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Second-order impulsive differential systems of mixed type: oscillation theorems

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Abstract

In this paper, we obtain necessary and sufficient conditions for the oscillation of solutions to a second-order neutral differential equation with impulses. Two examples are provided to show the effectiveness and feasibility of the main results. Our main tool is Lebesgue's dominated convergence theorem.

MSC: 34C10; 34K11

Keywords: Oscillation; Delay; Neutral; Lebesgue's dominated convergence theorem

1 Introduction

Nowadays impulsive differential equations are attracting a lot of attention. They appear in the study of several real world problems (see, for instance, [1, 2, 15]). In general, it is well known that several natural phenomena are driven by differential equations, but the description of some real world problems subjected to sudden changes in their stated became very interesting from the mathematical point of view because they should be described considering systems of differential equations with impulses. Examples of the aforementioned phenomena are related to mechanical systems, biological systems, population dynamics, pharmacokinetics, theoretical physics, biotechnology processes, chemistry, engineering, and control theory.

We also stress that the modeling of these phenomena is suitably formulated by evolutive partial differential equations; moreover, moment problem approaches appear also as a natural instrument in control theory of neutral type systems; see [16, 20, 34] and [13], respectively.

The literature related to impulsive differential equations is very wide. Here we mention some recent developments in this field.

In [28], Shen and Wang considered impulsive differential equations of the following form:

$$\begin{cases} u'(\iota) + r(\iota)u(\iota - \nu) = 0, & \iota \neq \phi_k, \iota \geq \iota_0, \\ u(\phi_k^+) - u(\phi_k^-) = I_k(u(\phi_k)), & k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $r \in C(\mathbb{R},\mathbb{R})$ and $I_k \in C(\mathbb{R},\mathbb{R})$ for $k \in \mathbb{N}$, and obtained sufficient conditions that ensure the oscillation and asymptotic behavior of the solutions of problem (1.1).

In [12], Graef et al. considered the problem

$$\begin{cases} (u(\iota) - q(\iota)u(\iota - \zeta))' + r(\iota)|u(\iota - \nu)|^{\lambda} \operatorname{sgn} u(\iota - \nu) = 0, & \iota \geq \iota_0, \\ u(\phi_k^+) = b_k u(\phi_k), & k \in \mathbb{N} \end{cases} \tag{1.2}$$

assuming that $q(\iota) \in PC([\iota_0,\infty),\mathbb{R}_+)$ (that is, $q(\iota)$ is piecewise continuous in $[\iota_0,\infty)$), obtained sufficient conditions for the oscillation of the solutions of problem (1.2).

In [27], Shen and Zou obtained oscillation criteria for first-order impulsive neutral delay differential equations of the form

$$\begin{cases} (u(\iota) - q(\iota)u(\iota - \zeta))' + r(\iota)u(\iota - \nu_1) - v(\iota)u(\iota - \nu_2) = 0, & \nu_1 \geq \nu_2 > 0, \\ u(\phi_k^+) = I_k(u(\phi_k)), & k \in \mathbb{N} \end{cases} \tag{1.3}$$

obtaining sufficient conditions that ensure the oscillation of the solutions of (1.3) under the assumptions that $q(\iota) \in PC([\iota_0,\infty),\mathbb{R}_+)$ and $b_k \leq \frac{I_k(u)}{u} \leq 1$.

Karpuz et al. in [14] extended the results contained in [27] by taking the nonhomogeneous counterpart of system (1.3) with variable delays.

Oscillation and nonoscillation properties for a class of second-order neutral impulsive differential equations with constant coefficients and constant delays were studied by Tripathy and Santra in [30], where the authors considered the problem

$$\begin{cases} (u(\iota) - qu(\iota - \zeta))'' + ru(\iota - \nu) = 0, & \iota \neq \phi_k, k \in \mathbb{N}, \\ \Delta(u(\phi_k) - qu(\phi_k - \zeta))' + \tilde{r}u(\phi_k - \nu) = 0, & k \in \mathbb{N}. \end{cases} \tag{1.4}$$

Other necessary and sufficient conditions for the oscillation of a class of second-order neutral impulsive systems were established in [32], where Tripathy and Santra studied systems of the form

$$\begin{cases} (p(\iota)(u(\iota) + q(\iota)u(\iota - \zeta))')' + r(\iota)g(u(\iota - \nu)), & \iota \neq \phi_k, k \in \mathbb{N}, \\ \Delta(p(\phi_k)(u(\phi_k) + q(\phi_k)u(\phi_k - \zeta))') + r(\phi_k)g(u(\phi_k - \nu)) = 0, & k \in \mathbb{N}. \end{cases} \tag{1.5}$$

In [32], in particular, the authors are interested in oscillating systems that, after a perturbation by instantaneous change of state, remain oscillating.

In [26], Santra and Tripathy investigated the oscillatory behavior of the solutions for first-order impulsive neutral delay differential equations of the form

$$\begin{cases} (u(\iota) - q(\iota)u(\iota - \zeta))' + r(\iota)g(u(\iota - \nu)) = 0, & \iota \neq \phi_k, \iota \geq \iota_0, \\ u(\phi_k^+) = I_k(u(\phi_k)), & k \in \mathbb{N}, \\ u(\phi_k^+ - \tau) = I_k(u(\phi_k - \tau)), & k \in \mathbb{N} \end{cases} \tag{1.6}$$

for different values of the neutral coefficient q.

We also mention the paper [24] in which Santra and Dix, using Lebesgue's dominated convergence theorem, obtained necessary and sufficient conditions for the oscillation of the solutions of the following second-order neutral differential equation with impulses:

$$\begin{cases} (p(\iota)(w'(\iota))^{\gamma})' + \sum_{j=1}^{m} r_j(\iota)g_j(u(\nu_j(\iota))) = 0, & \iota \geq \iota_0, \iota \neq \phi_k, k \in \mathbb{N}, \\ \Delta(p(\phi_k)(w'(\phi_k))^{\gamma}) + \sum_{j=1}^{m} \widetilde{r}_j(\phi_k)g_j(u(\nu_j(\phi_k))) = 0, \end{cases} \tag{1.7}$$

where

$$w(\iota) = u(\iota) + q(\iota)u\bigl(\zeta(\iota)\bigr), \qquad \Delta u(a) = \lim_{s\to a^+} u(s) - \lim_{s\to a^-} u(s).$$

In line with the contents of [24], Tripathy and Santra in [31] examined oscillation and nonoscillation properties for the solutions of the following class of forced impulsive nonlinear neutral differential systems:

$$\begin{cases} (p(\iota)(u(\iota) + q(\iota)u(\iota - \zeta))')' + r(\iota)g(u(\iota - \nu)) = f(\iota), & \iota \neq \phi_k, k \in \mathbb{N}, \\ \Delta(p(\phi_k)(u(\phi_k) + q(\phi_k)u(\phi_k - \zeta))') + \tilde{r}(\phi_k)g(u(\phi_k - \nu)) = \tilde{f}(\phi_k), & k \in \mathbb{N} \end{cases} \tag{1.8}$$

for different values of $q(\iota)$ and obtained sufficient conditions for the existence of positive bounded solutions of system (1.8).

Finally, we mention the recent work [33] in which Tripathy and Santra obtained some characterizations for the oscillation of solutions of the following second-order neutral impulsive differential system:

$$\begin{cases} (p(\iota)(w'(\iota))^{\gamma})' + \sum_{j=1}^{m} r_j(\iota)x^{\alpha_j}(\nu_j(\iota)) = 0, & \iota \geq \iota_0, t \neq \phi_k, \\ \Delta(p(\phi_k)(w'(\phi_k))^{\gamma}) + \sum_{j=1}^{m} h_j(\phi_k)x^{\alpha_j}(\nu_j(\phi_k)) = 0, & k \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $w(\iota) = u(\iota) + q(\iota)u(\zeta(\iota))$ and $-1 < q(\iota) \leq 0$.

For further details on neutral impulsive differential equations and for recent results related to the oscillation theory for ordinary differential equations, we refer the reader to the papers [3–6, 8, 9, 11, 21–23, 25, 29, 35] and to the references therein. In particular, the study of oscillation of half-linear/Emden–Fowler (neutral) differential equations with deviating arguments (delayed or advanced arguments or mixed arguments) has numerous applications in physics and engineering (e.g., half-linear/Emden–Fowler differential equations arise in a variety of real world problems such as in the study of p-Laplace equations, non-Newtonian fluid theory, the turbulent flow of a polytropic gas in a porous medium, and so forth); see, e.g., the papers [7, 10, 16–20] for more details.

Motivated by the aforementioned findings, in this paper we prove necessary and sufficient conditions for the oscillation of solutions to a second-order nonlinear impulsive differential system of the form

$$\bigl(p(\iota)\bigl(w'(\iota)\bigr)^{\alpha}\bigr)' + r(\iota)g\bigl(u\bigl(\nu(\iota)\bigr)\bigr) = 0, \quad \iota \geq \iota_0, \iota \neq \phi_k, k \in \mathbb{N}, \tag{1.10}$$

$$\Delta\bigl(p(\phi_k)\bigl(w'(\phi_k)\bigr)^{\alpha}\bigr) + \tilde{r}(\phi_k)g\bigl(u\bigl(\nu(\phi_k)\bigr)\bigr) = 0, \tag{1.11}$$

where

$$w(\iota) = u(\iota) + q(\iota)u\bigl(\zeta(\iota)\bigr), \qquad \Delta u(a) = \lim_{s\to a^+} u(s) - \lim_{s\to a^-} u(s),$$

the functions g, r, $\tilde{r}$, p, q, ν, ζ are continuous and satisfy the conditions stated below; the sequence $\{\phi_k\}$ satisfies $0 < \phi_1 < \phi_2 < \cdots < \phi_k <\to \infty$ as $k \to \infty$; and α is the quotient of two positive odd integers.

In this paper we use the following assumptions:

(a) $\nu \in C([0,\infty),\mathbb{R})$, $\zeta \in C^2([0,\infty),\mathbb{R})$, $\nu(\iota) < \iota$, $\zeta(\iota) < \iota$, $\lim_{\iota\to\infty} \nu(\iota) = \infty$, $\lim_{\iota\to\infty} \zeta(\iota) = \infty$.

(b) $\nu \in C([0,\infty),\mathbb{R})$, $\zeta \in C^2([0,\infty),\mathbb{R})$, $\nu(\iota) > \iota$, $\zeta(\iota) < \iota$, $\lim_{\iota\to\infty} \zeta(\iota) = \infty$.

(c) $p \in C^1([0,\infty),\mathbb{R})$, $r, \tilde{r} \in C([0,\infty),\mathbb{R})$; $0 < p(\iota)$, $0 \le r(\iota)$, $0 \le \tilde{r}(\iota)$ for all $\iota \ge 0$; $\sum r(\iota)$ is not identically zero in any interval $[b,\infty)$.

(d) $q \in C^2([0,\infty),\mathbb{R}_+)$ with $0 \le q(\iota) \le a < 1$;

(e) $g \in C(\mathbb{R},\mathbb{R})$ is nondecreasing and $g(\iota)\iota > 0$ for $\iota \neq 0$.

(f) $\lim_{\iota\to\infty} P(\iota) = \infty$, where $P(\iota) = \int_0^\iota p^{-1/\alpha}(s)\,\mathrm{d}s$.

2 Preliminary results

For the sake of simplicity, we set

$$R_1(\iota) = r(\iota)g\big((1-a)w\big(\nu(\iota)\big)\big);$$
$$R_{(1,k)} = \tilde{r}(\phi_k)g\big((1-a)w\big(\nu(\phi_k)\big)\big).$$

Lemma 2.1 *Suppose that* (a)–(f) *hold for* $\iota \ge \iota_0$, *and let* u *be an eventually positive solution of* (1.10)–(1.11). *Then* w *satisfies*

$$0 < w(\iota), \qquad w'(\iota) > 0, \quad \textit{and} \quad \big(p(\iota)\big(w'(\iota)\big)^\alpha\big)' \le 0 \quad \textit{for } \iota \ge \iota_1. \tag{2.1}$$

Proof Let u be an eventually positive solution. Then $w(\iota) > 0$ and there exists $\iota_0 \ge 0$ such that $u(\iota) > 0$, $u(\nu(\iota)) > 0$, $u(\zeta(\iota)) > 0$ for all $\iota \ge \iota_0$. Then (1.10)–(1.11) gives that

$$\begin{aligned} &\big(p(\iota)\big(w'(\iota)\big)^\alpha\big)' = -r(\iota)g\big(u\big(\nu(\iota)\big)\big) \le 0 \quad \text{for } \iota \neq \phi_k, \\ &\Delta\big(p(\phi_k)\big(w'(\phi_k)\big)^\alpha\big) = -\tilde{r}(\phi_k)g\big(u\big(\nu(\phi_k)\big)\big) \le 0 \quad \text{for } k \in \mathbb{N}, \end{aligned} \tag{2.2}$$

which shows that $p(\iota)(w'(\iota))^\alpha$ is nonincreasing for $\iota \ge \iota_0$, including jumps of discontinuity. Next we claim that for $w > 0$, $p(\iota)(w'(\iota))^\alpha$ is positive for $\iota \ge \iota_1 > \iota_0$. If not, letting $p(\iota)(w'(\iota))^\alpha \le 0$ for $\iota \ge \iota_1$, we can choose $c > 0$ such that

$$p(\iota)\big(w'(\iota)\big)^\alpha \le -c,$$

that is,

$$w'(\iota) \le (-c)^{1/\alpha} p^{-1/\alpha}(\iota).$$

Integrating both sides from ι_1 to ι, we get

$$w(\iota) - w(\iota_1) - \sum_{k=1}^{\infty} w'(\phi_k) \le (-c)^{1/\alpha}\big(P(\iota) - P(\iota_1)\big).$$

Taking limit on both sides as $\iota \to \infty$, we have $\lim_{\iota\to\infty} w(\iota) \le -\infty$, which leads to a contradiction to $w(\iota) > 0$. Hence, $p(\iota)(w'(\iota))^{\alpha} > 0$ for $\iota \ge \iota_1$, that is, $w'(\iota) > 0$ for $\iota \ge \iota_1$. This completes the proof. □

Lemma 2.2 *Suppose that* (a)–(f) *hold for* $\iota \ge \iota_0$, *and let* u *be an eventually positive solution of* (1.10)–(1.11). *Then* w *satisfies*

$$u(\iota) \ge (1-a)w(\iota) \quad \textit{for } \iota \ge \iota_1. \tag{2.3}$$

Proof Assume that u is an eventually positive solution of (1.10)–(1.11). Then $w(\iota) > 0$ and there exists $\iota \ge \iota_1 > \iota_0$ such that

$$\begin{aligned} u(\iota) &= w(\iota) - q(\iota)u\bigl(\zeta(\iota)\bigr) \\ &\ge w(\iota) - q(\iota)w\bigl(\zeta(\iota)\bigr) \\ &\ge w(\iota) - q(\iota)w(\iota) \\ &= \bigl(1-q(\iota)\bigr)w(\iota) \\ &\ge (1-a)w(\iota). \end{aligned}$$

Hence w satisfies (2.3) for $\iota \ge \iota_1$. □

3 Main results

In Theorem 3.1 we use a constant β, the quotient of two odd positive integers with $\beta > \alpha$, for which

$$\frac{g(\iota)}{\iota^{\beta}} \text{ is nondecreasing for } 0 < \iota. \tag{3.1}$$

The existence of such a constant can be established by taking $g(\iota) = |\iota|^{\delta}\operatorname{sgn}(\iota)$ with $\beta < \delta$.

Theorem 3.1 *Let* (b)–(f) *and* (3.1) *hold for* $\iota \ge \iota_0$. *Then every solution of* (1.10)–(1.11) *is oscillatory if and only if*

$$\int_0^{\infty} p^{-1/\alpha}(s)\biggl[\int_s^{\infty} r(\psi)\,\mathrm{d}\psi + \sum_{\phi_k \ge s} \tilde{r}(\phi_k)\biggr]^{1/\alpha}\,\mathrm{d}s = \infty. \tag{3.2}$$

Proof Let u be an eventually positive solution of (1.10)–(1.11). Then $w(\iota) > 0$ and there exists $\iota_0 \ge 0$ such that $u(\iota) > 0$, $u(\nu(\iota)) > 0$, $u(\zeta(\iota)) > 0$ for all $\iota \ge \iota_0$. Thus, Lemmas 2.1 and 2.2 hold for $\iota \ge \iota_1$. By Lemma 2.1, there exists $\iota_2 > \iota_1$ such that $w'(\iota) > 0$ for all $\iota \ge \iota_2$. Then there exist $\iota_3 > \iota_2$ and $c > 0$ such that $w(\iota) \ge c$ for all $\iota \ge \iota_3$. Next, using Lemma 2.2, we get $u(\iota) \ge (1-a)w(\iota)$ for all $\iota \ge \iota_3$ and (1.10)–(1.11) become

$$\begin{aligned} &\bigl(p(\iota)\bigl(w'(\iota)\bigr)^{\alpha}\bigr)' + R_1(\iota) \le 0 \quad \text{for } \iota \ne \phi_k, \\ &\Delta\bigl(p(\phi_k)\bigl(w'(\phi_k)\bigr)^{\alpha}\bigr) + R_{(1,k)} \le 0 \quad \text{for } k = 1,2,\ldots. \end{aligned} \tag{3.3}$$

Integrating (3.3) from ι to ∞, we get

$$\bigl[p(s)\bigl(w'(s)\bigr)^{\alpha}\bigr]_{\iota}^{\infty} + \int_{\iota}^{\infty} R_1(s)\,\mathrm{d}s + \sum_{\phi_k \ge \iota} R_{(1,k)} \le 0.$$

Since $p(\iota)(w'(\iota))^\alpha$ is positive and nondecreasing, $\lim_{\iota\to\infty} p(\iota)(w'(\iota))^\alpha$ exists, and it is finite and positive. Then

$$p(\iota)\big(w'(\iota)\big)^\alpha \geq \int_\iota^\infty R_1(s)\,\mathrm{d}s + \sum_{\phi_k\geq\iota} R_{(1,k)},$$

that is,

$$w'(\iota) \geq p^{-1/\alpha}(\iota)\bigg[\int_\iota^\infty R_1(s)\,\mathrm{d}s + \sum_{\phi_k\geq\iota} R_{(1,k)}\bigg]^{1/\alpha}. \tag{3.4}$$

Since

$$\begin{aligned} g\big[(1-a)w\big(\nu(\iota)\big)\big] &= \frac{g[(1-a)w(\nu(\iota))]}{(1-a)^\beta w^\beta(\nu(\iota))}(1-a)^\beta w^\beta\big(\nu(\iota)\big) \\ &\geq \frac{g[c(1-a)]}{c^\beta(1-a)^\beta}(1-a)^\beta w^\beta\big(\nu(\iota)\big) \\ &= \frac{g[c(1-a)]}{c^\beta} w^\beta\big(\nu(\iota)\big), \end{aligned} \tag{3.5}$$

then we use (3.5) in (3.4) to get

$$\begin{aligned} w'(\iota) \geq p^{-1/\alpha}(\iota)\bigg[&\int_\iota^\infty r(s)\frac{g[c(1-a)]}{c^\beta} w^\beta\big(\nu(s)\big)\,\mathrm{d}s \\ &+ \sum_{\phi_k\geq\iota} \tilde{r}(\phi_k)\frac{g[c(1-a)]}{c^\beta} w^\beta\big(\nu(\phi_k)\big)\bigg]^{1/\alpha}. \end{aligned}$$

Next, if we set $K = \frac{g_0[c(1-a)]}{c^\beta}$, where $g_0[c(1-a)] = \min\{g[c(1-a)]\}$, the above inequality becomes

$$w'(\iota) \geq K^{1/\alpha}p^{-1/\alpha}(\iota)\bigg[\int_\iota^\infty r(s)w^\beta\big(\nu(s)\big)\,\mathrm{d}s + \sum_{\phi_k\geq\iota} \tilde{r}(\phi_k)w^\beta\big(\nu(\phi_k)\big)\bigg]^{1/\alpha}.$$

Using (b) and the fact that $w(\iota)$ is nondecreasing, we have

$$w'(\iota) \geq K^{1/\alpha}p^{-1/\alpha}(\iota)\bigg[\int_\iota^\infty r(s)\,\mathrm{d}s + \sum_{\phi_k\geq\iota} \tilde{r}(\phi_k)\bigg]^{1/\alpha} w^{\beta/\alpha}(\iota),$$

i.e.,

$$\frac{w'(\iota)}{w^{\beta/\alpha}(\iota)} \geq K^{1/\alpha}p^{-1/\alpha}(\iota)\bigg[\int_\iota^\infty r(s)\,\mathrm{d}s + \sum_{\phi_k\geq\iota} \tilde{r}(\phi_k)\bigg]^{1/\alpha}.$$

Integrating both sides from ι_3 to ∞, we get

$$K^{1/\alpha}\int_{\iota_3}^\infty p^{-1/\alpha}(s)\bigg[\int_s^\infty r(\psi)\,\mathrm{d}\psi + \sum_{\phi_k\geq\iota} \tilde{r}(\phi_k)\bigg]^{1/\alpha}\,\mathrm{d}s \leq \int_{\iota_3}^\infty \frac{w'(s)}{w^{\beta/\alpha}(s)}\,\mathrm{d}s < \infty$$

due to $\beta > \alpha$, which is a contradiction to (3.2) and hence the sufficiency part of the theorem is proved.

Next we prove the necessary part by a contrapositive argument. If (3.2) does not hold, then for every $\varepsilon > 0$ there exists $\iota \geq \iota_0$, for which

$$\int_\iota^\infty p^{-1/\alpha}(s)\left[\int_s^\infty r(\psi)\,\mathrm{d}\psi + \sum_{\phi_k \geq s} \tilde{r}(\phi_k)\right]^{1/\alpha} \mathrm{d}s < \varepsilon \quad \text{for } \iota \geq Y,$$

where $2\varepsilon = [\max\{g(\frac{1}{1-a})\}]^{-1/\alpha} > 0$.

Let us define the set

$$V = \left\{u \in C\big([0,\infty)\big) : \frac{1}{2} \leq u(\iota) \leq \frac{1}{1-a} \text{ for all } \iota \geq Y\right\}$$

and $\Phi : V \to V$ as

$$(\Phi u)(\iota) = \begin{cases} 0 & \text{if } \iota \leq Y, \\ \frac{1+a}{2(1-a)} - q(\iota)u(\zeta(\iota)) & \\ \quad + \int_\iota^\iota p^{-1/\alpha}(s)[\int_s^\infty r(\psi)g(u(\nu(\psi)))\,\mathrm{d}\psi & \\ \quad + \sum_{\phi_k \geq s} \tilde{r}(\phi_k)g(u(\nu(\phi_k)))]^{1/\alpha}\,\mathrm{d}s & \text{if } \iota > Y. \end{cases}$$

Now we prove that $(\Phi u)(\iota) \in V$. For $u(\iota) \in V$,

$$\begin{aligned} (\Phi u)(\iota) &\leq \frac{1+a}{2(1-a)} + \int_T^\iota p^{-1/\alpha}(s)\left[\int_s^\infty r(\psi)g\left(\frac{1}{1-a}\right)\mathrm{d}\psi\right. \\ &\quad \left. + \sum_{\phi_k \geq s} \tilde{r}(\phi_k)g\left(\frac{1}{1-a}\right)\right]^{1/\alpha} \mathrm{d}s \\ &\leq \frac{1+a}{2(1-a)} + \left[\max\left\{g\left(\frac{1}{1-a}\right)\right\}\right]^{1/\alpha} .\varepsilon \\ &= \frac{1+a}{2(1-a)} + \frac{1}{2} = \frac{1}{1-a}, \end{aligned}$$

and further, for $u(\iota) \in V$,

$$(\Phi u)(\iota) \geq \frac{1+a}{2(1-a)} - q(\iota).\frac{1}{1-a} + 0 \geq \frac{1+a}{2(1-a)} - \frac{a}{1-a} = \frac{1}{2}.$$

Hence Φ maps from V to V.

Now we are going to find a fixed point for Φ in V, which will give an eventually positive solution of (1.10)–(1.11).

First we define a sequence of functions in V by

$$\begin{aligned} u_0(\iota) &= 0 \quad \text{for } \iota \geq_0, \\ u_1(\iota) &= (\Phi u_0)(\iota) = \begin{cases} 0 & \text{if } \iota < Y, \\ \frac{1}{2} & \text{if } \iota \geq Y, \end{cases} \end{aligned}$$

$$u_{n+1}(\iota) = (\Phi u_n)(\iota) \quad \text{for } n \geq 1, \iota \geq Y.$$

Here we see $u_1(\iota) \geq u_0(\iota)$ for each fixed ι and $\frac{1}{2} \leq u_{n-1}(\iota) \leq u_n(\iota) \leq \frac{1}{1-a}$ for $\iota \geq Y$ for all $n \geq 1$. Thus u_n converges point-wise to a function u. By Lebesgue's dominated convergence theorem u is a fixed point of Φ in V, which shows that it has a nonoscillatory solution. This completes the proof of the theorem. □

In Theorem 3.2 we take a constant β, the quotient of two odd positive integers with $\beta < \alpha$, for which

$$\frac{g(\iota)}{\iota^\beta} \text{ is nonincreasing for } 0 < \iota. \tag{3.6}$$

The existence of such a constant can be established by taking $g(\iota) = |\iota|^\delta \operatorname{sgn}(\iota)$ with $\beta > \delta$. The assumption upon β can be withdrawn by taking $|u|^\beta \operatorname{sgn}(u)$ instead of u^β.

Theorem 3.2 *Let* (a), (c)–(f), *and* (3.6) *hold for* $\iota \geq \iota_0$. *Then every solution of* (1.10)–(1.11) *is oscillatory if*

$$\begin{aligned}\frac{1}{(2c)^\beta}\Bigg[\int_0^\infty r(\psi) g\big[c(1-a)P\big(\nu(\psi)\big)\big]\,\mathrm{d}\psi \\ + \sum_{k=1}^{\infty} \tilde{r}(\phi_k) g\big[c(1-a)P\big(\nu(\phi_k)\big)\big]\Bigg] = \infty \quad \forall c \neq 0.\end{aligned} \tag{3.7}$$

Proof Let $u(\iota)$ be an eventually positive solution of (1.10)–(1.11). Then, proceeding as in the proof of Theorem 3.1, we have $\iota_2 > \iota_1 > \iota_0$ such that inequality (3.4) holds for all $\iota \geq \iota_2$. Using (e), there exists $\iota_3 > \iota_2$ for which $P(\iota) - P(\iota_3) \geq \frac{1}{2}P(\iota)$ for $\iota \geq \iota_3$. Integrating (3.4) from ι_3 to ι, we have

$$\begin{aligned} w(\iota) - w(\iota_3) &\geq \int_{\iota_3}^{\iota} p^{-1/\alpha}(s)\Bigg[\int_s^\infty R_1(\kappa)\,\mathrm{d}\kappa + \sum_{\phi_k \geq s} R_{(1,k)}\Bigg]^{1/\alpha}\,\mathrm{d}s \\ &\geq \int_{\iota_3}^{\iota} p^{-1/\alpha}(s)\Bigg[\int_\iota^\infty R_1(\kappa)\,\mathrm{d}\kappa + \sum_{\phi_k \geq \iota} R_{(1,k)}\Bigg]^{1/\alpha}\,\mathrm{d}s, \end{aligned}$$

that is,

$$\begin{aligned} w(\iota) &\geq \big(P(\iota) - P(\iota_3)\big)\Bigg[\int_\iota^\infty R_1(\kappa)\,\mathrm{d}\kappa + \sum_{\phi_k \geq \iota} R_{(1,k)}\Bigg]^{1/\alpha} \\ &\geq \frac{1}{2}P(\iota)\Bigg[\int_\iota^\infty R_1(\kappa)\,\mathrm{d}\kappa + \sum_{\phi_k \geq \iota} R_{(1,k)}\Bigg]^{1/\alpha}. \end{aligned} \tag{3.8}$$

Since $p(\iota)(w'(\iota))^\alpha$ is nonincreasing and positive, then there exist $c > 0$ and $\iota_4 > \iota_3$ such that $p(\iota)(w'(\iota))^\alpha \leq c^\alpha$ for $\iota \geq \iota_4$. Integrating the relation $w'(\iota) \leq cp^{-1/\alpha}(\iota)$ from ι_4 to ι, we have

$$w(\iota) - w(\iota_4) \leq c\big(P(\iota) - P(\iota_4)\big),$$

that is,

$$w(\iota) \le cP(\iota) \quad \text{for } \iota \ge \iota_4. \tag{3.9}$$

Using (3.6) and (3.9), we obtain

$$\begin{aligned} g\big[(1-a)w\big(\nu(\iota)\big)\big] &= \frac{g[(1-a)w(\nu(\iota))]}{(1-a)^\beta w^\beta(\nu(\iota))}(1-a)^\beta w^\beta\big(\nu(\iota)\big) \\ &\ge \frac{g[c(1-a)P(\nu(\iota))]}{c^\beta(1-a)^\beta P^\beta(\nu(\iota))}(1-a)^\beta w^\beta\big(\nu(\iota)\big) \\ &= \frac{g[c(1-a)P(\nu(\iota))]}{c^\beta P^\beta(\nu(\iota))} w^\beta\big(\nu(\iota)\big) \quad \forall \iota \ge \iota_4. \end{aligned} \tag{3.10}$$

Using (3.10) in (3.8), we obtain

$$\begin{aligned} w(\iota) \ge \frac{1}{2}P(\iota)\bigg[&\int_\iota^\infty r(\kappa)\frac{g[c(1-a)P(\nu(\kappa))]}{c^\beta P^\beta(\nu(\kappa))} w^\beta\big(\nu(\kappa)\big)\,\mathrm{d}\kappa \\ &+ \sum_{\phi_k \ge \iota} \tilde{r}(\phi_k)\frac{g[c(1-a)P(\nu(\phi_k))]}{c^\beta P^\beta(\nu(\phi_k))} w^\beta\big(\nu(\phi_k)\big)\bigg]^{1/\alpha}. \end{aligned}$$

Hence,

$$w(\iota) \ge \frac{1}{2}P(\iota)U^{1/\alpha}(\iota) \quad \text{for } \iota \ge \iota_4,$$

where

$$\begin{aligned} U(\iota) = \frac{1}{c^\beta}\bigg[&\int_\iota^\infty r(\kappa)g\big[c(1-a)P\big(\nu(\kappa)\big)\big]\frac{w^\beta(\nu(\kappa))}{P^\beta(\nu(\kappa))}\,\mathrm{d}\kappa \\ &+ \sum_{\phi_k \ge \iota} \tilde{r}(\phi_k)g\big[c(1-a)P\big(\nu(\phi_k)\big)\big]\frac{w^\beta(\nu(\phi_k))}{P^\beta(\nu(\phi_k))}\bigg]. \end{aligned}$$

Now,

$$\begin{aligned} U'(\iota) &= -\frac{1}{c^\beta}r(\iota)g\big[c(1-a)P\big(\nu(\iota)\big)\big]\frac{w^\beta(\nu(\iota))}{P^\beta(\nu(\iota))} \\ &\le -\frac{1}{(2c)^\beta}r(\iota)g\big[c(1-a)P\big(\nu(\iota)\big)\big]U^{\beta/\alpha}\big(\nu(\iota)\big) \le 0 \end{aligned} \tag{3.11}$$

and

$$\Delta U(\phi_k) = -\frac{1}{(2c)^\beta}r(\phi_k)g\big[c(1-a)P\big(\nu(\phi_k)\big)\big]U^{\beta/\alpha}\big(\nu(\phi_k)\big) \le 0, \tag{3.12}$$

which shows that $U(\iota)$ is nonincreasing on $[\iota_4, \infty)$ and $\lim_{\iota\to\infty} U(\iota)$ exists. Using (3.11) and (a), we find

$$\big[U^{1-\beta/\alpha}(\iota)\big]' = (1-\beta/\alpha)U^{-\beta/\alpha}(\iota)U'(\iota)$$

$$\begin{aligned}&\le -\frac{1-\beta/\alpha}{(2c)^\beta} r(\iota)g\bigl[c(1-a)P\bigl(\nu(\iota)\bigr)\bigr]U^{\beta/\alpha}\bigl(\nu(\iota)\bigr)U^{-\beta/\alpha}(\iota)\\&\le -\frac{1-\beta/\alpha}{(2c)^\beta} r(\iota)g\bigl[c(1-a)P\bigl(\nu(\iota)\bigr)\bigr].\end{aligned}\tag{3.13}$$

To estimate the discontinuity of $U^{1-\beta/\alpha}$, we use a Taylor polynomial of order 1 from the function $h(u) = u^{1-\beta/\alpha}$, with $0 < \beta < \alpha$, about $u = a$:

$$b^{1-\beta/\alpha} - a^{1-\beta/\alpha} \le (1-\beta/\alpha)a^{-\beta/\alpha}(b-a).$$

Then

$$\begin{aligned}\Delta U^{1-\beta/\alpha}(\phi_k) &\le (1-\beta/\alpha)U^{-\beta/\alpha}(\phi_k)\Delta U(\phi_k)\\&\le -\frac{1-\beta/\alpha}{(2c)^\beta} r(\phi_k)g\bigl[c(1-a)P\bigl(\nu(\phi_k)\bigr)\bigr].\end{aligned}$$

Now, integrating (3.13) from ι_4 to ι, we have

$$\begin{aligned}&\bigl[U^{1-\beta/\alpha}(s)\bigr]_{\iota_4}^{\iota} - \sum_{\phi_k\ge\iota}\Delta\bigl[U^{1-\beta/\alpha}(\phi_k)\bigr]\\&\quad\le -\frac{1-\beta/\alpha}{(2c)^\beta}\int_{\iota_4}^{\iota} r(s)g\bigl[c(1-a)P\bigl(\nu(s)\bigr)\bigr]\,\mathrm{d}s,\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1-\beta/\alpha}{(2c)^\beta}\Biggl[\int_0^\infty r(s)g\bigl[c(1-a)P\bigl(\nu(s)\bigr)\bigr]\,\mathrm{d}s + \sum_{k=1}^{\infty}\tilde{r}(\phi_k)g\bigl[c(1-a)P\bigl(\nu(\phi_k)\bigr)\bigr]\Biggr]\\&\quad\le -\bigl[U^{1-\beta/\alpha}(s)\bigr]_{\iota_4}^{\iota} < U^{1-\beta/\alpha}(\iota_4) < \infty,\end{aligned}$$

which contradicts (3.7). This completes the proof. □

Example 3.1 Consider the neutral differential equations

$$\Bigl(\Bigl(\bigl(u(\iota)+e^{-\iota}u\bigl(\zeta(\iota)\bigr)\bigr)'\Bigr)^{1/3}\Bigr)' + \iota\bigl(u(\iota-2)\bigr)^{7/3} = 0,\tag{3.14}$$

$$\Bigl(\Bigl(\bigl(u\bigl(3^k\bigr)-e^{-3^k}x\bigl(\zeta\bigl(3^k\bigr)\bigr)\bigr)'\Bigr)^{1/3}\Bigr)' + (\iota+2)\bigl(u\bigl(3^k-2\bigr)\bigr)^{7/3} = 0.\tag{3.15}$$

Here $\alpha = 1/3$, $p(\iota) = 1$, $0 < q(\iota) = e^{-\iota} < 1$ $\nu(\iota) = \iota - 2$, $\phi_k = 3^k$ for $k \in \mathbb{N}$, $g(\iota) = \iota^{7/3}$. For $\beta = 5/3$, we have $\delta = 7/3 > \beta = 5/3 > \alpha = 1/3$ and $g(\iota)/\iota^\beta = \iota^{2/3}$, which are increasing functions. Now we check (3.2). We have

$$\begin{aligned}&\int_{\iota_0}^{\infty}\Biggl[\frac{1}{p(s)}\Biggl[\int_s^\infty r(\psi)\,d\psi + \sum_{\phi_k\ge s}\tilde{r}(\phi_k)\Biggr]\Biggr]^{1/\alpha}\mathrm{d}s\\&\quad\ge\int_{\iota_0}^{\infty}\biggl[\frac{1}{p(s)}\biggl[\int_s^\infty r(\psi)\,d\psi\biggr]\biggr]^{1/\alpha}\mathrm{d}s\\&\quad=\int_2^\infty\biggl[\int_s^\infty\psi\,d\psi\biggr]^3\mathrm{d}s = \infty.\end{aligned}$$

So, all the conditions of Theorem 3.1 hold. Thus, each solution of (3.14)–(3.15) is oscillatory.

Example 3.2 Consider the neutral differential equations

$$\left(e^{-\iota}\left(\left(u(\iota)+e^{-\iota}u\left(\zeta(\iota)\right)\right)'\right)^{11/3}\right)'+\frac{1}{\iota+1}\left(u(\iota-2)\right)^{1/3}=0, \tag{3.16}$$

$$\left(e^{-k}\left(\left(u(k)+e^{-k}u\left(\zeta(k)\right)\right)'\right)^{11/3}\right)'+\frac{1}{\iota+4}\left(u(k-2)\right)^{1/3}=0. \tag{3.17}$$

Here $\alpha = 11/3$, $p(\iota) = e^{-\iota}$, $0 < q(\iota) = e^{-\iota} < 1$, $\nu(\iota) = \iota - 2$, $\phi_k = k$ for $k \in \mathbb{N}$, $P(\iota) = \int_0^\iota e^{3s/11}\,ds = \frac{11}{3}(e^{3\iota/11} - 1)$, $g(\iota) = \iota^{1/3}$. For $\beta = 7/3$, we have $\delta = 1/3 < \beta = 7/3 < \alpha = 11/3$ and $g(\iota)/\iota^\beta = \iota^{-2}$, which are decreasing functions. Now we check (3.7). We have

$$\begin{aligned}
&\frac{1}{(2c)^\beta}\left[\int_0^\infty r(\psi)gc(1-a)P\left(\nu(\psi)\right)\right]\mathrm{d}\psi+\sum_{k=1}^{\infty}\tilde{r}(\phi_k)g\left[c(1-a)P\left(\nu(\phi_k)\right)\right]] \\
&\quad\geq \frac{1}{(2c)^{7/3}}\int_0^\infty r(\psi)g\left[c(1-a)P\left(\nu(\psi)\right)\right]\mathrm{d}\psi \\
&\quad= \frac{1}{(2c)^{7/3}}\int_0^\infty \frac{1}{\psi+1}\left[c(1-a)\frac{11}{3}\left(e^{3(\psi-2)/11}-1\right)\right]^{1/3}\mathrm{d}\psi=\infty \quad \forall c>0.
\end{aligned}$$

So, all the conditions of Theorem 3.2 hold, and therefore each solution of (3.16)–(3.17) is oscillatory.

4 Conclusions

In this work, we have undertaken the problem by taking a second-order highly nonlinear neutral impulsive differential system and established necessary and sufficient conditions for the oscillation of (1.10)–(1.11) when the neutral coefficient lies in $[0,1)$. It would be of interest to investigate the oscillation of (1.10)–(1.11) with different neutral coefficients; see, e.g., the papers [17–19] for more details. Furthermore, it is also interesting to analyze the oscillation of (1.10)–(1.11) with a nonlinear neutral term; see, e.g., the paper [10] for more details.

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