Thermodynamic limit in the two-qubit quantum Rabi model with spin-spin coupling

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The occurrence of a second-order superradiant quantum phase transition is brought to light in a quantum system consisting of two interacting qubits coupled to the same quantized field mode. We introduce an appropriate thermodynamic limit for the integrable two-qubit quantum Rabi model with spin-spin interaction. Namely, this limit is determined by the infinite ratios of the spin-spin and the spin-mode couplings to the mode frequency, regardless of the spin-to-mode frequency ratios.

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I. INTRODUCTION

Over the past decades quantum phase transitions [1–3] have attracted a great deal of attention in the condensed matter community [4]. Traditionally, quantum phase transitions (QPTs) are intended to occur in the thermodynamic limit, that is when the number of elements of the system is very large [3,4]. However, critical phenomena, for instance jumps of some physical observable, can also occur in systems with few degrees of freedom [5,6]. In this case, one speaks of few-body quantum phase transitions [7–9], a topic which is arousing the interest of a growing number of researchers [10–14]. The emergence of such peculiar transitions in the matter-radiation interaction is strongly related to the possibility of achieving the strong [15], ultrastrong [16–18], and deep strong coupling regimes [19–21].

The quantum Rabi model (QRM) [22–24] is the simplest paradigmatic few-body system exhibiting the occurrence of critical phenomena [5–14]. In the past three decades the QRM has been implemented in different regions of its parameter space, proving to be an adaptable theoretical resource [25–28] to investigate several radiation-matter scenarios as for example trapped ions [29–31], cavity QED [32,33], and circuit QED systems [34–36].

The dynamical behaviors [10,11,37], as well as the phase diagrams [35] exhibited by the QRM and the Dicke model [38], suggest the possible existence of a correspondence between the few-body and the regular QPTs. By analyzing the scaling of the critical exponents, the QPT exhibited by the QRM can be indeed connected to many-body and thermodynamic cases [13]. When one deals with the QRM, the standard thermodynamic limit must be *de facto* replaced by the so called classical oscillator limit [5,6], which consists in the ideal physical regime identified by the vanishing spin-to-field frequency ratio. It is worth noticing that the QRM presents critical phenomena also in the finite-frequency regime [7–9] and recently a Beretzinski-Kosterlitz-Thouless QPT was found to occur in a dissipative QRM [39]. These properties of the QRM thus spur to search for new physical scenarios where few-body systems can manifest, in special parameter space regions, intriguing critical phenomena conceptually traceable back to the standard QPTs in the thermodynamic limit.

It seems therefore reasonable to foresee a comparable dynamic richness for the extended and generalized versions of the QRM, like the two-qubit [40–45] and multiqubit [46,47] QRM, the two-photon [48,49] and multiphoton [50] QRM, and the multilevel QRM [51,52]. Recently, investigating a two-qubit QRM, where a nontrivial qubit-qubit interaction is considered, a first-order QPT in the finite-frequency limit has been brought to light [53]. We remark that introducing a two-qubit coupling in the Hamiltonian model complies with the main goal of quantum computation and quantum information, namely the implementation of two-qubit quantum logic gates [54–56].

In this article we deal with a two-qubit QRM. We demonstrate that, crossing the critical value of an appropriately defined adimensional control parameter, the tripartite (qubit-qubit-radiation mode) system undergoes a second-order superradiant QPT. The peculiarity of such a QPT lies in the nature of the classical limit involved. In this case, indeed, what goes to infinity is the ratio of both the qubit-qubit and the qubit-mode couplings to the oscillator frequency. The frequencies of the qubits remain instead free parameters and can take values close to the oscillator's frequency. A new physical condition for reaching the thermodynamic limit is then brought to light in the framework of the two-qubit QRM. In addition, as a difference with the most common scenario characterized by a transverse magnetic field, the considered qubit-qubit coupling gives rise to the system's symmetry, leading to an integrable and exactly solvable model.

II. MODEL

Consider the following Hamiltonian model (in units of $\hbar$):

$$H = \omega a^\dagger a + \epsilon_1\sigma_1^z + \epsilon_2\sigma_2^z + \gamma\sigma_1^x\sigma_2^x + \left(\lambda_1\sigma_1^z + \lambda_2\sigma_2^z\right)(a + a^\dagger), \quad (1)$$

which describes two interacting, biased (ϵ_1 and ϵ_2) qubits coupled to a single bosonic field mode. ω is the characteristic frequency of the mode, while γ and λ_i ($i = 1, 2$) are the qubit-qubit and the (ith) qubit-mode couplings, respectively. σ_k^l ($k = 1, 2$, $l = x, y, z$) are qubit Pauli operators, while a and $a^\dagger$ are the annihilation and creation boson operators, respectively.

Since $\sigma_1^z\sigma_2^z$ is a constant of motion, the Hamiltonian can be unitarily transformed into $H = H_+ \oplus H_-$, with (see Appendix A)

$$H_\pm = \omega\, a^\dagger a + \epsilon_\pm\sigma_\pm^z + \gamma\sigma_\pm^x + \lambda_\pm(a^\dagger + a)\sigma_\pm^z, \quad (2)$$

where $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$, $\lambda_\pm = \lambda_1 \pm \lambda_2$, and $\sigma_\pm^l$ ($l = x, y, z$) are standard two-level Pauli operators. H_+ (H_-) is the effective Hamiltonian governing the dynamics of the two-qubit-mode system within the dynamically invariant subspace $\mathscr{H}_+$ ($\mathscr{H}_-$), which is spanned by $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \otimes \{|n\rangle\}_{n\in 0}^{\infty}$ ($\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\} \otimes \{|n\rangle\}_{n\in 0}^{\infty}$), where we defined $\sigma^z|\uparrow\rangle = +|\uparrow\rangle$, $\sigma^z|\downarrow\rangle = -|\downarrow\rangle$, and $a^\dagger a|n\rangle = n|n\rangle$ [53,57]. The field operators appearing in H_+ and H_-, although identically written for convenience as those in H, are formally different. In fact, the operator $(a + a^\dagger)$ in H_+ and H_- must be intended as $(a + a^\dagger) \otimes (|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$ and $(a + a^\dagger) \otimes (|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|)$, respectively. We remark that H_+ and H_- result to be single-spin QRM Hamiltonians, where the qubit-qubit coupling γ plays the role of the strength of a fictitious transverse field.

It must be noted that two qubits holistically behave as a two-level system within each invariant subspace, meaning that $\sigma_\pm^l$ ($l = x, y, z$) are the Pauli operators of such effective two-level systems. Thus the original system (two interacting qubits coupled to the same quantized field mode) can be exactly reduced to two independent effective single-spin quantum Rabi problems defined in $\mathscr{H}_+$ and $\mathscr{H}_-$.

It is worth noticing that, since such a reduction of $\mathscr{H}$ into $\mathscr{H}_+$ and $\mathscr{H}_-$ is independent of the Hamiltonian parameters, this approach keeps its validity in the weak, strong, ultrastrong, and deep-strong regimes (see Refs. [18,58,59] for the classification of the qubit-mode coupling regimes). Moreover, we remark that this reduction can be exactly achieved even in the case of time-dependent control classical fields, that is $\epsilon_1(t)$ and $\epsilon_2(t)$ [60–62].

III. BIASED QPT

The parameters $\epsilon_\pm$ and $\lambda_\pm$, characterizing the two effective Hamiltonians (2), suggest considering the case of counterbiased qubits, i.e., $\epsilon_1 = -\epsilon_2 = \epsilon/2$, equally coupled to the field mode, namely $\lambda_1 = \lambda_2 = \lambda/2$. We stress that these conditions correspond to realistic geometric symmetries governing the coupling in the tripartite system and demand an experimentally feasible external classical control. The advantage of such particular conditions is that the two effective Hamiltonians (2) assume the simple forms

$$H_+ = \omega\, a^\dagger a + \gamma\sigma_+^x + \lambda(a^\dagger + a)\sigma_+^z,$$
$$H_- = \omega\, a^\dagger a + \epsilon\sigma_-^z + \gamma\sigma_-^x. \quad (3)$$

We see that H_+ possesses the form of the standard QRM Hamiltonian, while H_- describes a (fictitious) two-level system decoupled from a bosonic field mode. Further, we highlight that through the qubit-qubit coupling γ we can tune the effective frequency of the two fictitious two-level systems, that is the energy splitting in the two invariant subspaces. For such a reason the interaction γ plays a fundamental role (mainly in $\mathscr{H}_+$), as shown in the following.

Let us analyze the particular regime reached by the tripartite system in the parameter-space region asymptotically defined by

$$\gamma/\omega \to \infty. \quad (4)$$

This condition ensures the classical oscillator limit [5,10], which resembles the role that the thermodynamic limit plays in many-body spin systems [5,6]. Such a kind of thermodynamic limit (not related to the size of the system) has already been introduced for the single-qubit QRM [5]. In that case, however, the classical regime arises from the diverging ratio between the spin and the oscillator frequencies. In our case the crucial ratio is the spin-spin coupling strength to the oscillator frequency (γ/ω), regardless of the magnitude of the qubit's frequency.

When Eq. (4) is fulfilled, the exact low-energy form of the Hamiltonian H_+ becomes (see Appendix B)

$$H_+^{np} = \omega\, a^\dagger a - \frac{\omega g^2}{4}(a^\dagger + a)^2 - \gamma, \quad (5)$$

for $g < 1$, and

$$H_+^{sp} = \omega\, a^\dagger a - \frac{\omega}{4g^4}(a^\dagger + a)^2 - \gamma\,\frac{g^2 + g^{-2}}{2}, \quad (6)$$

for $g > 1$, with $g = \sqrt{2}\lambda/\sqrt{\omega\gamma}$. The fictitious single-spin QRM H_+ in Eq. (B4) exhibits a second-order QPT [10], guided by the parameter g, between a normal (np) and a superradiant (sp) phase. We remark that, besides $\gamma/\omega \to \infty$, the limit $\lambda/\omega \to \infty$ must be contextually considered. This condition is required in order to have a nonvanishing control parameter g, driving the system towards the QPT.

We observe that the occurrence of a QPT in the tripartite two-qubit QRM has a transparent interpretation in view of the exact separation of the Hilbert space in two invariant subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$. Based on this mapping, in the following we investigate the effects of the normal-superradiant QPT on the ground state of the interacting-qubits QRM.

The lowest-energy states and the corresponding eigenenergies of H_+^{np} and H_+^{sp} expressed in terms of the two-qubit-mode system read (see Appendix B)

$$E_{0+}^{np} = \omega\frac{\sqrt{1-g^2}-1}{2} - \gamma,$$
$$\left|\Psi_{0+}^{np}\right\rangle = \mathcal{S}[r_{np}(g)]|0\rangle \otimes \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}, \tag{7}$$

for $g < 1$, and

$$E_{0+}^{sp} = \omega\frac{\sqrt{1-g^{-4}}-1}{2} - \gamma\frac{g^2+g^{-2}}{2},$$
$$\left|\Psi_{0+}^{sp}\right\rangle = \mathcal{S}[r_{sp}(g)]|0\rangle \otimes \left[\left(\frac{\sqrt{1+g^{-2}} \mp \sqrt{1-g^{-2}}}{2}\right)|\uparrow\uparrow\rangle - \left(\frac{\sqrt{1+g^{-2}} \pm \sqrt{1-g^{-2}}}{2}\right)|\downarrow\downarrow\rangle\right], \tag{8}$$

when $g > 1$. Here $r_{np} = -\ln(1-g^2)/4$, $r_{sp} = -\ln(1-g^{-4})/4$, and $\mathcal{S}(x) = \exp\{(x/2)(a^{\dagger 2} - a^2)\}$. Note that the ground state of H_+^{sp} in Eq. (8) is twofold degenerate.

Let us suppose that the qubits' energy is comparable with that of the oscillator and then much smaller than the spin-spin coupling $\epsilon \ll \gamma$. In such a condition the lowest energy of H_- and the related eigenstate read

$$E_0^- = -\gamma,$$
$$|\Psi_0^-\rangle = |0\rangle \otimes \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \tag{9}$$

If we introduce the rescaled energy $\widetilde{E} = E \cdot \omega/\gamma$, in the limit $\gamma/\omega \to \infty$, we get

$$\widetilde{E}_0^+ = \begin{cases} \widetilde{E}_{0+}^{np} = -\omega, & g < 1, \\ \widetilde{E}_{0+}^{sp} = -\omega\,(g^2+g^{-2})/2, & g > 1, \end{cases}$$
$$\widetilde{E}_0^- = -\omega. \tag{10}$$

We see that for $g < 1$ the ground state of the tripartite system has a double degeneracy and, in general, it possesses nonvanishing projections in $\mathscr{H}_+$ and $\mathscr{H}_-$. For $g > 1$, instead, the ground state of the same system is $|\Psi_{0+}^{sp}\rangle$ and belongs to the subspace $\mathscr{H}_+$ since $\widetilde{E}_{0+}^{sp} < \widetilde{E}_0^-$, as clearly shown in Fig. 1(a), where the three rescaled energies in Eq. (10) are plotted in units of ω.

It is worth pointing out that, given the factorization of the ground states in qubit and bosonic parts (for $g < 1$ and for $g > 1$), the mean photon number $N = \langle\Psi_0|a^\dagger a|\Psi_0\rangle$, the two-qubit magnetization $M = \langle\Psi_0|(\sigma_1^z + \sigma_2^z)/2|\Psi_0\rangle$, and the concurrence C [63] can be easily obtained (here $|\Psi_0\rangle$ indicates the generic ground state, independently of the phase, either normal or superradiant). In the following these three quantities, both for the normal ($g < 1$) and the superradiant ($g > 1$) phases, are given:

$$\widetilde{N}_{np} = 0, \quad \widetilde{N}_{sp} = \frac{g^2 - g^{-2}}{4},$$
$$C_{np} = 1, \quad C_{sp} = g^{-2}, \tag{11}$$
$$M_{np} = 0, \quad M_{sp} = \pm\sqrt{1-g^{-2}},$$

where $\widetilde{N} = N \cdot (\omega/\gamma)$ is the rescaled mean photon number. The rescaling is necessary in order to highlight its difference in the two phases. In fact, for $g > 1$ the mean photon number is infinite, as usual for the superradiant phase. The dependence of these three quantities on the dimensionless parameter g is shown in Figs. 1(b), 2(a), and 2(b). We note that the mean two-qubit magnetization M depends on the nature of the ground state in the superradiant phase. The two curves (black and blue) in Fig. 2(b) are indeed due to the degeneracy of the ground state in the superradiant phase [see Eq. (8)]. This means that the associated statistical ensemble should be described by a density matrix with equal weights for the two states in Eq. (8). Thus it would be characterized by a vanishing mean value of the two-qubit magnetization.

Conversely, the mean photon number and the concurrence have uniquely determined values in both phases. The drastic change of the dependence of the mean photon number on g is at the origin of the superradiant nature of the QPT. Moreover, the two phases are characterized by a different level of

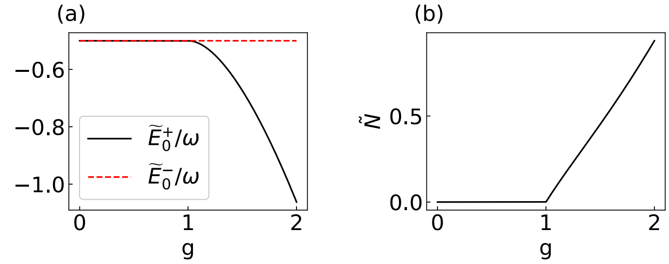

FIG. 1. (a) Dependence of the first two lowest eigenenergies $\widetilde{E}_0^+/\omega$ and $\widetilde{E}_0^-/\omega$ on the control parameter $g = g_1 = g_2$, for counter-biased qubits ($\epsilon_1 = -\epsilon_2$) and equal spin-mode couplings ($\lambda_1 = \lambda_2$). The black curve in the superradiant phase ($g > 1$) corresponds to two degenerate eigenstates [see Eq. (8)]. (b) Dependence of the rescaled mean photon number on g, for the ground state of the two-qubit QRM, with $\epsilon/\gamma \to 0$, $\omega/\gamma \to 0$, and $\omega/\lambda \to 0$.

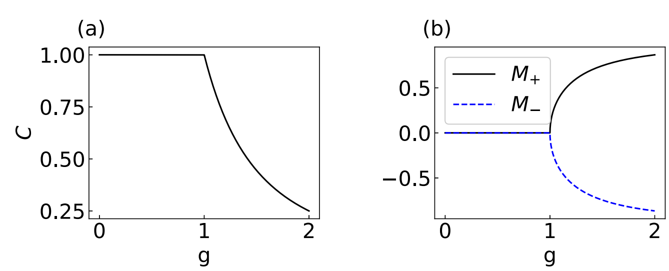

FIG. 2. Dependence of (a) the two-qubit concurrence and (b) the two-qubit magnetization on g, for the ground state of the two-qubit QRM, with $\epsilon/\gamma \to 0$, $\omega/\gamma \to 0$, and $\omega/\lambda \to 0$. The two curves (black and blue) for $g > 1$ (superradiant phase) are related to the two states in Eq. (8).

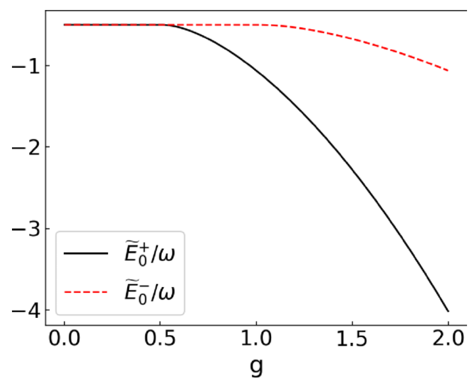

FIG. 3. Dependence of the first two lowest eigenenergies $\widetilde{E}_0^+/\omega$ and $\widetilde{E}_0^-/\omega$ on the control parameter $g = g_- = g_+/2$ (i.e., $g = 2g_1/3 = 2g_2$), for unbiased qubits ($\epsilon_1 = \epsilon_2 = 0$) and different spin-mode couplings ($\lambda_1 = 3\lambda_2$, implying $\lambda_+/2 = \lambda_- = \lambda$).

entanglement exhibited by the two qubits. Precisely, the normal phase is characterized by a maximum level of entanglement ($C_{np} = 1$), while the superradiant phase exhibits a concurrence which decreases as g increases. Therefore, besides the mean photon number, also the level of entanglement between the two qubits can serve as a signature of the occurrence of the superradiant QPT, like in other spin systems [64–68]. We thus see the occurrence of a second-order QPT from a normal to a superradiant phase for the two-qubit quantum Rabi system. This second-order QPT is characterized by two physical quantities (N and C), which exhibit a critical behavior: they are not differentiable at the critical point, g_c, of the control parameter, where the transition occurs.

IV. UNBIASED QPT

If we relax the constraints imposed to the parameters appearing in the Hamiltonian model (A1), and consider the more general case $|\epsilon_1| \neq |\epsilon_2|$, a bias term (namely $\epsilon_+\sigma^z$) appears in H_+. We emphasize that, in this instance, this more general Hamiltonian cannot be approximated by the expressions in Eqs. (5) and (6) valid for the two regimes.

In light of the previous observation, we change the physical scenario assuming $\epsilon_1 = \epsilon_2 = 0$ (implying $\epsilon_+ = \epsilon_- = 0$). Such a condition, even keeping $\lambda_1 \neq \lambda_2$, is still compatible with the procedure leading to the two Hamiltonians H_+^{np} and H_+^{sp} [Eqs. (5) and (6)]. This time, the same approximation procedure can be performed for H_-. In this case the Hamiltonians H_+^{np} (H_+^{sp}) and H_-^{np} (H_-^{sp}) have a similar structure and it is easy to verify that the same occurs for both the corresponding eigensolutions [see Eqs. (B8) and (8)]. The two lowest energies are plotted in Fig. 3 for $\lambda_+/2 = \lambda_- = \lambda$, which implies $g_+/2 = g_- = g$ (with $g_\pm = \sqrt{2}\lambda_\pm/\sqrt{\omega\gamma}$). We see that the normal-superradiant phase transition occurs in both the subspaces. Nevertheless, the condition $\lambda_+ \neq \lambda_-$, which implies $g_+ \neq g_-$, produces a different dependence of the lowest energy level of H_+^{sp} and H_-^{sp} on the control parameter. Therefore, in general, $g_c^+ \neq g_c^-$, with $g_c^\pm$ being the critical values of g for which the QPT occurs in the subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$, respectively. In the specific considered example, i.e., $\lambda_+/2 = \lambda_- = \lambda$ (implying $g_+/2 = g_- = g$), we have $g_c^+ = 0.5$ and $g_c^- = 1$, as shown in Fig. 3. This fact implies that, in this instance, for $g < g_c^+$ the ground state of the two-qubit-mode system has nonvanishing projections on both $\mathscr{H}_+$ and $\mathscr{H}_-$, while for $g > g_c^+$ it uniquely belongs to the subspace $\mathscr{H}_+$. Such a transition is still characterized by the physical quantities C, M, and N, which present the same dependence on g seen before. Finally, more in general, we can say that the subspace where the ground state is placed after the QPT depends on the quantity $\min(g_c^+, g_c^-)$.

V. CONCLUSIVE REMARKS

The central result of this work consists in the physical condition expressed by Eq. (4), that determines the thermodynamic limit under which a two-qubit QRM system undergoes a second-order superradiant QPT. The thermodynamic limit [5] (considering an infinite number of spins in many-body systems [38]) can be achieved in finite-size systems [5,6], like the QRM, by pushing to infinity the ratio of qubit frequency to mode frequency [5,10].

In this article, we have shown that, in order to reach the thermodynamic limit [5] in a two-qubit QRM, the ultrastrong interaction regime must be fulfilled by both the qubit-qubit and the qubit-mode couplings ($\gamma/\omega \to \infty$, $\lambda/\omega \to \infty$), regardless of the qubit-to-mode frequency ratio. This circumstance highlights that in more complex systems, where the Hamiltonian is characterized by several parameters, there exist different conditions making it possible to reach the thermodynamic limit. In our case, for example, including the additional interaction term $\Gamma\sigma_1^y\sigma_2^y$ in H gives rise to an extra transverse field in H_+, namely $-\Gamma\sigma^x$. Moreover, if the Dzialoshinskii-Moryia coupling terms, $\delta\sigma_1^x\sigma_1^y$ and $\Delta\sigma_1^y\sigma_1^x$, were considered, the further effective transverse field $(\delta + \Delta)\sigma^y$ would appear in H_+. In both cases, the symmetry of the model is preserved and we can define new thermodynamic limit(s) depending on the different coupling parameters.

For experimental implementation of the unveiled physics the best candidates are trapped-ion [69] and superconducting-circuit architectures [70]. Multiqubit systems of trapped ions or superconducting devices coupled to harmonic modes have been considered for a long time in a variety of academic and industrial laboratories [71–73]. In particular, a detailed proposal of implementation is under investigation, following the lines of Ref. [52], where an unconventional fluxonium design of a multilevel superconducting device allows to solve the related longstanding problem of the detection of virtual phothons in the USC regime [74], leveraging relaxation mechanisms for the measurement [75,76]. Hybrid semiconductor/superconductor devices also implement the QRM and its descendants, allowing for a large degree of tunability [77].

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APPENDIX A: MODEL AND ITS SYMMETRIES

Let us consider the following Hamiltonian model:

$$H = \omega a^\dagger a + \epsilon_1\sigma_1^z + \epsilon_2\sigma_2^z + \gamma\sigma_1^x\sigma_2^x + \left(\lambda_1\sigma_1^z + \lambda_2\sigma_2^z\right)(a + a^\dagger), \tag{A1}$$

describing two interacting qubits subjected to local longitudinal (along the $\hat{z}$ direction) fields and coupled to a common single quantized field mode. σ_k^x, σ_k^y, and σ_k^z ($k = 1, 2$) are the Pauli matrices, while a and $a^\dagger$ are the annihilation and creation boson operators, respectively.

It is possible to verify that $\sigma_1^z\sigma_2^z$ is a constant of motion since the Hamiltonian is invariant under a π rotation of each spin around the z axis [53,57], that is,

$$e^{-i\pi\sigma_1^z/2} \otimes e^{-i\pi\sigma_2^z/2} H\, e^{i\pi\sigma_1^z/2} \otimes e^{i\pi\sigma_2^z/2} = \sigma_1^z\sigma_2^z H\, \sigma_1^z\sigma_2^z = H. \tag{A2}$$

In this way, the total Hilbert space $\mathscr{H}$ is composed of two orthogonal, dynamically invariant subspaces, namely $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, related to the two eigenvalues (± 1) of the constant of motion. As a consequence the Schrödinger equation relative to H can be exactly decoupled into two *mathematically independent* Schrödinger equations related to the two invariant subspaces.

By transforming the Hamiltonian H through the following unitary and Hermitian operator

$$\mathbb{U} = \tfrac{1}{2}\left[\mathbb{1} + \sigma_1^z + \sigma_2^x - \sigma_1^z\sigma_2^x\right], \tag{A3}$$

which accomplishes the transformation $\mathbb{U}^\dagger\sigma_1^z\sigma_2^z\mathbb{U} = \mathbb{U}\sigma_1^z\sigma_2^z\mathbb{U} = \hat{\mathbb{1}}_1 \otimes \sigma_2^z$, we get

$$\begin{aligned}\widetilde{H} = \mathbb{U}^\dagger H \mathbb{U} &= \epsilon_1\sigma_1^z + \epsilon_2\sigma_1^z\sigma_2^z + \omega\, a^\dagger a \\ &+ \gamma\sigma_1^x + \lambda_1(a + a^\dagger)\sigma_1^z + \lambda_2(a + a^\dagger)\sigma_1^z\sigma_2^z.\end{aligned} \tag{A4}$$

Since σ_2^z is a constant of motion of $\widetilde{H}$, it may be treated as a parameter and then we may write two effective Hamiltonians, which describe the two-qubit QRM dynamics within each dynamically invariant subspace. This circumstance is related to the fact that the two qubits effectively behave as a two-level system in each dynamically invariant subspace. Specifically, we get

$$H_+ = (\epsilon_1 + \epsilon_2)\sigma_+^z + \gamma\sigma_+^x + \omega\, a^\dagger a + (\lambda_1 + \lambda_2)(a^\dagger + a)\sigma_+^z, \tag{A5}$$

for $\sigma_2^z = 1$, and

$$H_- = (\epsilon_1 - \epsilon_2)\sigma_-^z + \gamma\sigma_-^x + \omega\, a^\dagger a + (\lambda_1 - \lambda_2)(a^\dagger + a)\sigma_-^z, \tag{A6}$$

for $\sigma_2^z = -1$. We notice that each effective Hamiltonian is nothing but a single-qubit QRM where the role of the transverse field is played by the spin-spin coupling. Therefore, we can obtain information about the two-qubit QRM system by exploiting the plethora of results existing in literature for the single-qubit QRM.

The Pauli matrices in H_+ and H_- are the operators of the fictitious two-level systems through which we effectively describe the two qubits in each invariant subspace. More precisely, it means that, in each Hilbert subspace, we can map the two-qubit standard basis states into a couple of fictitious two-level states. Specifically, we map the states $|\uparrow\uparrow\rangle$ ($|\uparrow\downarrow\rangle$) and $|\downarrow\downarrow\rangle$ ($|\downarrow\uparrow\rangle$) into the two fictitious states $|\uparrow\rangle_+$ ($|\uparrow\rangle_-$) and $|\downarrow\rangle_+$ ($|\downarrow\rangle_-$), where $\{|\uparrow\rangle_+, |\downarrow\rangle_+\}$ and $\{|\uparrow\rangle_-, |\downarrow\rangle_-\}$ are the couple of states of the fictitious two-level systems defined in the subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$, respectively. Here we have defined $\sigma^z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma^z|\downarrow\rangle = -|\downarrow\rangle$.

It is worth pointing out that the two effective Hamiltonians represent the projections of the Hamiltonian H into the two infinite-dimensional invariant subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$, that is, $H_+ = P_+HP_+$ and $H_- = P_-HP_-$, accomplished by the two corresponding projection operators P_+ and P_-, respectively. The identity $H = P_+HP_+ \oplus P_-HP_- = H_+ \oplus H_-$ can be derived by considering that $(P_+ + P_-)H(P_+ + P_-) = H$ and that the existence of the constant of motion implies $P_+HP_- = P_-HP_+ = 0$. In view of the previous observation, as commented in the main text, the field operators appearing in H_+ and H_- are formally different from those in H. Namely, the operator $(a + a^\dagger)$ in H_+ and H_- must be intended as $(a + a^\dagger) \otimes (|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$ and $(a + a^\dagger) \otimes (|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|)$, respectively.

APPENDIX B: LOW-ENERGY HAMILTONIAN IN THE CLASSICAL OSCILLATOR LIMIT

Considering the case of counterbiased qubits, that is, $\epsilon_1 = -\epsilon_2 = \epsilon/2$, and an equal coupling with the mode for the two qubits, i.e., $\lambda_1 = \lambda_2 = \lambda/2$, the two effective Hamiltonians, defined in the two invariant subspaces, read

$$\begin{aligned} H_+ &= \omega\, a^\dagger a + \gamma\sigma_+^x + \lambda(a^\dagger + a)\sigma_+^z, \\ H_- &= \omega\, a^\dagger a + \epsilon\sigma_-^z + \gamma\sigma_-^x. \end{aligned} \tag{B1}$$

The spectrum of H_- is easily derivable:

$$E_{n\pm}^- = \pm\sqrt{\epsilon^2 + \gamma^2} + n\omega, \tag{B2}$$

and the related eigenstates, considering the two-qubit–single-qubit mapping, read

$$|\Psi_{n\pm}^-\rangle = \frac{(\epsilon \pm \sqrt{\epsilon^2 + \gamma^2})|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle}{N} \otimes \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \tag{B3}$$

with $N = \sqrt{2(\epsilon^2 + \gamma^2 \pm \epsilon\sqrt{\epsilon^2 + \gamma^2})}$ the normalization factor and $a^\dagger a|n\rangle = n|n\rangle$, $n \in \mathrm{N}$.

As H_+ is concerned, we first transform the Hamiltonian by rotating it $\pi/2$ (clockwise) with respect to the y axis, obtaining

$$\widetilde{H}_+ = \omega\, a^\dagger a + \gamma\sigma_+^z - \lambda(a^\dagger + a)\sigma_+^x. \tag{B4}$$

In this way, by exploiting the procedure reported in Ref. [10], we can claim that the exact forms of this Hamiltonian, for low energies and in the classical oscillator limit $\omega/\gamma \to 0$, with $\omega/\lambda \to 0$, are

$$H_+^{np} = \omega\, a^\dagger a - \frac{\omega g^2}{4}(a^\dagger + a)^2 - \gamma \tag{B5}$$

and

$$H_+^{sp} = \omega a^\dagger a - \frac{\omega}{4g^4}(a^\dagger + a)^2 - \gamma\frac{g^2+g^{-2}}{2}, \tag{B6}$$

for $g<1$ and $g>1$, respectively, with $g=\sqrt{2}\lambda/\sqrt{\omega\gamma}$. We remark that such expressions result from an approximation procedure, consisting in keeping the terms up to ω/γ [10]. However, the error (discrepancy) between the exact (numerically evaluated) solution and the approximated one vanishes as $\omega/\gamma \to 0$ [10]. Therefore, in the classical oscillator limit ($\omega \to 0$), the spectrum of H_+^{np} (H_+^{sp}) is exactly the low-energy spectrum of H for $g<1$ ($g>1$) [10].

The eigenvalues of H_+^{np} and H_+^{sp} are proved to be [10]

$$\begin{aligned} E_{n+}^{np} &= \omega\sqrt{1-g^2}\left(n+\frac{1}{2}\right) - \frac{\omega}{2} - \gamma, \\ E_{n+}^{sp} &= \omega\sqrt{1-g^{-4}}\left(n+\frac{1}{2}\right) - \frac{\omega+\gamma(g^2+g^{-2})}{2}, \end{aligned} \tag{B7}$$

whose rescaled expressions (obtained by multiplying by ω/γ and taking the limit for $\omega/\gamma \to 0$) are exactly those given in Eq. (10).

Basing on Ref. [10], and after rotating and mapping back according to our procedure, the corresponding eigenstates of H, related to the subspace $\mathscr{H}_+$, read

$$\begin{aligned} \left|\Psi_{n+}^{np}\right\rangle &= \mathcal{S}[r_{np}(g)]|n\rangle \otimes \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}, \\ \left|\Psi_{n+}^{sp}\right\rangle &= \mathcal{S}[r_{sp}(g)]|n\rangle \otimes \left[\left(\frac{\sqrt{1+g^{-2}} \mp \sqrt{1-g^{-2}}}{2}\right)|\uparrow\uparrow\rangle \right. \\ &\quad \left. - \left(\frac{\sqrt{1+g^{-2}} \pm \sqrt{1-g^{-2}}}{2}\right)|\downarrow\downarrow\rangle\right], \end{aligned} \tag{B8}$$

with $r_{np} = -\ln(1-g^2)/4$, $r_{sp} = -\ln(1-g^{-4})/4$, and $\mathcal{S}(x) = \exp\{(x/2)[(a^\dagger)^2 - a^2]\}$. We stress that the above states constitute half of the set of eigenstates of $\widetilde{H}_+$, precisely those corresponding to the low-energy subspace identified by the projection of the (fictitious) spin onto the down-state $|\widetilde{\downarrow}\rangle \equiv (|\downarrow\rangle + |\uparrow\rangle)/\sqrt{2}$ [in terms of the two actual qubits, according to the mapping (and the y rotation performed), we have $|\widetilde{\downarrow}\rangle \equiv |\widetilde{\downarrow\downarrow}\rangle \equiv (|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$]. Such a subspace is demonstrated to be an invariant subspace of $\widetilde{H}_+$, after the latter is unitarily transformed and the terms up to the first order in ω/γ are taken into account [10]. The same procedure can be analogously repeated for both H_+ and H_- in the case of unbiased qubits ($\epsilon_1 = \epsilon_2 = 0$) and for different qubit-mode couplings ($\lambda_1 \neq \lambda_2$).

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