfrak{a}_t(Y|\mathcal{G})$ are both $\geq W_0$ for a given $W_0 \geq 0$ satisfying (41) and (42). For all $W \in B$ such that $W \leq W_0$, combining monotonicity of the randomized conditional expectile risk measures with definition (43) we have $\rho_W(X|\mathcal{G}) \leq \rho_{W_0}(X|\mathcal{G}) \leq 0$ and $\rho_W(Y|\mathcal{G}) \leq \rho_{W_0}(Y|\mathcal{G}) \leq 0$. An appeal to the conditional positive homogeneity of $\rho_W(\cdot|\mathcal{G})$ now entails

$$\rho_W(\Lambda X|\mathcal{G}) = \Lambda \rho_W(X|\mathcal{G}) \leq 0$$

and

$$\rho_W((1-\Lambda)Y|\mathcal{G}) = (1-\Lambda)\rho_W(Y|\mathcal{G}) \leq 0,$$

for every random variable $0 \leq \Lambda \leq 1$. Moreover, by subadditivity of $\rho_W(\cdot|\mathcal{G})$ we have

$$\rho_W(\Lambda X + (1-\Lambda)Y|\mathcal{G}) \leq 0$$

for every $W < W_0$ and it follows $\text{ess sup} \{W \in B \mid \rho_W(\Lambda X + (1-\Lambda)Y|\mathcal{G}) \leq 0\} \geq W_0$. As a by-product, definition (43) yields $\mathfrak{a}_t(\Lambda X + (1-\Lambda)Y|\mathcal{G}) \geq W_0$ and quasi-concavity is proved. Conditional scale invariance of $\mathfrak{a}_t(\cdot|\mathcal{G})$ follows on B

by conditional positive homogeneity of $\rho_W(\cdot | \mathcal{G})$. To show the continuity, pick a sequence $(X_n)_{n \in \mathbb{N}} \subset L^1(\mathcal{F})$ such that $X_n \uparrow X$ and assume $\text{ai}(X_n | \mathcal{G}) \geq W$ for every $n \in \mathbb{N}$ and $W \in B$. Since randomized conditional expectile risk measures are continuous from above, $\rho_W(X_n | \mathcal{G}) \downarrow \rho_W(X | \mathcal{G})$ so that $\rho_W(X | \mathcal{G}) \leq 0$, provided that $\rho_W(X_n | \mathcal{G}) \leq 0$ and this combined with monotonicity of the conditional acceptability index eventually implies $\text{ai}(X | \mathcal{G}) \geq W$ as desired. \square

As we have seen in the unconditional case, it is also possible to recover randomized conditional expectile risk measures by conditional acceptability indices. This is guaranteed by the following.

Lemma 6 *Let $B = \{A \in L^1(\mathcal{G}) \mid A \leq \frac{1}{2}\}$ and define*

$$\rho_W(X | \mathcal{G}) := \text{ess inf} \left\{ M \in L^1(\mathcal{G}) \mid \text{ai}(X + M | \mathcal{G}) \geq W \right\},$$

for every $W \in B$, (44)

where A and W are related by Eq. (41) and $\text{ai}(\cdot | \mathcal{G})$ is a conditional acceptability index. Then, $\rho_W(X | \mathcal{G})$ satisfies properties 1–4 of Definition 2 together with subsequent properties 5,6.

Proof Assume B as above. First, we check that the mapping $W \mapsto \rho_W(\cdot | \mathcal{G})$ defined in (44) is increasing [recall Eq. (40)]. It suffices considering $0 \leq W_1 \leq W_2$ and taking the essential infimum of both sets

$$\left\{ M \in L^1(\mathcal{G}) \mid \text{ai}(X + M | \mathcal{G}) \geq W_2 \right\} \subset \left\{ M \in L^1(\mathcal{G}) \mid \text{ai}(X + M | \mathcal{G}) \geq W_1 \right\}.$$

To show conditional cash invariance, monotonicity, conditional positive homogeneity and subadditivity we proceed as in the Proof of Lemma 2, provided that all scalars x, m, λ, c are replaced by $W, M, \Lambda, C \in L^1(\mathcal{G})$, where in addition Λ is a nonnegative random variable, and the infimum is replaced by the essential infimum. To check the continuity from above of randomized conditional expectile risk measures, it suffices to proceed as in the last part of the Proof of Lemma 2 by replacing the scalar ϵ with the random variable $M \in L^1(\mathcal{G})$ and considering the continuity of $\text{ai}(\cdot | \mathcal{G})$. \square

Eventually, by Lemmas 5 and 6 together with Bellini et al. (2018, Theorem 2.4) and its proof we have our proposed conditional version of the unconditional set of scenarios (14) in Sect. 4:

Corollary 5 *Let B be defined as in Lemma 5. The conditional expectile-based acceptability index can be represented as*

$$\text{ai}(X | \mathcal{G}) = \text{ess sup} \left\{ W \in B \mid \text{ess min}_{\varphi \in S_W} E(\varphi X | \mathcal{G}) \geq 0 \right\}, \tag{45}$$

where

$$S_W = \left\{ \varphi \in L^\infty(\mathcal{F}) \mid \varphi > 0, E(\varphi | \mathcal{G}) = 1, \frac{\text{ess sup}(\varphi | \mathcal{G})}{\text{ess inf}(\varphi | \mathcal{G})} \leq W \right\}$$

is such that $0 \leq W_1 \leq W_2$ implies $\mathcal{S}_{W_1} \subset \mathcal{S}_{W_2}$.

The last condition in the definition of \mathcal{S}_W makes use of the conditional supremum $\text{ess sup}(\varphi | \mathcal{G})$ of each scenario, see Bellini et al. (2018) and the references therein. It is worth noting that choosing the trivial sigma-algebra $\mathcal{G} = \{\emptyset, \Omega\}$, all the above representation results yield the unconditional representation of acceptability indices based on expectile risk measures up to replacing the continuity properties of $\text{at}(\cdot | \mathcal{G})$ and $\rho_W(\cdot | \mathcal{G})$ by the original continuity properties in Sect. 3.

8 Conclusions

Given the recent increasing attention to the use of expectiles as coherent risk measures, we investigate their link to performance measurement beyond the classical gain-loss and Omega ratios. Hinged on the concept of acceptability indices of performance, we are able to drive the dual representation of expectile-based coherent risk measures to the construction of two new expectile-based performance ratios. Given the elicibility of the proposed expectile-based performance measures proved in Sect. 6.1, we suggest a proper scoring function that can easily implemented in practical forecast selection problems. Moreover, we generalize the notion of conditional expectiles and give one more representation of such performance measures when new information arrives and the probability level implicit in the piecewise loss function giving expectiles is updated accordingly.

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Declarations

Conflicts of interest The corresponding author states that there is no conflict of interest.

Ethical statement The manuscript is not submitted to other journals. The manuscript has not been published elsewhere. Based on the well-developed framework of Acceptability Indices of Performance, the manuscript does provides a novel application using expectiles as coherent risk measures. Thus, it elaborates on this and provides additional new characterizations of expectile-based performance measures. The contribution presented is the author's own.

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