Journal of Mathematical Psychology 115 (2023) 102791

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

Modal preference structures[☆]

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ARTICLE INFO

Article history:
Received 13 January 2023
Received in revised form 24 March 2023
Accepted 6 May 2023
Available online xxxx

Keywords:
Transitivity
Completeness
Transitive coherence
Mixed completeness
NaP-preference
GNaP-preference
Group decision
Modal preference structure

ABSTRACT

A total preorder is a transitive and complete binary relation on a set. A modal preference structure of rank n is a string composed of 2 to the exponent n binary relations on a set such that there is a family of total preorders that gives all relations by taking intersections and unions. Total preorders are structures of rank zero, NaP-preferences (Giarlotta and Greco, 2013) are structures of rank one, and GNaP-preferences (Carpentiere et al., 2022) are structures of rank two. We characterize modal preference structures of any rank by properties of transitive coherence and mixed completeness. Moreover, we show how to construct structures of a given rank from others of lower rank. Modal preference structures arise in economics and psychology, in the process of aggregating hierarchical judgements of groups of agents, where each of the n coordinates represents a feature/stage of the decision procedure.

1. Introduction

Whenever the preference structure of a decision maker (DM) is modeled by means of a binary relation on a set of alternatives, the properties of transitivity and completeness are usually regarded as *the* basic tenets of economic rationality (Kreps, 2013; Mas-Colell et al., 1995). This is the reason why total preorders – reflexive, transitive, and complete binary relations – are often considered a prototypical way to encode DMs' preferences. This attitude is also justified by the fact that, under some assumptions on the separability of the induced topology, representability of a total preorder by a continuous utility function is guaranteed (Bridges & Mehta, 1995).

Over time, transitivity and completeness have been questioned by several scholars from various points of view: see, among many contributions on the topic, the seminal work of Tversky (1969), who reports on an experiment that reliably induces people to violate the transitivity axiom.[1] On one hand, this criticism spiked a fervent research toward the use of binary relations that satisfy weaker forms of transitivity, such as *semiorders* (Fishburn, 1968, 1997; Luce, 1956; Pirlot & Vincke, 1997), *interval orders* (Fishburn, 1970, 1973, 1985), and, more generally, (m, n)-*Ferrers relations* (Giarlotta & Watson, 2014, 2018; Öztürk, 2008). On the other hand, after completeness was elusively doubted by von Neumann and Morgenstern (1944, p. 19–20) and then strongly opposed by Aumann (1962), a plethora of contributions studying incomplete modelizations of preferences followed in many fields of research: for a survey, see Carpentiere et al. (2022, Section 1) and references therein.

A quite recent line of research considers pairs – and, more generally, lists – of binary relations, in which the two basic tenets of economic rationality are satisfied 'collectively' (that is, by the family of binary relations as a whole) instead of 'individually' (that is, by every binary relation). An instance of this approach is a *necessary and possible preference* or, for brevity, a *NaP-preference* (Giarlotta & Greco, 2013): this is a pair of nested binary relations on a nonempty set of alternatives such that the smaller is transitive, the larger is complete, and the two preferences are mutually linked by properties of 'transitive coherence' and 'mixed completeness'. The connection between the two relations ensures that any NaP-preference is representable by means of a family of total preorders, whose intersection and union gives the two components, respectively.

Using a similar approach and motivation, Carpentiere et al. (2022) have recently extended the notion of a NaP-preference by that of a *GNaP-preference (generalized NaP-preference)*: this

[☆] This paper is dedicated to the memory of Peter Fishburn, a truly exceptional applied mathematician, and a pioneer in several fields of research, such as order theory, graph theory, decision theory, preference modeling, choice theory, mathematical psychology, and mathematical economics. The elegance and the depth of his results, constructions, and proofs is largely acknowledged by the most brilliant scholars in these fields.

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[1] For a different perspective in mathematical psychology, see Regenwetter et al. (2011).

https://doi.org/10.1016/j.jmp.2023.102791
0022-2496/

is a quadruple of binary relations on the same set, arranged in a diamond-shaped structure, such that the smallest is transitive, the largest is complete, and the four relations are mutually connected by properties of transitive coherence and mixed completeness. Similarly to NaP-preferences, any GNaP-preference is characterized by the existence of a family of total preorders, which is indexed over a Cartesian product of two sets in a way that the four relations can be retrieved by taking suitable intersections and unions of these preorders.

In this paper, we describe and characterize a general way to employ multiple binary relations to aggregate rational (i.e., transitive and complete) preferences expressed by groups of agents. Specifically, reversing the approach used for NaP-preferences and GNaP-preferences, we define a *modal preference structure (MPS)* as a 2^n-tuple of binary relations on a given nonempty set of alternatives such that there exists a family of prototypical preferences – total preorders – indexed over a Cartesian product of n sets, with the property that all 2^n binary relations are suitable intersections/unions of the preorders in the family.

Our goal is to show that any MPS is characterized by four 'rational' properties of consistency, which are natural extensions of those that define NaP-preferences and GNaP-preferences. These properties guarantee that the two axioms of transitivity and completeness are collectively – but not necessarily individually – satisfied by the 2^n binary relations arranged in a boolean structure. In fact, for any given MPS, not only the smallest relation is transitive and the largest is complete, but suitable pairs of binary relations in the structure are mutually linked by forms of transitive coherence and mixed completeness.

Similarly to NaP-preferences and GNaP-preferences, modal preference structures are designed with the purpose of better describing the process of preference modeling. In fact, any family of total preorders that represents an MPS is indexed over a Cartesian product of n sets, where each set describes a feature of the decision procedure. The term 'modal' is used because each feature can be satisfied either 'universally' (i.e., for all values in the range) or 'existentially' (i.e., for at least a value in the range). An additional reason that supports such a terminology is the close connection with modal logic. In fact, modal logic extends pre-existing settings by adding two unary operations $\Box$ and $\diamond$, which are interpreted, respectively, as 'necessary' and 'possible' operators. (For instance, the formula $\Box P$ is interpreted as 'necessarily P', and $\Box P$ holds when P holds for every value assigned to free variables.) Therefore, a close connection with universal and existential quantifiers is apparent. More generally, multi-modal systems, where multiple necessary and possible operators are defined, are semantically similar to modal preference structures, as they are both applied to systems in which a 'necessary and possible' analysis is required (Fitting & Mendelsohn, 1998).

Here we also show that the collection of all modal preference structures on a given set of alternatives is closed under an operation of 'tensor product'. This algebraic feature means two things: (1) any MPS can be constructed from below, that is, by taking the tensor product of MPSs having lower complexity; and (2) the tensor product of a family of MPSs having the same complexity yields an MPS of higher complexity. In particular, any MPS of arbitrary complexity can be always obtained from MPSs having the lowest complexity – total preorders – by suitably constructing the sets whose Cartesian product indexes the total preorders. In more practical terms, this means that, given (i) a family of rational economic agents (endowed with total preorders) and (ii) a hierarchical structure guiding the aggregation process of their preferences (a rooted tree), the resulting MPS describes an 'organized synthesis' of all pieces of preferential information related to the problem at hand. This may be particularly useful in applications, whenever an overwhelming number of actors is involved in the decision procedure, and a well-structured synthesis of their judgements need be presented to DM(s) to facilitate the final decision.

The paper is organized as follows. In Section 2 we recall the notions of NaP-preferences and GNaP-preferences. In Section 3 we introduce modal preference structures and give some examples. Sections 4 and 5 collect some preliminary results, based on the two notions of suitable lists and interpolating preorders. The characterization of modal preference structures is proved in Section 6. In Section 7 we define the operation of tensor product and discuss feasible applications. Section 8 suggests several possible directions of future research.

2. NaP-preferences and GNaP-preferences

Let X be a nonempty set of alternatives. A reflexive binary relation $\succsim$ on X is called a *weak preference* on X, and $x \succsim y$ is interpreted as "alternative x is weakly preferred to alternative y". The *strict preference* $\succ$, the *indifference* $\sim$, and the *incomparability* $\perp$ associated to $\succsim$ are the binary relations on X defined by, respectively, $x \succ y$ if $x \succsim y$ and $\neg(y \succsim x)$, $x \sim y$ if $x \succsim y$ and $y \succsim x$, and $x \perp y$ if $\neg(x \succsim y)$ and $\neg(y \succsim x)$.[2]

A *preorder* $\succsim$ on X is a reflexive and transitive binary relation on X. A preorder $\succsim$ is *total* (or *complete*) if for all distinct $x, y \in X$, at least one between $x \succsim y$ and $y \succsim x$ holds. Under suitable separability assumptions on the order topology associated to a total preorder $\succsim$, there is a continuous function $u : X \to \mathbb{R}$ such that $x \succsim y$ if and only if $u(x) \geqslant u(y)$, for all $x, y \in X$.[3]

A *necessary and possible preference* (*NaP-preference*) on X is a pair $\left(\succsim^N, \succsim^P\right)$ of binary relations on X satisfying the following properties (Giarlotta & Greco, 2013)[4]:

(NP1: *core rationality*) $\succsim^N$ is a preorder;

(NP2: *chain structure*) $\succsim^N \subseteq \succsim^P$;

(NP3: *transitive coherence*) $\succsim^N \circ \succsim^P \subseteq \succsim^P$ and $\succsim^P \circ \succsim^N \subseteq \succsim^P$;

(NP4: *mixed completeness*) $x \succsim^N y$ or $y \succsim^P x$ for all $x, y \in X$.

Originally introduced in the field of *multiple criteria decision analysis (MCDA)* via the so-called *robust ordinal regression* (Greco et al., 2008), NaP-preferences have been an object of careful study, both in applications and in theory: see the survey by Giarlotta (2019) for a vast account of research on the topic. From a decision-theoretic perspective, NaP-preferences consistently combine *Knightian preferences* (Bewley, 1986) and *justifiable preferences* (Lehrer & Teper, 2011). Under the Axiom of Choice (AC), the following characterization of NaP-preferences holds:

Theorem (*Giarlotta & Greco, 2013, Theorem 3.4*). (AC) *A pair $\left(\succsim^N, \succsim^P\right)$ of binary relations on X is a NaP-preference if and only if there exist a nonempty set H and a family $\{\succsim_h : h \in H\}$ of total preorders on X such that $\succsim^N := \bigcap_{h\in H} \succsim_h$ and $\succsim^P := \bigcup_{h\in H} \succsim_h$.*

Note that the above theorem generalizes a well-known result by Donaldson and Weymark (1998), which says that any preorder is the intersection of a family of total preorders.

In the same line of research – that is, modeling preferences by pairs of binary relations – some recent contributions are the

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[2] Note that $\succ$ is the asymmetric part of $\succsim$, $\sim$ is the symmetric part of $\succsim$, and $\perp$ is the symmetric part of the complement of $\succsim$. Moreover, $\succsim$ is the union of $\succ$ and $\sim$.

[3] See Bridges and Mehta (1995) for a survey on the representability of a total preorder by utility functions.

[4] The inclusion $\succsim^N \circ \succsim^P \subseteq \succsim^P$ means that $x \succsim^N y \succsim^P z$ implies $x \succsim^P z$, for all $x, y, z \in X$. A similar meaning has the inclusion $\succsim^P \circ \succsim^N \subseteq \succsim^P$. The terminology 'transitive coherence' originates from the fact that a binary relation $\succsim$ is transitive if and only if $\succsim \circ \succsim \subseteq \succsim$.

following: (1) preference structures (Nishimura & Ok, 2019) and their relationship with top-cycle choice rules (Evren et al., 2019); (2) consistency and decisiveness of a double-minded DM (Uyanik & Khan, 2019), also in connection to the continuity postulate (Uyanik & Khan, 2022) and the intermediate value property (Ghosh et al., 2019); (3) objective and subjective rationality in a multiple prior model (Gilboa et al., 2010); (4) interactions between mental and behavioral preferences (Cerreia-Vioglio et al., 2020).

Very recently, Carpentiere et al. (2022) have extended the notion of NaP-preference by that of *generalized NaP-preference (GNaP-preference)*: this is a quadruple $\left(\succsim^{NN}, \succsim^{NP}, \succsim^{PN}, \succsim^{PP}\right)$ of binary relations on X satisfying the following properties:

(GNP1: *core rationality*) $\succsim^{NN}$ is a preorder;

(GNP2: *diamond structure*)

- $\succsim^{NN} \subseteq \left(\succsim^{NP} \cap \succsim^{PN}\right) \subseteq \left(\succsim^{NP} \cup \succsim^{PN}\right) \subseteq \succsim^{PP}$;

(GNP3: *transitive coherence*)

- $\succsim^{NN} \circ \succsim^{NP} \subseteq \succsim^{NP}$ and $\succsim^{NP} \circ \succsim^{NN} \subseteq \succsim^{NP}$,
- $\succsim^{NN} \circ \succsim^{PN} \subseteq \succsim^{PN}$ and $\succsim^{PN} \circ \succsim^{NN} \subseteq \succsim^{PN}$,
- $\succsim^{NP} \circ \succsim^{PN} \subseteq \succsim^{PP}$ and $\succsim^{PN} \circ \succsim^{NP} \subseteq \succsim^{PP}$,
- $\succsim^{NN} \circ \succsim^{PP} \subseteq \succsim^{PP}$ and $\succsim^{PP} \circ \succsim^{NN} \subseteq \succsim^{PP}$;

(GNP4: *mixed completeness*) for all $x, y \in X$,

- $x \succsim^{NN} y \ \vee \ y \succsim^{PP} x$,
- $x \succsim^{NP} y \ \vee \ y \succsim^{PN} x$.

As for NaP-preferences, also GNaP-preferences were originally introduced in MCDA, with the goal of developing a sound multiple criteria methodology applicable to group decision making (Greco et al., 2012). However, the axiomatic treatment of the topic was missing from the mentioned paper, and only came ten years later. Under the Axiom of Choice, GNaP-preferences can be characterized in a way similar – *mutatis mutandis* – to NaP-preferences:

Theorem (*Carpentiere et al., 2022, Theorem 4*)**.** (AC) *A quadruple* $\left(\succsim^{NN}, \succsim^{NP}, \succsim^{PN}, \succsim^{PP}\right)$ *of binary relations on* X *is a GNaP-preference if and only if there exist nonempty sets* H, K *and a family* $\{\succsim_{hk} : (h, k) \in H \times K\}$ *of total preorders on* X *such that*

$$\succsim^{NN} = \bigcap_{h\in H}\bigcap_{k\in K} \succsim_{hk}\,, \quad \succsim^{NP} = \bigcap_{h\in H}\bigcup_{k\in K} \succsim_{hk}\,,$$
$$\succsim^{PN} = \bigcup_{h\in H}\bigcap_{k\in K} \succsim_{hk}\,, \quad \succsim^{PP} = \bigcup_{h\in H}\bigcup_{k\in K} \succsim_{hk}\,.$$

The proof of the above characterization requires considerably more work than the corresponding result for NaP-preferences. The reason is that several technicalities in the proof – e.g., the interplay between the two index sets, the construction of suitable 'interpolating' preorders, etc. – remain hidden when only two relations are considered, and exclusively arise when at least four relations are arranged into a boolean structure.

3. Modal preference structures

Here we define a general type of preference structure, which comprises NaP-preferences and GNaP-preferences as special cases.

Definition 1. Let $n \geqslant 1$ be an integer, and $Q = \{\forall, \exists\}$ the set of quantifiers. Elements of Q^n are denoted by $\S = (\S^1, \ldots, \S^n)$, and are called *strings* (of quantifiers); unless confusion may arise, we simplify notation and use $\S^1 \ldots \S^n$ for a string in Q^n. A *modal preference structure (MPS) of rank n* is a 2^n-tuple $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ of reflexive binary relations on X such that there is a family of (index) sets $\mathscr{K} = (K_1, \ldots, K_n)$ and a family

$$\mathscr{T} = \left\{\succsim_{k_1\ldots k_n} : (k_1, \ldots, k_n) \in K_1 \times \cdots \times K_n\right\}$$

of total preorders on X with the property that, for all $x, y \in X$ and $\S = \S^1 \ldots \S^n \in Q^n$,

$$x \succsim^{\S^1\ldots\S^n} y \quad \Longleftrightarrow \quad (\S^1 k_1 \in K_1)\ldots(\S^n k_n \in K_n)\, x \succsim_{k_1\ldots k_n} y\,. \tag{1}$$

By definition, a total preorder is an MPS of rank 0.

Remark 1. Definition 1 does not explicitly say in which order the binary relations are listed in an MPS. This fact does not affect our analysis by any means. However, for the sake of clarity, it is better to decide from the outset how these binary relations are arranged into a 2^n-tuple. To that end, we use an order that suggests their level of refinement in the boolean structure that will be associated to them (see Fig. 1). Specifically, if we use the identifications '$\forall \equiv 0$' and '$\exists \equiv 1$', then the binary relations are listed in increasing order according to the sum of their indices, and, in case of an equal sum, the lexicographic order on $\{0, 1\}^2$ is applied. Thus, for instance, an MPS of rank 3 is listed as follows:

$$\left(\succsim^{\forall\forall\forall}, \succsim^{\forall\forall\exists}, \succsim^{\forall\exists\forall}, \succsim^{\exists\forall\forall}, \succsim^{\forall\exists\exists}, \succsim^{\exists\forall\exists}, \succsim^{\exists\exists\forall}, \succsim^{\exists\exists\exists}\right)$$
$$= \left(\succsim^{000}, \succsim^{001}, \succsim^{010}, \succsim^{100}, \succsim^{011}, \succsim^{101}, \succsim^{110}, \succsim^{111}\right).$$

Example 1. By Theorem 3.4 in Giarlotta and Greco (2013), a modal preference structure of rank 1 is a NaP-preference $\left(\succsim^{N}, \succsim^{P}\right) = \left(\succsim^{\forall}, \succsim^{\exists}\right) = \left(\succsim^{0}, \succsim^{1}\right)$.

Example 2. By Theorem 4 in Carpentiere et al. (2022), an MPS of rank 2 is a GNaP-preference $\left(\succsim^{NN}, \succsim^{NP}, \succsim^{PN}, \succsim^{PP}\right) = \left(\succsim^{\forall\forall}, \succsim^{\forall\exists}, \succsim^{\exists\forall}, \succsim^{\exists\exists}\right) = \left(\succsim^{00}, \succsim^{01}, \succsim^{10}, \succsim^{11}\right)$.

The idea behind the notion of an MPS is natural. Each set K_i in the Cartesian product $K_1 \times \cdots \times K_n$ encodes a 'mode' of the preference structure. Specifically, each K_i models a relevant feature/stage of the decision procedure, which can be witnessed either universally or existentially — that is, either necessarily (for any value in its range) or possibly (for at least one value in its range). The next example illustrates a possible semantics of MPSs in a concrete scenario.

Example 3 (*Aggregating Rankings of Projects*)**.** A multinational corporation has to select a key investment from five feasible projects. The structure of the company comprises many departments, some of which are directly involved in the decision procedure, namely finance (F), marketing (M), research and development (R&D), and human resources (HR). Each department is split into sub-departments, which in turn have their own officers. Selecting the correct project is crucial for the future of the company, and so the CEO decides that every officer of each sub-department must give her/his own opinion.

To that end, all officers are asked to rank the five candidate projects in descending order, with *ex-aequo* allowed; that is, they must provide total preorders on the set of projects. Note that the amount of preferences to be evaluated by the CEO may turn out overwhelmingly large. For instance, if each department has 4 sub-departments, and each sub-department has 3 officers, then overall there are 48 total preorders that concur in the evaluation.

It is advisable – if not mandatory – that the CEO is provided with a synthetic and organized view of all these pieces of preferential information emanating from the lower levels of the organized chart of the corporation. A modal preference structure of rank 3 may perform this required process of 'organized synthesis' in an effective and sound way. Let us explain how.

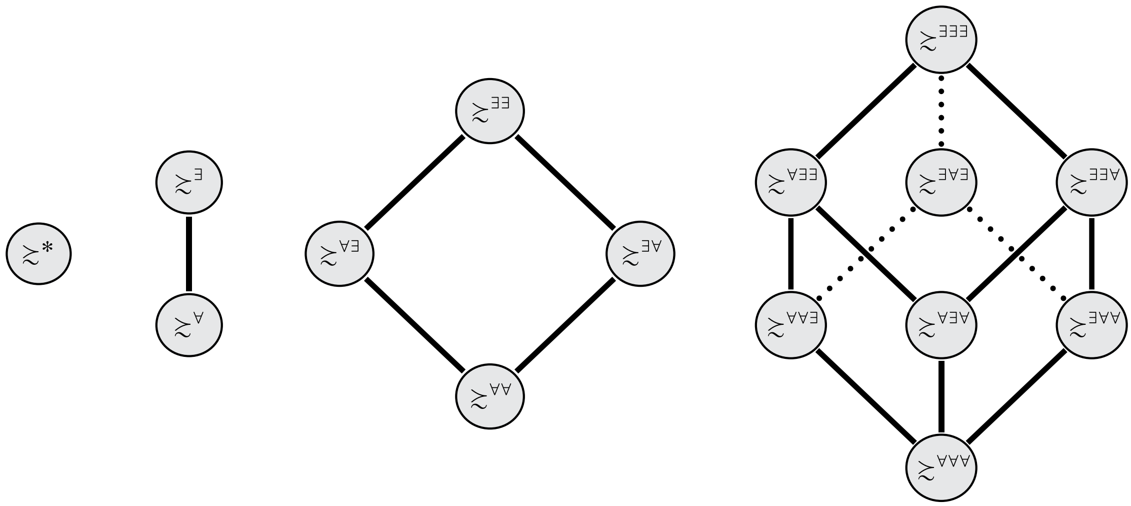

Fig. 1. The monotonicity property of a modal preference structure of rank $n = 0, 1, 2, 3$. On the left, a modal preference structure of rank 0 (a total preorder). In the middle, two modal preference structures of rank 1 (a NaP-preference) and 2 (a GNaP-preference), respectively. On the right, a modal preference structure of rank 3, whose components are the vertices of a cube embedded in the 3-dimensional space.

The first stage of the process of aggregation is done at the level of sub-departments. Specifically, for each of them, all officers' rankings (total preorders) are collected together. As a consequence, by Theorem 3.4 in Giarlotta and Greco (2013), any sub-department is associated with a NaP-preference on the set of the five available projects. The second stage of the process consists of collecting the opinions of all sub-departments for any of the four departments. This stage requires to perform the 'tensor product' of the NaP-preferences corresponding to its sub-departments.[5] By Lemma 8 in Carpentiere et al. (2022), this tensor product is a GNaP-preference. In other words, each of the four departments involved in the process is associated with a GNaP-preference on the set of projects. The third and last stage of this synthesis of evaluations is to aggregate the opinions of the four departments. This is done by taking the tensor product of the four associated GNaP-preferences, which, by Lemma 9 in Section 7, gives rise to a modal preference structure of rank 3.

The final result of this aggregation process is an 8-tuple of binary relations on the set of available projects, where the semantics of each of the eight preference relations is quite simple. For instance, $x \succsim^{\forall\exists\forall} y$ means that for all departments F, M, R&D, HR, there is a sub-department such that all of its officers rank project x at least as good as project y. Note that the aggregation process naturally yields the index sets K_1, K_2 and K_3: in fact, K_i is the $(4-i)$-th stage, $i = 1, 2, 3$, of the synthesis. Overall, the process of aggregation of the total preorders appears as an organized, synthetic view of all individual preferences. The 'rationality' of the model is retained by the properties of transitive coherence and mixed completeness (later defined as properties **M3** and **M4**), whereas an excessive complexity is removed from the hierarchical decisional structure.

It is worth observing that this type of process applies to any organized structure that can be represented as a rooted tree of arbitrary (finite) length and form. In this tree, all terminal nodes represent rational agents (i.e., total preorders) or rational groups of agents (i.e., NaP-preferences, GNaP-preferences, etc.), whereas the unique root represents the decisional unit (a single DM or a group of DMs). We shall formally elaborate on this point in Section 7 (see also Example 11).

The main result of this paper is a characterization of modal preference structures. In order to state it, we first introduce some preliminary notions.

Definition 2. We define a partial binary operation '$+$' and a partial order '$\leqslant$' on Q^n. To start, we define $+$ for strings of length two. Let $+ : Q^2 \setminus \{\exists\exists\} \to Q$ be the map[6]

$$\forall + \forall := \forall\,, \quad \forall + \exists := \exists\,, \quad \exists + \forall := \exists\,.$$

We extend $+$ to a partial operation on Q^n as follows: for all strings $\S = \S^1 \dots \S^n$ and $\S' = \S^{1\prime} \dots \S^{n\prime}$ in Q^n such that $\S^i + \S^{i\prime}$ is defined for all $i \in \{1, \dots, n\}$,[7] let

$$\S + \S' := \left(\S^i + \S^{i\prime}\right)_{i=1}^{n}.$$

Furthermore, let $\leqslant$ be the *Pareto ordering* on Q^n derived from the linear order $\leqslant$ on Q such that $\forall < \exists$. For every string $\S = \S^1 \dots \S^n$ in Q^n, we shall denote by $\overline{\S} = \overline{\S}^1 \dots \overline{\S}^n \in Q^n$ the *opposite* string, that is, the string $\overline{\S}$ such that $\overline{\S}^i \neq \S^i$ for all $i \in \{1, \dots, n\}$.

Thus, for instance, we have $\forall\exists\forall\forall\exists + \forall\forall\exists\forall\forall = \forall\exists\exists\forall\exists$ in Q^5, whereas $\forall\exists\forall\exists + \forall\forall\forall\exists$ is undefined in Q^4. Moreover, the strict inequality $\exists\forall\forall\exists\forall\forall < \exists\exists\forall\exists\exists\forall$ holds in Q^6, whereas $\forall\exists\forall$ and $\forall\forall\exists$ are incomparable in Q^3.

Remark 2. The notation employed for the partial operation '$+$' and the partial order '$\leqslant$' becomes clear as soon as one identifies the quantifier $\forall$ with the digit 0, and the quantifier $\exists$ with the digit 1: for instance, we have $0 + 0 = 0$, $0 < 1$, etc.

Remark 3. The partial operation $+$ on Q^n is associative, commutative, and has $\forall\forall\dots\forall$ as identity. Thus $(Q^n, +)$ is a partial abelian semigroup with unity. Note also that $\S + \overline{\S} = \exists\exists\dots\exists$ for each $\S \in Q^n$.

Remark 4. Observe that $\leqslant$ is the partial order on Q^n defined by

$$\S \leqslant \S' \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall i \in \{1, \dots, n\}) \, \left(\S^i \neq \S^{i\prime} \implies \S^i = \forall\right)$$

[5] The general operation of tensor product for modal preference structures will be introduced in Section 7. However, at the moment we only need the notion of tensor product of NaP-preferences, which is already given in Carpentiere et al. (2022, Section 5).

[6] Recall that the string '$\exists\exists$' is an abbreviation for the pair $(\exists, \exists)$. The partial operation $+$ is not defined for the pair $\exists\exists$, because $\exists + \exists \equiv 1 + 1 = 2 \notin \{0, 1\} \equiv \{\forall, \exists\}$.

[7] That is, there is no coordinate k, with $1 \leqslant k \leqslant n$, such that $\S^k = \S^{k\prime} = \exists$.

for all $\S = \S^1 \ldots \S^n$ and $\S' = \S^{1'} \ldots \S^{n'}$ in Q^n. Indeed, we have $\S \leqslant \S'$ if and only if $\S^i \leqslant \S^{i'}$ for all $i \in \{1, \ldots, n\}$. Note also that the two strings $\S$ and $\bar{\S}$ are incomparable according to the Pareto ordering on Q^n.

Next, we list some properties of the components of any 2^n-tuple $\left(\succsim^{\S} : \S \in Q^n\right)$ of binary relations on X (here $\S, \S' \in Q^n$ and $x, y \in X$ are arbitrary):

(M1: core transitivity) $\succsim^{\forall\forall\ldots\forall}$ is a preorder;

(M2: monotonicity) if $\S \leqslant \S'$, then $\succsim^{\S} \subseteq \succsim^{\S'}$;

(M3: transitive coherence) $\succsim^{\S} \circ \succsim^{\S'} \subseteq \succsim^{\S+\S'}$ if $\S + \S'$ is defined;

(M4: mixed completeness) $x \succsim^{\S} y$ or $y \succsim^{\bar{\S}} x$.

Note that if we set $Q^0 := \{*\}$ by definition, then for $n = 0$ the four properties M1–M4 collapse to requiring that transitivity and completeness hold for the unique preference relation $\succsim^*$ in the 2^0-tuple; that is, $\succsim^*$ is a total preorder on X.

Moreover, observe that property **M3** implies that $\succsim^{\forall\forall\ldots\forall}$ is transitive. Therefore, we could reduce **M1** to reflexivity, or remove the case of $\succsim^{\forall\forall\ldots\forall} \circ \succsim^{\forall\forall\ldots\forall}$ from **M3**. However, in order to keep the symmetry between MPSs and NaP-preferences, and also to avoid overly complicating the formulation of property **M3**, we prefer to endure a slightly redundancy of the axioms.

In the next two examples, the notation $\succsim^{\S} \mathbin{\widehat{\circ}} \succsim^{\S'}$ stands for either $\succsim^{\S} \circ \succsim^{\S'}$ or $\succsim^{\S'} \circ \succsim^{\S}$.

Example 4. For $n = 1$, we obtain a pair $\left(\succsim^{\forall}, \succsim^{\exists}\right)$ of binary relations on X, and properties M1–M4 reduce to the following, where $x, y \in X$ are arbitrary:

(M1) $\succsim^{\forall}$ is a preorder;

(M2) $\succsim^{\forall} \subseteq \succsim^{\exists}$;

(M3) $\succsim^{\forall} \mathbin{\widehat{\circ}} \succsim^{\exists} \subseteq \succsim^{\exists}$;

(M4) $x \succsim^{\forall} y \ \vee \ y \succsim^{\exists} x$.

Thus, $\left(\succsim^{\forall}, \succsim^{\exists}\right)$ is a NaP-preference, with $\succsim^{\forall} = \succsim^{N}$ and $\succsim^{\exists} = \succsim^{P}$.

Example 5. For $n = 2$, we have a quadruple $\left(\succsim^{\forall\forall}, \succsim^{\forall\exists}, \succsim^{\exists\forall}, \succsim^{\exists\exists}\right)$ of binary relations on X, and properties M1–M4 reduce to the following, where $x, y \in X$ are arbitrary:

(M1) $\succsim^{\forall\forall}$ is a preorder;

(M2) $\succsim^{\forall\forall} \subseteq \left(\succsim^{\forall\exists} \cap \succsim^{\exists\forall}\right) \subseteq \left(\succsim^{\forall\exists} \cup \succsim^{\exists\forall}\right) \subseteq \succsim^{\exists\exists}$;

(M3) $\succsim^{\forall\forall} \mathbin{\widehat{\circ}} \succsim^{\forall\exists} \subseteq \succsim^{\forall\exists}$, $\succsim^{\forall\forall} \mathbin{\widehat{\circ}} \succsim^{\exists\forall} \subseteq \succsim^{\exists\forall}$, $\succsim^{\forall\exists} \mathbin{\widehat{\circ}} \succsim^{\exists\forall} \subseteq \succsim^{\exists\exists}$, and $\succsim^{\forall\forall} \mathbin{\widehat{\circ}} \succsim^{\exists\exists} \subseteq \succsim^{\exists\exists}$;

(M4) $\left(x \succsim^{\forall\forall} y \ \vee \ y \succsim^{\exists\exists} x\right)$ and $\left(x \succsim^{\forall\exists} y \ \vee \ y \succsim^{\exists\forall} x\right)$.

Thus, $\left(\succsim^{\forall\forall}, \succsim^{\forall\exists}, \succsim^{\exists\forall}, \succsim^{\exists\exists}\right) = \left(\succsim^{NN}, \succsim^{NP}, \succsim^{PN}, \succsim^{PP}\right)$ is a GNaP-preference on X.

To get an idea of the content of properties M1–M4, Fig. 1 graphically represents property M2 of modal preference structures of rank 0, 1, 2, and 3. For a graphical representation of the properties of transitive coherence and mixed completeness in the case of MPSs of rank 2 and 3, we refer the reader to Figs. 1–3 in Carpentiere et al. (2022).

The following conjecture is stated in Carpentiere et al. (2022):

Conjecture 1. *For any integer $n \geqslant 1$, the following statements are equivalent for a 2^n-tuple $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ of binary relations on X:*

(i) *$\mathscr{M}$ is a modal preference structure on X;*
(ii) *$\mathscr{M}$ satisfies properties* M1–M4.

The next three sections are devoted to the technical proof of Conjecture 1. The necessity part is straightforward. On the contrary, the sufficiency part requires several preparatory results to obtain a witnessing family of total preorders, which emanates from properties M1–M4. The next two sections collect such preliminaries, whereas Section 6 proves the claim.

Specifically, in Section 4 we list some general combinatorial results, which require no assumption on the preference structure and solely depend on properties M1–M4. In Section 5 we apply these results to our setting with the goal of getting the required total preorders. Finally, in Section 6 we prove Conjecture 1 by fixing the set $Q^n \times X^2 \times \{0, 1\} = K_i$ for all $i = 1, \ldots, n$, and then showing that the equivalence (1) in Definition 1 holds.

4. Suitable lists

Here we define the notion of a 'suitable list', which allows one to use the operation $+$ on selected lists of quantifiers. Informally, a suitable list is a collections of pairs, each of them composed of a string of n quantifiers and a bit (0 or 1), satisfying a certain property. If such a property is satisfied, then the addition is defined for selected strings. The strings that we sum are chosen by looking at the bit associated to each original string.

Definition 3. A *building list* – for brevity a *list* – in Q^n is a family $((\S_t, b_t))_{t=1,\ldots,m}$ of pairs in $Q^n \times \{0, 1\}$, where m is a positive integer. A building list $((\S_t, b_t))_{t=1,\ldots,m}$ is *suitable* whenever for any $j \in \{1, \ldots, n\}$, there is at most one $h \in \{1, \ldots, m\}$ such that the pair $(\S_h^j, b_h)$ is equal to either $(\forall, 0)$ or $(\exists, 1)$. (A component $\S_h^j$ of the string $\S_h$ that generates such a situation is called a *bad occurrence*.)

In other words, if we arrange the m strings of n quantifiers in an $m \times n$ matrix, then suitability requires that for each coordinate of the given strings, there is at most one bad occurrence. Note that suitability is independent of the order in which the building list is presented.

The next example exhibits an instance of a suitable list.

Example 6. Arrange $m = 4$ given strings of quantifiers in $Q^n = Q^4$ in a 4×4 matrix, one per row, and add a column for the corresponding bits as follows:

$$\left[\begin{array}{cccc|c} \forall & \exists & \forall & \exists & 1 \\ \exists & \forall & \forall & \forall & 1 \\ \exists & \exists & \forall & \exists & 0 \\ \exists & \exists & \exists & \exists & 0 \end{array}\right] \equiv \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

(In the matrix on the right, we are using the identifications '$\forall \equiv 0$' and '$\exists \equiv 1$' mentioned in Remark 2, in order to provide a more graphic way to verify the suitability of a building list.) It is straightforward to check that the list $((\S_t, b_t))_{t=1,2,3,4}$ in Q^4 is suitable. For instance, in the first column we only have one bad occurrence, which is in the second row (emphasized in magenta in both matrices). A similar reasoning applies for the other three columns (where again the three bad occurrences are emphasized in magenta).

Observe that every property which can be encoded using 0's and 1's can also be used to define a suitable list. The next definition applies to a very general setting.

Definition 4. Let C be a nonempty set, and $((A_t, k_t))_{t=1,\ldots,m}$ a nonempty family of elements in $2^C \times C$. For all $t \in \{1, \ldots, m\}$, fix $\S_t \in Q^n$ and define

$$b_t := \begin{cases} 1 & \text{if } k_t \in A_t \\ 0 & \text{if } k_t \notin A_t. \end{cases}$$

We call $((\S_t, b_t))_{t=1,\dots,m}$ the list *induced by* $\{A_t : t \in \{1,\dots,m\}\}$, $\{k_t : t \in \{1,\dots,m\}\}$, and $\{\S_t : t \in \{1,\dots,m\}\}$. Whenever the setting is clear, we omit mentioning the generating sets.

The next example exhibits an algebraic instance of a (suitable) list induced by prime numbers.

Example 7. For $m = n \geqslant 1$, let $\{p_1,\dots,p_m\}$ be the set of the first m prime numbers. Take $B := \mathbb{N}$, and for each $t \in \{1,\dots,m\}$, let $A_t := p_t\mathbb{N}$ be the set of all multiples of p_t. (Thus, for instance, A_1 is the set of all even numbers.) Suppose $\S^t \in Q^m$ and $k_t \in \mathbb{N}$ are given for all $t \in \{1,\dots,m\}$. The list induced by the A_t's, k_t's, and the $\S_t$'s is $((\S_t, b_t))_{t=1,\dots,m}$, where $b_t = 1$ if p_t divides k_t, and $b_t = 0$ otherwise. This list becomes suitable whenever for every j in $\{1,\dots,m\}$, there is at most one t such that either $\S^j_t = \exists$ and p_t divides k_t, or $\S^j_t = \forall$ and p_t does not divide k_t. Now suppose $\S_t \in Q^m = Q^n$ is the string whose components are all '$\forall$' except for the tth, which is '$\exists$'. In other words, the entries of the $n \times n$ matrix obtained by listing $\S_1, \S_2, \dots, \S_m$ by row are always '$\forall$', except on the main diagonal, where there are '$\exists$'. The list $((\S_t, b_t))_{t=1,\dots,m}$ is suitable if and only if p_t divides k_t for all t's. In particular, the list $((\S_t, b_t))_{t=1,\dots,m}$ induced by $\{p_t\mathbb{N} : t \in \{1,\dots,m\}\}$ and $\{k_t : (\forall t \in \{1,\dots,m\})\ k_t = p_t c_t \wedge c_t \in \mathbb{N}\}$ is suitable.

In Section 5, we shall consider building lists in Q^n induced by a family of binary relations and a family of ordered pairs of elements. That is, the set C in Definition 4 is X^2, the sets $A_t \subseteq C$ are binary relations on X, and the elements $k_t \in C$ are ordered pairs: see Lemmas 5 and 6.

Next, we prove some algebraic and order-theoretic properties of suitable lists.

Lemma 1. *Let $S = ((\S_t, b_t))_{t=1,\dots,m}$ be a list in Q^n. If S is suitable, then the following properties hold:*

(1) *for all distinct $p, q \in \{1,\dots,m\}$ such that $b_p = b_q = 1$, the sum $\S_p + \S_q$ is defined;*
(2) *for all distinct $p, q \in \{1,\dots,m\}$ such that $b_p = b_q = 0$, the sum $\overline{\S_p} + \overline{\S_q}$ is defined;*
(3) *for all distinct $p, q \in \{1,\dots,m\}$ such that $b_p = 1$ and $b_q = 0$, the sum $\S_p + \overline{\S_q}$ is defined;*
(4) *for all $d, s \geq 0$ such that $d + s = t$, the sum $\S_{i_1} + \cdots + \S_{i_d} + \overline{\S_{j_1}} + \cdots + \overline{\S_{j_s}}$, where $b_{i_l} = 1$ for each l such that $1 \leq l \leq d$, and $b_{j_r} = 0$ for all r with $1 \leq r \leq s$, is defined.*[8]

Moreover, if every string $\S_t$ is distinct from both $\forall\ldots\forall$ and $\exists\ldots\exists$, then the addends of the sums are different in each case.[9]

Proof. We prove (1)–(4) by way of contradiction.

For (1), suppose there exist distinct $p, q \in \{1,\dots,m\}$ such that $b_p = b_q = 1$ and $\S_p + \S_q$ is not defined. It follows that there exists $j \in \{1,\dots,n\}$ such that $\S^j_p = \S^j_q = \exists$. This is impossible, because $b_p = b_q = 1$ implies that both $\S^j_p$ and $\S^j_q$ are bad occurrences, contradicting the suitability of S. Moreover, if $\S_p$ is equal to $\S_q$ and different from $\forall\ldots\forall$ and $\exists\ldots\exists$, then there is $j \in \{1,\dots,n\}$ such that $\S^j_q = \S^j_p = \exists$, which again contradicts suitability. Part (2) is similar to (1).

Next, we prove (3). Suppose there are distinct p, q in $\{1,\dots,m\}$ such that $\S_p + \overline{\S_q}$ is not defined, $b_p = 1$, and $b_q = 0$. It follows that there is $j \in \{1,\dots,n\}$ such that $\S^j_p = \exists$ and $\S^j_q = \forall$. This is a contradiction, because S is suitable. The proof that the addends of the sum are different whenever both strings of quantifiers are different from $\exists\ldots\exists$ and $\forall\ldots\forall$ is similar to that of (1).

Finally, we prove (4). Suppose the sum is not defined. We obtain $\S$ and $\S'$ such that either $\S + \S'$ or $\S + \overline{\S'}$ or $\overline{\S} + \overline{\S'}$ is not defined. Now apply (1), (2), or (3) to get a contradiction. □

Lemma 2. *Let $((\S_t, b_t))_{t=1,\dots,m}$ be a building list in $Q^n \setminus \{\forall\ldots\forall, \exists\ldots\exists\}$ such that $b_h = 0$ for some $h \in \{1,\dots,m\}$. If S is suitable, then the following properties hold:*

(1) *for all $l \in \{1,\dots,m\}$ such that $b_l = 1$, we have $\S_l \leqslant \S_h$;*
(2) *for all $l \in \{1,\dots,m\}$ such that $l \neq h$ and $b_l = 0$, we have $\overline{\S_l} \leqslant \S_h$;*
(3) *if $p \neq q$ and $b_p = b_q = 1$, then $\S_p + \S_q \leqslant \S_h$;*
(4) *if p, q, h are all distinct and $b_p = b_q = 0$, then $\overline{\S_p} + \overline{\S_q} \leqslant \S_h$;*
(5) *if $p \neq q \neq h$, $b_p = 1$, and $b_q = 0$, then $\S_p + \overline{\S_q} \leqslant \S_h$;*
(6) *for all $d, s \geq 0$ such that $d + s = t$, if $b_{i_l} = 1$ and $i_l \neq h$ for each l with $1 \leq l \leq d$, and $b_{j_r} = 0$ and $j_r \neq h$ for all r with $1 \leq r \leq s$, then $\S_{i_1} + \cdots + \S_{i_d} + \overline{\S_{j_1}} + \cdots + \overline{\S_{j_s}} \leq \S_h$.*[10]

Proof. For (1), let $l \in \{1,\dots,m\}$ be such that $b_l = 1$. Suppose there is $j \in \{1,\dots,n\}$ such that $\S^j_l = \exists$. Since $b_h = 0$, suitability yields $\S^j_h = \exists$, otherwise we would have two bad occurrences in column j. By the arbitrariness of j, we obtain $\S_l \leqslant \S_h$, as claimed.

For (2), suppose there exists $j \in \{1,\dots,n\}$ such that $\overline{\S^j_l} = \exists$, hence $\S^j_l = \forall$. Again apply suitability to get $\S^j_h = \exists$. The claim follows.

Next, we prove (3). By Lemma 1(1), $\S_p + \S_q$ is defined. Suppose there is $j \in \{1,\dots,n\}$ such that $(\S_p + \S_q)^j = \exists$. It follows that either $\S^j_p = \exists$ or $\S^j_q = \exists$. Now part (1) yields $\S^j_h = \exists$, and we are done.

The arguments to prove (4) and (5) are similar. Part (6) is proved by induction. □

The following simple consequences of Lemmas 1 and 2, which hold under additional properties of the list, will be useful in the next section.

Corollary 1. *Let $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ be a 2^n-tuple of binary relations on X satisfying property M2, and m an integer such that $1 \leqslant m \leqslant n$. Moreover, let $((\S_t, b_t))_{t=1,\dots,m}$ be a building list in Q^n such that $(\S_h, 0) \in S$ for some $h \in \{1,\dots,m\}$. Suppose that S is ordered in way that $b_t = 1$ if and only if $t \leq d$ for some nonnegative integer d such that $(\S_h, 0)$ is the last entry in the list. If S is suitable, then*

$$\succsim^{\S_1 + \cdots + \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_{m-1}}} \subseteq \succsim^{\S_m}.$$

Proof. The sum $\S_1 + \cdots + \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_{m-1}}$ is defined by Lemma 1(4), and the inequality $\S_1 + \cdots + \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_{m-1}} \leqslant \S_m$ holds by Lemma 2. The claim follows from property M2. □

Remark 5. By Remark 3, Corollary 1 also holds if the addends of $\S_1 + \cdots \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_{m-1}}$ are listed in a different order.

Corollary 2. *Let $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ be a 2^n-tuple of binary relations on X satisfying properties M2 and M3, and m an integer such that $1 \leqslant m \leqslant n$. Moreover, let $((\S_t, b_t))_{t=1,\dots,m}$ be a list in Q^n, which is ordered in a way such that $b_t = 1$ if and only if $t \leq d$ for some nonnegative integer d. If S is suitable, then*

$$\succsim^{\S_1} \circ \ldots \circ \succsim^{\S_d} \circ \succsim^{\overline{\S_{d+1}}} \circ \ldots \circ \succsim^{\overline{\S_m}} \subseteq \succsim^{\S_1 + \cdots + \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_m}} \subseteq \succsim^{\exists\ldots\exists}.$$

[8] If $d = 0$, we abuse notation, and let the sum be $\overline{\S_{j_1}} + \cdots + \overline{\S_s}$. Similarly, if $s = 0$, we consider $\S_{i_1} + \cdots + \S_{i_d}$.

[9] For instance, in case (3) we have $\S_p \neq \overline{\S_q}$.

[10] As in Lemma 1, we are slightly abusing notation: see Footnote 8.

Proof. By Lemma 1(4), the sum $\S_1 + \cdots + \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_m}$ is defined. Now the first inclusion follows from property M3 applied $(m-1)$ times, whereas the second inclusion is an immediate consequence of property M2. □

Remark 6. Corollary 2 also holds for every permutation of the composition $\succsim^{\S_1} \circ \ldots \circ \succsim^{\S_d} \circ \succsim^{\overline{\S_{d+1}}} \circ \ldots \circ \succsim^{\overline{\S_m}}$.

5. Interpolating preorders

In this section we define two notions, and prove some related properties. These notions are:

(1) a *parsimonious extension* of a preference relation, which is a set-theoretic superset of a given relation having the same asymmetric part;
(2) an *interpolating relation* for a pair of nested preferences, which is a binary relation that is set-theoretically in between the two components of the original pair.

Definition 5. Let $\succsim$ be a weak preference on X. A binary relation $\succsim'$ on X such that $\succsim \subseteq \succsim'$ and $x \succsim' y$ implies $x \succsim y$ or $x \perp y$ is called a *parsimonious extension* of $\succsim$.

Note that a parsimonious extension never makes a strict preference of x over y into an indifference between x and y.

Definition 6. Let $(\succsim_1, \succsim_2)$ be a pair of weak preferences on X such that $\succsim_2$ extends $\succsim_1$. A binary relation $\succsim$ on X is said to be $(\succsim_1, \succsim_2)$*-interpolating* if $\succsim_1 \subseteq \succsim \subseteq \succsim_2$.[11]

To start, we obtain interpolating preorders for NaP-preferences.

Lemma 3. *Let $\left(\succsim^N, \succsim^P\right)$ be a NaP-preference on X. If $\succsim$ is a $\left(\succsim^N, \succsim^P\right)$-interpolating preorder and $\succsim'$ is a preorder that parsimoniously extends $\succsim$, then also $\succsim'$ is $\left(\succsim^N, \succsim^P\right)$-interpolating.*

Proof. Let $\succsim$ be a preorder such that $\succsim^N \subseteq \succsim \subseteq \succsim^P$, and $\succsim'$ a preorder that parsimoniously extends $\succsim$. Clearly $\succsim^N \subseteq \succsim'$. To prove that also $\succsim' \subseteq \succsim^P$ holds, suppose $x \succsim' y$, hence either $x \succsim y$ or $x \perp y$. If $x \succsim y$, then $x \succsim^P y$ by hypothesis. Otherwise, we have $x \perp y$, and mixed completeness yields $x \succsim^P y$. □

In order to obtain interpolating preorders that are *total*, we need the Axiom of Choice in its equivalent form of Zorn's Lemma.

Lemma 4 (*AC*)**.** *Let $\left(\succsim^N, \succsim^P\right)$ be a NaP-preference on X. For any $\left(\succsim^N, \succsim^P\right)$-interpolating preorder $\succsim$, there is a parsimonious extension $\succsim'$ of $\succsim$ that is a $\left(\succsim^N, \succsim^P\right)$-interpolating total preorder.*

Proof. Let $\succsim$ be a preorder on X such that $\succsim^N \subseteq \succsim \subseteq \succsim^P$. By Lemma 3, we only need to prove the existence of a parsimonious extension $\succsim'$ of $\succsim$ that is a total preorder.

Let $\mathscr{P}$ be the set of all parsimonious extension of $\succsim$ that are preorders. Note that $\mathscr{P} \neq \varnothing$, because $\succsim$ belongs to it. Suppose $\mathcal{C}$ is a $\subseteq$-chain of elements of $\mathscr{P}$. Observe that $\bigcup \mathcal{C}$ is a parsimonious extension of $\succsim$, and is a preorder. By Zorn's lemma, we obtain a maximal element $\succsim'$ of $\mathscr{P}$. To complete the proof, we show that $\succsim'$ is total.

Toward a contradiction, suppose there are $x, y \in X$ such that neither $x \succsim' y$ nor $y \succsim' x$ holds; in particular, $x \perp y$. Define $\succsim''$ as the transitive closure of $\succsim' \cup \{(x, y)\}$. We shall prove $\succsim'' \in \mathscr{P}$, which contradicts the maximality of $\succsim'$. It suffices to show that $\succsim''$ is parsimonious. To that end, suppose $z, w \in X$ are such that $z \succsim'' w$ and $\neg(z \perp w)$. Transitivity of $\succsim''$ implies that $\neg(w \succ z)$, since otherwise $z \succsim'' w \succ z$ would give $z \succ z$. Now $\neg(z \perp w)$ and $\neg(w \succ z)$ yield $z \succsim w$, which proves that $\succsim''$ is parsimonious. □

The next two lemmas establish the existence of suitable interpolating preorders for the first and the last components of any 2^n-tuple of binary relations satisfying properties M1–M4.

Lemma 5 (Interpolation Lemma)**.** *Let $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ be a 2^n-tuple of binary relations on X satisfying properties M1–M4, and m an integer such that $1 \leqslant m \leqslant n$. For all $t \in \{1, \ldots, m\}$, let $\S^t = \S^t_1 \ldots \S^t_n \in Q^n$ and $k_t = (x_t, y_t) \in X^2$. Consider the list $((\S_t, b_t))_{t=1,\ldots,m}$ induced by $\{\succsim^{\S_t} : t \in \{1, \ldots, m\}\}$, $\{k_t : t \in \{1, \ldots, m\}\}$, and $\{\S_t : t \in \{1, \ldots, m\}\}$. If S is suitable, then there is an $\left(\succsim^{\forall\ldots\forall}, \succsim^{\exists\ldots\exists}\right)$-interpolating preorder $\succsim_{k_1\ldots k_m}$ on X satisfying the following properties for all $t \in \{1, \ldots, m\}$:*

(i) $k_t \in \succsim^{\S_t} \implies k_t \in \succsim_{k_1\ldots k_m}$;
(ii) $k_t \notin \succsim^{\S_t} \implies \left(k_t \notin \succsim_{k_1\ldots k_m} \wedge (y_t, x_t) \in \succsim_{k_1\ldots k_m}\right)$.

Proof. For brevity, denote by $[m]$ the set $\{1, \ldots, m\}$. There are seven possible cases for the list S, which are mutually exclusive and exhaustive:

(1) for all $t \in [m]$, $(\S_t, b_t) \in \left\{(\forall\ldots\forall, 1), (\exists\ldots\exists, 0)\right\}$;
(2) for all $t \in [m]$, $(\S_t, b_t) \in \left\{(\forall\ldots\forall, 0), (\exists\ldots\exists, 1)\right\}$;
(3) there is $h \in [m]$ such that $(\S_h, b_h) \in \left\{(\forall\ldots\forall, 1), (\exists\ldots\exists, 0)\right\}$, there is $k \in [m]$ such that $(\S_k, b_k) \notin \left\{(\forall\ldots\forall, 1), (\exists\ldots\exists, 0)\right\}$, for all $t \in [m]$, $(\S_t, b_t) \notin \left\{(\forall\ldots\forall, 0), (\exists\ldots\exists, 1)\right\}$;
(4) there is $h \in [m]$ such that $(\S_h, b_h) \in \left\{(\forall\ldots\forall, 0), (\exists\ldots\exists, 1)\right\}$, there is $k \in [m]$ such that $(\S_k, b_k) \notin \left\{(\forall\ldots\forall, 0), (\exists\ldots\exists, 1)\right\}$, for all $t \in [m]$, $(\S_t, b_t) \notin \left\{(\forall\ldots\forall, 1), (\exists\ldots\exists, 0)\right\}$;
(5) there is $h \in [m]$ such that $(\S_h, b_h) \in \left\{(\forall\ldots\forall, 1), (\exists\ldots\exists, 0)\right\}$, there is $k \in [m]$ such that $\S_k \notin \left\{\forall\ldots\forall, \exists\ldots\exists\right\}$, there is $l \in [m]$ such that $(\S_l, b_l) \in \left\{(\forall\ldots\forall, 0), (\exists\ldots\exists, 1)\right\}$;
(6) there is $h \in [m]$ such that $(\S_h, b_h) \in \left\{(\forall\ldots\forall, 1), (\exists\ldots\exists, 0)\right\}$, there is no $k \in [m]$ such that $\S_k \notin \left\{\forall\ldots\forall, \exists\ldots\exists\right\}$, there is $l \in [m]$ such that $(\S_l, b_l) \in \left\{(\forall\ldots\forall, 0), (\exists\ldots\exists, 1)\right\}$;
(7) for all $t \in [m]$, $\S_t \notin \left\{\forall\ldots\forall, \exists\ldots\exists\right\}$.

Here we only examine the general case, namely (7), and leave the others to the reader.[12] Suppose S is suitable. Define an extension $\succsim$ of $\succsim^{\forall\ldots\forall}$ by setting

$$\succsim \;:=\; \succsim^{\forall\ldots\forall} \;\cup \bigcup_{t\in[m]} \left\{(x_t, y_t) : k_t \in \succsim^{\S_t}\right\} \;\cup \bigcup_{t\in[m]} \left\{(y_t, x_t) : k_t \notin \succsim^{\S_t}\right\}. \tag{2}$$

By construction, $\succsim$ extends $\succsim^{\forall\ldots\forall}$ and contains all pairs in any $\succsim^{\S_t}$; moreover, if a pair (x_t, y_t) is not in $\succsim^{\S_t}$, then its reverse pair (y_t, x_t) belongs to $\succsim$. Let $\succsim_{k_1\ldots k_m}$ be the transitive closure of $\succsim$. To complete the proof, it suffices to show

(A) $\succsim_{k_1\ldots k_m}$ is included in $\succsim^{\exists\ldots\exists}$, and
(B) for all $t \in [m]$, $\neg(x_t \succsim^{\S_t} y_t)$ implies $\neg(x_t \succsim_{k_1\ldots k_m} y_t)$.

First, we prove (A). By Corollary 2 and Remark 6, we have

$$\succsim^{\S_1} \circ \ldots \circ \succsim^{\S_d} \circ \succsim^{\overline{\S_{d+1}}} \circ \ldots \circ \succsim^{\overline{\S_m}} \subseteq \succsim^{\S_1 + \cdots + \S_d + \overline{\S_{d+1}} + \cdots + \overline{\S_m}} \subseteq \succsim^{\exists\ldots\exists} \tag{3}$$

[11] The term 'interpolating' may appear overinflated at the moment. However, the employed terminology becomes more appropriate as soon as we make this interpolating preference contain or avoid some ordered pairs: see the *Interpolation Lemma* (Lemma 5) and the *Total Interpolation Lemma* (Lemma 6) below.

[12] Cases (1) and (2) are routine, cases (4) and (5) are impossible by the suitability of S, and the nontrivial parts of cases (3) and (6) are subsumed by case (7).

for any permutation of the composition. Suppose $x, y \in X$ are such that $x \succsim_{k_1 \ldots k_m} y$. By the definition of transitive closure, there is a $\succsim$-chain of elements starting at x and ending at y; let

$$x \succsim c_1 \succsim \ldots \succsim c_v \succsim y$$

be a minimal one. It follows that any binary relation $\succsim^{\S_{h_l}}$ or $\overline{\succsim^{\S_{h_l}}}$ appears at most once in such a chain. (For instance, if there are $p, q \in [m]$, with $p < q$, and $\succsim^{\S_t}$ such that $c_p \succsim^{\S_t} c_{p+1}$, $c_q \succsim^{\S_t} c_{q+1}$, $c_p = c_q = x_t$, and $c_{p+1} = c_{q+1} = y_t$, then we obtain $c_p \succsim^{\S_t} c_{q+1}$, and so we can shorten the chain, contradicting maximality.) Hence, we get $(x, y) \in \succsim^{\S_{h_1}} \circ \ldots \circ \succsim^{\S_{h_p}} \circ \overline{\succsim^{\S_{h_{p+1}}}} \circ \ldots \circ \overline{\succsim^{\S_{h_r}}}$ (or a permutation of such a composition), and we are done.

Finally, we prove (B). Toward a contradiction, suppose there are distinct elements $x, y \in X$ and an index h such that $\neg(x_h \succsim^{\S_h} y_h)$ but $x_h \succsim_{k_1 \ldots k_m} y_h$. We claim that $\neg(x_h \succsim y_h)$. Indeed, if $x_h \succsim y_h$, then there is t such that either $(x_h, y_h) = (x_t, y_t)$ or $(x_h, y_h) = (y_t, x_t)$ holds. In the first case, we get $x_h \succsim^{\S_t} y_h$ by definition of $\succsim$, whence $\succsim^{\S_t} \subseteq \succsim^{\S_h}$ by Lemma 2(1) and property M2, contradicting the hypothesis. By a similar argument, also the second case is impossible. From $\neg(x_h \succsim y_h)$ we conclude that the pair (x_h, y_h) has been added to $\succsim_{k_1 \ldots k_m}$ in the process of taking the transitive closure of $\succsim$. Therefore, there is a nontrivial $\succsim$-chain of the type

$$x_h \succsim c_1 \succsim \ldots \succsim c_t \succsim y_h.$$

As in the proof of (A), we obtain a minimal chain in which every binary relation $\succsim^{\S_{h_l}}$ or $\overline{\succsim^{\S_{h_l}}}$ appears at most once. Using (3), Corollary 1 and Remark 5 yield $\succsim^{\S_{t_1} + \cdots + \S_{t_p} + \overline{\S_{t_{p+1}}} + \cdots + \overline{\S_{t_r}}} \subseteq \succsim^{\S_h}$, which is impossible because $\neg(x_h \succsim^{\S_h} y_h)$. □

Again, in order to extend the existence result obtained in Lemma 5 to total preorders, we need to appeal to the Axiom of Choice.

Lemma 6 (Total Interpolation Lemma)**.** (AC) *Let $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ be a 2^n-tuple of binary relations on X satisfying properties M1–M4, and m be an integer such that $1 \leqslant m \leqslant n$. For all $t \in \{1, \ldots, m\}$, let $k_t \in X^2$ and $\S^t = \S^t_1 \ldots \S^t_n \in Q^n$. Consider the building list $((\S_t, b_t))_{t=1,\ldots,m}$ induced by $\{\succsim^{\S^t} : t \in \{1, \ldots, m\}\}$, $\{k_t : t \in \{1, \ldots, m\}\}$, and $\{\S_t : t \in \{1, \ldots, m\}\}$. If S is suitable, then there is a $\left(\succsim^{\forall \ldots \forall}, \succsim^{\exists \ldots \exists}\right)$-interpolating total preorder $\succsim_{k_1 \ldots k_m}$ on X satisfying the following properties for all $h \in \{1, \ldots, m\}$:*

(i) $k_h \in \succsim^{\S_h} \implies k_h \in \succsim_{k_1 \ldots k_m}$;
(ii) $k_h \notin \succsim^{\S_h} \implies k_h \notin \succsim_{k_1 \ldots k_m}$.

Proof. Suppose S is suitable. By Lemma 5, there is a preorder $\succsim_{k_1 \ldots k_m}$ satisfying (i) and (ii); moreover, if $\neg(x_h \succsim^{\S_h} y_h)$ for some h in $\{1, \ldots, m\}$, then $y_h \succsim_{k_1 \ldots k_m} x_h$. Using Definition 5 and Lemma 4, we obtain a total preorder satisfying (i) and (ii). (To show that, simply observe that $(\succsim^{\forall \ldots \forall}, \succsim^{\exists \ldots \exists})$ is a NaP-preference, and the inclusions $\succsim^{\forall \ldots \forall} \subseteq \succsim_{k_1 \ldots k_m} \subseteq \succsim^{\exists \ldots \exists}$ hold.) □

6. Characterization

Here we prove that Conjecture 1 holds true.

Definition 7. We call $Z = Q^n \times X^2 \times \{0, 1\}$ the set of *tasks*.

Let $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ be a modal preference structure on X having rank n. For each $z \in Z$, there is a unique associated string $\S_z = \S^1_z \ldots \S^n_z$, and so a unique binary relation $\succsim^{\S_z} = \succsim^{\S^1_z \ldots \S^n_z}$ in $\mathscr{M}$. Moreover, we have a unique bit $b_z \in \{0, 1\}$, and a unique pair $k_z = (x, y)$. In what follows, we show that every subset V of Z induces a list in Q^n. This is done in two steps.

For each $v \in V$, denote by $\S_v$ the associated string of quantifiers, and by k_v the associated pair in X^2. As first step, we use V to obtain sets $A_v \subseteq X^2$ and pairs $k_v \in X^2$, for each $v \in Z$. (Note that A_v is equal to $\succsim^{\S_v}$ for all $v \in V$.) As second step, we define the list induced by $\{A_v : v \in V\}$, $\{k_v : v \in V\}$, and $\{\S_v : v \in V\}$.[13]

Lemma 7. *Let $V = \{v_1, \ldots, v_n\}$ be a subset of $Z = Q^n \times X^2 \times \{0, 1\}$, and $((\S_{v_t}, b_{v_t}))_{t=1,\ldots,n}$ the list induced by V. Consider the following conditions, where $t \in \{1, \ldots, n\}$:*

(1) *if $k_{v_t} \in \succsim^{\S_{v_t}}$, then for all $j \in \{1, \ldots, n\}$, $\S^j_{v_t} = \exists$ implies $v_t = v_j$;*
(2) *if $k_{v_t} \notin \succsim^{\S_{v_t}}$, then for all $j \in \{1, \ldots, n\}$, $\S^j_{v_t} = \forall$ implies $v_t = v_j$.*

Then the subset D of $\left\{k_{v_1}, \ldots, k_{v_n}\right\}$ obtained by removing from D all k_{v_t}'s such that either (1) or (2) fails, along with the set $\left\{\succsim^{\S_d} : d \in D\right\}$, induces a suitable list S.

Proof. Toward a contradiction, suppose that S is not suitable. Thus there are distinct $k_v, k_{v'} \in D$ and $j \in \{1, \ldots, n\}$ such that $(\S^j_{k_v}, b_{k_v})$ and $(\S^j_{k_{v'}}, b_{k_{v'}})$ belong to $\{(\forall, 0), (\exists, 1)\}$. Three possible cases arise. We show that each case leads to a contradiction.

To start, we examine the case $(\S^j_{k_v}, b_{k_v}) = (\S^j_{k_{v'}}, b_{k_{v'}}) = (\exists, 1)$, hence $\S^j_{k_v} = \S^j_{k_{v'}} = \exists$, $k_v \in \succsim^{\S_v}$, and $k_{v'} \in \succsim^{\S_{v'}}$. By condition (1) in the hypothesis, we derive $v = v'$, and so $k_v = k_{v'}$, which contradicts the assumption. The analysis of case $(\S^j_{k_v}, b_{k_v}) = (\S^j_{k_{v'}}, b_{k_{v'}}) = (\forall, 0)$ is similar, using condition (2).

Next, we examine the case $(\S^j_{k_v}, b_{k_v}) = (\exists, 1)$ and $(\S^j_{k_{v'}}, b_{k_{v'}}) = (\forall, 0)$, hence $\S^j_{k_v} = \exists$, $\S^j_{k_{v'}} = \forall$, $k_v \in \succsim^{\S_v}$, and $k_{v'} \notin \succsim^{\S_{v'}}$. By condition (1), we get $v = v_j$. Similarly, by condition (2), we get $v' = v_j$. However, this implies $v = v'$, which is impossible. □

The notion of task plays a key role in the proof, because it allows us to obtain an indexing that is more complex than the usual indexing by integers.

Definition 8. If $k \in X^2$ is in some $\succsim^{\S^1 \ldots \S^n}$ with $\succsim^{\S^1 \ldots \S^n} \in \mathfrak{M}$ for some $\mathfrak{M}$ modal preference structure, then a string of tasks $z_1 \ldots z_n$, where if $\S^i = \exists$ then $k_{z_i} = k$ and $b_{z_i} = 1$, is said to be *induced* by k. Furthermore, if k is in some set $X^2 \setminus \succsim^{\S^1 \ldots \S^n}$, then a string $k_{z_1} \ldots k_{z_n}$, where if $x_i = \forall$ then $k_{z_i} = k$ and $b_{z_i} = 0$, is still said to be induced by k.

Lemma 8 (*AC*)**.** *Let $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ be a 2^n-tuple of binary relations on X satisfying properties M1–M4. For any $(z_1, \ldots, z_n) \in Z^n$, there is a $\left(\succsim^{\forall \ldots \forall}, \succsim^{\exists \ldots \exists}\right)$-interpolating total preorder $\succsim_{z_1 \ldots z_n}$ on X with the following properties:*

(1) *if $k_{z_t} \in \succsim^{\S_{z_t}}$ and for all $j \in \{1, \ldots, n\}$, $\S^j_{z_t} = \exists$ implies $k_{z_t} = k_{z_j}$, then $k_{z_t} \in \succsim_{z_1 \ldots z_n}$;*
(2) *if $k_{z_t} \notin \succsim^{\S_{z_t}}$ and for all $j \in \{1, \ldots, n\}$, $\S^j_{z_t} = \forall$ implies $k_{z_t} = k_{z_j}$, then $k_{z_t} \notin \succsim_{z_1 \ldots h_n}$.*

Moreover, if $z_1 \ldots z_n$ is a string of tasks induced by some k belonging to some $\succsim^{x_1 \ldots x_n}$ (resp. some $X \setminus \succsim^{x_1 \ldots x_n}$), then there is a total preorder $\succsim_{z_1 \ldots z_n}$ such that $k \in \succsim_{z_1 \ldots z_n}$ (resp. $k \notin \succsim_{z_1 \ldots z_n}$).

Proof. Apply Lemma 7 to obtain a suitable list, and then apply Lemma 6 to get the required total preorders. To prove the last statement, note that k satisfies either (1) or (2). □

[13] Note that here we are extending Definition 4 to a more general notion. In fact, Definition 4 uses a set of integers to index the two generating sets, whereas here we are using a generic set V for the indexing.

We are ready.

Ttheorem 1 (*AC*). *For any integer $n \geqslant 1$, the following statements are equivalent for a 2^n-tuple $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ of binary relations on X:*

(i) *$\mathscr{M}$ is a modal preference structure on X;*
(ii) *$\mathscr{M}$ satisfies properties M1–M4.*

Proof. In what follows, we denote $Q^n_{-\exists} := Q^n \setminus \{\exists\ldots\exists\}$ and $Q^n_{-\forall} := Q^n \setminus \{\forall\ldots\forall\}$.

(i)$\Longrightarrow$(ii): Suppose $\mathscr{M}$ be a modal preference structure. Therefore, there is a family

$$\mathscr{T} = \left\{\succsim_{k_1\ldots k_n} : (k_1, \ldots, k_n) \in K_1 \times \cdots \times K_n\right\}$$

of total preorders on X with the property that, for all $x, y \in X$ and $\S = \S^1\ldots\S^n \in Q^n$,

$$x \succsim^{\S^1\ldots\S^n} y \quad\Longleftrightarrow\quad (\S^1 k_1 \in K_1)\ldots(\S^n k_n \in K_n)\, x \succsim_{k_1\ldots k_n} y.$$

Below we prove that properties M1–M4 hold. Let $\S = \S^1\ldots\S^n$ and $\S' = \S^{1\prime}\ldots\S^{n\prime}$ be two lists of n quantifiers.

(M1) By definition of modal preference structure, $\succsim^{\forall\ldots\forall}$ is the intersection of all total preorders in $\mathscr{T}$, and therefore it is a preorder.

(M2) Suppose $\S \leqslant \S'$. We show that $\succsim^{\S}$ is contained in $\succsim^{\S'}$. Let $x, y \in X$ be such that $x \succsim^{\S} y$. From $(\S^1 k_1 \in K_1)\ldots(\S^n k_n \in K_n) x \succsim_{k_1\ldots k_n} y$ we get $(\S^{1'} k_1 \in K_1)\ldots(\S^{n'} k_n \in K_n) x \succsim_{k_1\ldots k_n} y$, and so $x \succsim^{\S'} y$.

(M3) Assume $\S^i + \S^{i'}$ is defined for all $i \in \{1, \ldots, n\}$. We prove $\succsim^{\S} \circ \succsim^{\S'} \subseteq \succsim^{\S+\S'}$. To that end, suppose $x \succsim^{\S} y \succsim^{\S'} z$. By hypothesis, we obtain

$$(\S^1 k_1 \in K_1)\ldots(\S^n k_n \in K_n)\, x \succsim_{k_1\ldots k_n} y$$
$$\text{and}\quad (\S^{'1} k_1 \in K_1)\ldots(\S^{n'} k_n \in K_n)\, y \succsim_{k_1\ldots k_n} z,$$

and so

$$((\S^1 + \S^{1'}) k_1 \in K_1)\ldots((\S^n + \S^{n'}) k_n \in K_n)\, x \succsim_{k_1\ldots k_n} y$$

as well as

$$((\S^1 + \S^{1'}) k_1 \in K_1)\ldots((\S^n + \S^{n'}) k_n \in K_n)\, y \succsim_{k_1\ldots k_n} z.$$

By the transitivity of the preorders $\succsim_{k_1\ldots k_n}$, we derive

$$((\S^1 + \S^{1'}) k_1 \in K_1)\ldots((\S^n + \S^{n'}) k_n \in K_n)\, x \succsim_{k_1\ldots k_n} z$$

and so $x \succsim^{\S+\S'} z$, as claimed.

(M4) Let $x, y \in X$ be such that $\neg(x \succsim^{\S} y)$. We prove $y \succsim^{\overline{\S}} x$. By hypothesis, the formula

$$(\S^1 k_1 \in K_1)\ldots(\S^n k_n \in K_n)\, x \succsim_{k_1\ldots k_n} y$$

fails, hence

$$(\overline{\S}^1 k_1 \in K_1)\ldots(\overline{\S}^n k_n \in K_n)\, \neg(x \succsim_{k_1\ldots k_n} y)$$

holds true. However, since $\succsim_{k_1\ldots k_n}$ are complete, we obtain that $y \succsim_{k_1\ldots k_n} x$, hence $y \succsim^{\overline{\S}} x$.

We conclude that (i) implies (ii).

(ii)$\Longrightarrow$(i): Suppose $\mathscr{M} = \left(\succsim^{\S} : \S \in Q^n\right)$ satisfies properties M1–M4. Set $K_i := Z$ for all $i \in \{1, \ldots, n\}$. For all $z_1, \ldots, z_n \in Z$, apply Lemma 8 to obtain the total preorder $\succsim_{z_1\ldots z_n}$.

Now suppose $x \succsim^{\S^1\ldots\S^n} y$. We need to prove $(\S^1 z_1 \in K_1)\ldots(\S^n z_n \in K_n)\, x \succsim_{z_1\ldots z_n} y$. To that end, if $\S^i = \exists$, fix $k_{z_i} = (x, y)$, and note that the claim holds if we show that for every task $z_1\ldots z_n$ induced by (x, y), we have $x \succsim_{z_1\ldots z_n} y$. Lemma 8 tells us that every preorder indexed by such a string contains (x, y), and we are done.

For the reverse implication, suppose

$$(\S^1 z_1 \in K_1)\ldots(\S^n z_n \in K_n)\, x \succsim_{z_1\ldots z_n} y.$$

Toward a contradiction, assume $\neg(x \succsim^{\S^1\ldots\S^n} y)$, hence $(x, y) \in X^2 \setminus \succsim^{\S^1\ldots\S^n}$. For every i such that $\S_i = \forall$, fix $k_{z_i} = (x, y)$. Observe that the defined string must satisfy $x \succsim_{z_1\ldots z_n} y$. However, Lemma 8 tells us that $(x, y) \notin \succsim_{z_1\ldots z_n}$, which is impossible. $\square$

7. Tensor product and layouts

In this section, we define an operation of 'tensor product' on the collection of all modal preference structures on a given set of alternatives. This operation delivers an MPS starting from MPSs having lower rank. The definition of tensor product is recursive, and proceeds in two steps.

The base step is 'homogeneous' and only goes 'one rank up', associating an MPS of rank $n+1$ to any finite family of MPSs of rank n. It turns out that the converse holds as well, namely any MPS of rank $n+1$ always arises – but not in a unique way – as a tensor product of a homogeneous family of MPSs having rank n.

For the recursive step, we need additional 'instructions', which guide the amalgamation process of the given family of MPSs into forming a new, more complex MPS. These instructions are embodied in a rooted tree such that its leaves are associated to the MPSs in the given family, and its root is associated to the new MPS of higher rank.

Definition 9. Denote by $\mathsf{MOD}_n(X)$ the collection of all modal preference structures of rank n on the set X. Let $\{\mathscr{M}_i : i \in I\}$ be a finite family of elements in $\mathsf{MOD}_n(X)$.[14] The *tensor product* of $\{\mathscr{M}_i : i \in I\}$ is the modal preference structure on X given by $\otimes_{i\in I}\mathscr{M}_i = \left(\succsim^{\S'} : \S' \in Q^{n+1}\right)$, where the generic binary relation $\succsim^{\S'}$ in $\otimes_{i\in I}\mathscr{M}_i$ is defined by

$$\succsim^{\S'} := \begin{cases} \bigcap_{i\in I} \succsim_i^{\S} & \text{if} \quad \S' = \forall\S \\ \bigcup_{i\in I} \succsim_i^{\S} & \text{if} \quad \S' = \exists\S. \end{cases}$$

(As usual, $\forall\S$ stands for the string $\forall\S^1\ldots\S^n$, and $\exists\S$ for the string $\exists\S^1\ldots\S^n$.) Equivalently, denoted by $\mathfrak{F}_n$ the collection of all finite families[15] of elements in $\mathsf{MOD}_n(X)$, the operation of tensor product can be seen as a collection of maps $\otimes_n$ such that

$$\otimes_n : \mathfrak{F}_n \longrightarrow \mathsf{MOD}_{n+1}(X), \quad \{\mathscr{M}_i : i \in I\} \longmapsto \otimes_{i\in I}\mathscr{M}_i,$$

with n ranging over the set of natural numbers. In what follows, the tensor product of the family $\{\mathscr{M}_1, \ldots, \mathscr{M}_k\}$ is also denoted by $\otimes_{i=1}^k \mathscr{M}_i$ or $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_k$.

The next examples illustrate Definition 9 in some cases.

[14] Note that we are not saying that $\{\mathscr{M}_i : i \in I\}$ is a subset of $\mathsf{MOD}_n(X)$: in fact, we allow MPSs in the family to be repeated, that is, it may happen that $\mathscr{M}_i = \mathscr{M}_j$ for some distinct $i, j \in I$. This is consistent with the fact that in applications some agents may display exactly the same preference structure.

[15] Again, these are not sets, because repetitions are allowed: see Footnote 14.

Example 8. Let $\mathscr{M}_1 = \left(\precsim_1^\forall, \precsim_1^\exists\right)$ and $\mathscr{M}_2 = \left(\precsim_2^\forall, \precsim_2^\exists\right)$ be two MPSs of rank 1 (NaP-preferences). The tensor product of $\{\mathscr{M}_1, \mathscr{M}_2\}$ is the GNaP-preference

$$\mathscr{M}_1 \otimes \mathscr{M}_2 = \left(\precsim^{\forall\forall}, \precsim^{\forall\exists}, \precsim^{\exists\forall}, \precsim^{\exists\exists}\right)$$

whose components are defined by

$$\precsim^{\forall\forall}:=\precsim_1^\forall \cap \precsim_2^\forall,\ \precsim^{\forall\exists}:=\precsim_1^\exists \cap \precsim_2^\exists,\ \precsim^{\exists\forall}:=\precsim_1^\forall \cup \precsim_2^\forall,\ \precsim^{\exists\exists}:=\precsim_1^\exists \cup \precsim_2^\exists.$$

In particular, if $\mathscr{M}_1$ and $\mathscr{M}_2$ have the same necessary component $\precsim_1^\forall = \precsim_2^\forall := \precsim^\forall$, then we get

$$\mathscr{M}_1 \otimes \mathscr{M}_2 = \left(\precsim^\forall, \precsim_1^\exists \cap \precsim_2^\exists, \precsim^\forall, \precsim_1^\exists \cup \precsim_2^\exists\right).$$

Similarly, if $\precsim_1^\exists = \precsim_2^\exists := \precsim^\exists$, then

$$\mathscr{M}_1 \otimes \mathscr{M}_2 = \left(\precsim_1^\forall \cap \precsim_2^\forall, \precsim^\exists, \precsim_1^\forall \cup \precsim_2^\forall, \precsim^\exists\right).$$

Finally, in the very special case that the two NaP-preferences are exactly the same, namely $\mathscr{M}_1 = \mathscr{M}_2 = \mathscr{M} = \left(\precsim^\forall, \precsim^\exists\right)$, then

$$\mathscr{M} \otimes \mathscr{M} = \left(\precsim^\forall, \precsim^\exists, \precsim^\forall, \precsim^\exists\right)$$

is the *rank-2-lifting* of $\mathscr{M}$.

Example 9. Define the following binary relations on $X = \mathbb{R}$:

- $x \precsim_1^\forall y \iff x \geq y+1 \ \vee\ x = y$;
- $x \precsim_2^\forall y \iff x \leq y - \frac{1}{2} \ \vee\ x = y$;
- $x \precsim_1^\exists y \iff x \geq y - 1$;
- $x \precsim_2^\exists y \iff x \leq y + \frac{1}{2}$.

One can check that both pairs $\mathscr{M}_i = (\precsim_i^\forall, \precsim_i^\exists)$ are NaP-preferences, and their possible components $\precsim_i^\exists$ are *Scott-Suppes representable semiorders* (Scott & Suppes, 1958). The tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2 = \left(\precsim^{\forall\forall}, \precsim^{\forall\exists}, \precsim^{\exists\forall}, \precsim^{\exists\exists}\right)$ has some peculiar properties:

- $\precsim^{\forall\forall} = \{(x,x) : x \in \mathbb{R}\}$;
- $\precsim^{\forall\exists} \subseteq \sim_1^\exists$ (the symmetric part of $\precsim_1^\exists$);
- $\precsim^{\exists\forall} = \{(x,y) : 2x^2 + 2y^2 - 4xy + x - 2y - 1 \geq 0\} \cup \{(x,x) : x \in \mathbb{R}\}$;
- $\precsim^{\exists\exists} = \mathbb{R} \times \mathbb{R}$.

These unusual results are semantically sound, because we are defining an MPS starting from two 'strongly discordant' NaP-preferences. Furthermore, the tensor product highlights some concordant properties which, at first sight, may be overlooked. Semantically, this example may model a group decision process involving two conflicting properties, e.g., price (evaluated by the Finance Department) and quality (evaluated by the Quality Control Department). Both attributes present 'gray areas' of indiscernibility, which are naturally modeled by semiorders.

The next result ensures that the operation of tensor product is well-defined.

Lemma 9. *The tensor product of a finite family of modal preference structures of rank n is a modal preference structure of rank n + 1.*

Proof. Let $\{\mathscr{M}_i : i \in I\}$ be a family of modal preference structures on X, where all structures $\mathscr{M}_i = \left\{\precsim_i^\S : \S \in Q^n\right\}$ have rank n. By definition, for each $i \in I$ there are nonempty sets $K_1^i, \ldots, K_n^i$ and a family $\mathscr{P}^i$ of total preorders on X indexed over $K_1^i \times \cdots \times K_n^i$ such that

$$x \precsim_i^{\S^1 \ldots \S^n} y \iff (\S^1 k_1 \in K_1^i) \ldots (\S^n k_n \in K_n^i)\, x \precsim_{k_1 \ldots k_n}^i y\,. \tag{4}$$

Since we can duplicate indices in K_j^i for each $j = 1, \ldots, n$, we may assume without loss of generality that $K_j^i = K_j^{i'}$ for all $i, i' \in I$. Therefore, each family $\mathscr{P}^i$ is indexed over the same Cartesian product $K_1 \times \cdots \times K_n$, and so (4) can be rewritten as

$$x \precsim_i^{\S^1 \ldots \S^n} y \iff (\S^1 k_1 \in K_1) \ldots (\S^n k_n \in K_n)\, x \precsim_{k_1 \ldots k_n}^i y\,. \tag{5}$$

For each $i = 1, \ldots, n$, denote by $\triangle_i$ the symbol $\bigcap_{k_i \in K_i}$ if $\S^i = \forall$, and $\bigcup_{k_i \in K_i}$ if $\S^i = \exists$. Thus we have $\precsim_i^{\S^1 \ldots \S^n} = \triangle_1 \ldots \triangle_n \precsim_{k_1 \ldots k_n}^i$. To complete the proof, we show that $\otimes_{i \in I} \mathscr{M}_i$ is an MPS of rank $n+1$. To that end, we define a family $\mathscr{P}'$ of total preorders, indexed by a Cartesian product $K_1' \times \cdots \times K_n' \times K_{n+1}'$, and show that the equivalence (5) holds for this new family. Let $\mathscr{P}'$ be the underlying set of total preorders in every $\mathscr{P}^i$ (that is, we eliminate repetitions); furthermore, for all $i = 1, \ldots, n$, let $K_i' = K_i$, and $K_{n+1}' = I$. It is lengthy but straightforward to show that the following equivalence holds:

$$x \precsim^{\S^{1'} \ldots \S^{n+1'}} y \iff (\S^{1'} k_1 \in K_1') \ldots (\S^{n+1'} k_{n+1} \in K_{n+1}')\, x \precsim_{k_1 \ldots k_{n+1}} y.$$

This completes the proof. □

Now we show that all maps $\otimes_n$ are onto, and so any MPS of positive rank can be built – even if not uniquely – from below.

Lemma 10. *Every modal preference structure of rank n + 1 is the product of a finite family of modal preference structures of rank n.*

Proof. Let $\mathscr{M}$ be a modal preference structure of rank $n+1$. By definition, there is a family $\mathscr{P}$ of total preorders indexed over $K_1 \times \cdots \times K_{n+1}$. For all $i = 1, \ldots, n+1$ and $\S \in Q^{n+1}$, denote by the symbol $\triangle_i^\S$ either $\bigcap_{k_i \in K_i}$ if $\S^i = \forall$, or $\bigcup_{k_i \in K_i}$ if $\S^i = \exists$. Note that every binary relation $\precsim^\S$ is equal to $\triangle_1^\S \ldots \triangle_{n+1}^\S \mathscr{P}$. Moreover, for each $k_1 \in K_1$, let

$$\mathscr{P}_{k_1} = \left\{\precsim_{k_1 \ldots k_{n+1}} : (k_2, \ldots, k_{n+1}) \in K_2 \times \cdots \times K_{n+1}\right\},$$

and observe that $\mathscr{M}_{k_1} := \left(\triangle_2^\S \ldots \triangle_{n+1}^\S \mathscr{P}_{k_1} : \S \in Q^n\right)$ is a modal preference structures of rank n. Now the tensor product of the family $\{\mathscr{M}_{k_1} : k_1 \in K_1\}$ gives $\otimes_{k_1 \in K_1} \{\mathscr{M}_{k_1} : k_1 \in K_1\} = \mathscr{M}$. □

The remainder of this section is devoted to extend the notion of tensor product to any finite family of MPSs on a given set of alternatives. This family need not be 'homogeneous' (i.e., all structures in it have identical rank): in fact, all modal preference structures in this family may well have pairwise distinct ranks. This added flexibility may be useful in applications, making the approach well suited to model hierarchically organized charts in a decision procedure.

In order to present the general notion of tensor product, first we recall some notions from graph theory. A *tree* is an acyclic connected graph $T = (N, E)$, where N is a set of *nodes*, and E is a set of *edges*. For simplicity (and slightly abusing notation), we shall identify a tree T with the set N of its nodes, and denote edges of T as pairs of the type $st \in E$, where $s, t \in T$. In this paper, we assume that the set of nodes is finite, even without explicit mention. A tree T is *rooted* if there is a distinguished node $a_T \in T$, called the *root* of T. A *path* in T is a list $P = (t_0, \ldots, t_n)$ of pairwise distinct nodes such that $t_i t_{i+1} \in E$ for all i; in this case, P has *length n*, and the nodes t_0 and t_n are its *endpoints*.[16] A *branch* of T is a maximal path, that is, a path such that there is no other path that properly extends it.

The *height* of a rooted tree T, denoted by $\mathrm{h}(T)$, is the maximum length of a branch.[17] Given a rooted tree T of height n, the set of its nodes can partitioned according to height. Specifically, the unique node of height n is the root a_T, that is, $\mathrm{h}(a_T) := n$;

[16] By definition, the path $P = (t_0)$ has length 0, and only has one endpoint.

[17] By definition, a rooted tree having a unique node (the root) has height 0.

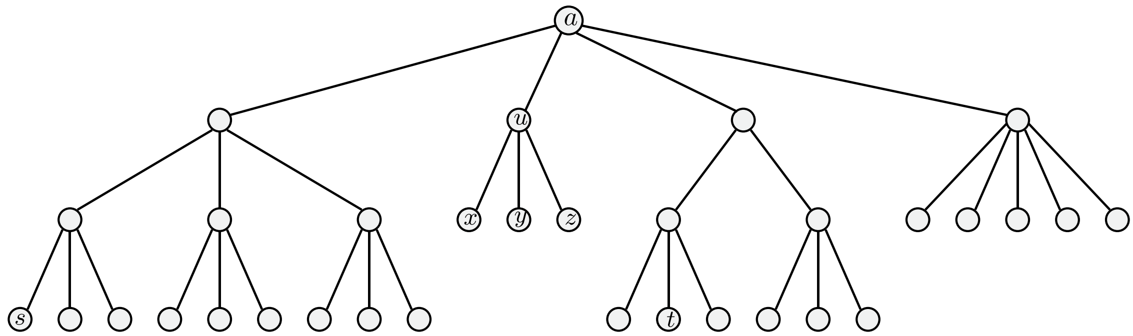

Fig. 2. A rooted tree T of height 3. The node a is the root of T, and has height 3. The nodes s, t are leaves of height 0, whereas x, y, z are leaves of height 1. The node u has height 2, and has $\{x, y, z\}$ as the set of all its predecessors and the root as unique successor. Note that T is not balanced.

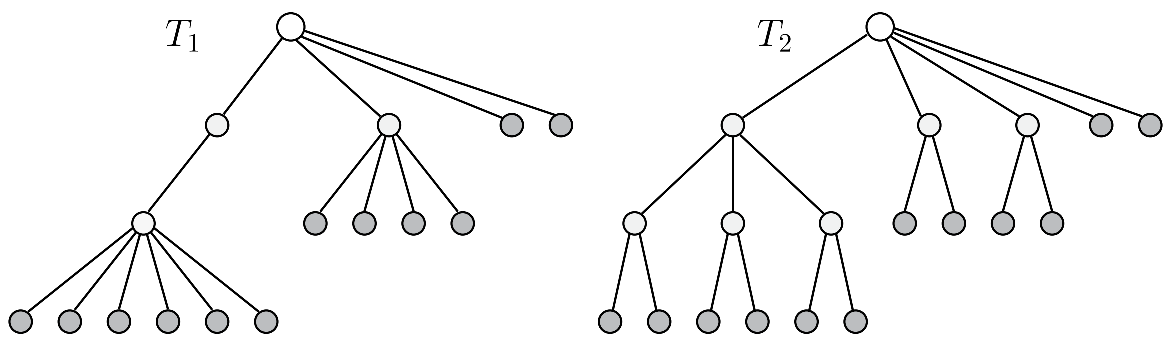

Fig. 3. Two different layouts T_1 and T_2 for the family $\mathscr{M}$ of modal preference structures in Example 10. The leaves of the two trees (in dark color) correspond to the preferences in $\mathscr{M}$. Note that in the layout T_1 the leftmost MPS of rank 2 is the rank-2-lifting (see Example 8) of the NaP-preference obtained as the tensor product of the six available total preorders.

moreover, for any $t \in T \setminus \{a_T\}$, set $\mathrm{h}(t) := n - p$, where p is the length of the (unique) path having a_T and t as endpoints. A *leaf* of T is a terminal node, that is, $t \in T$ such that there is no edge $st \in E$ with $\mathrm{h}(s) < \mathrm{h}(t)$; we denote by $\mathrm{Leaf}(T)$ the set of all leaves of T.[18] A rooted tree is *balanced* if all leaves have height 0 (equivalently, all branches have the same length).

Since all edges $st \in E$ in a rooted tree are such that $|\mathrm{h}(s) - \mathrm{h}(t)| = 1$, we use the convention that the first node of the edge has lower height, that is, $\mathrm{h}(s) = \mathrm{h}(t) - 1$: in this case, we call s a *predecessor* of t, and t a *successor* of s. For any node $t \in T$, we denote by $\mathrm{Pred}(t)$ and $\mathrm{Succ}(t)$ the sets of all predecessors and successors of t, respectively. By definition of rooted tree, any node may have several predecessors, but has at most one successor. Fig. 2 illustrates some of the notions defined above.

Next, we associate a family of rooted trees to any collection of modal preference structures.

Definition 10. Suppose a family $\mathfrak{M} = \{\mathscr{M}_i : i \in I\}$ of MPSs on X is given. Let $\mathrm{r}(\mathscr{M}_i)$ be the rank of $\mathscr{M}_i$, and $\mathrm{r}_{\max}$ and $\mathrm{r}_{\min}$ the maximum and the minimum rank of the elements of $\mathfrak{M}$. Moreover, for each integer k such that $\mathrm{r}_{\min} \leqslant k \leqslant \mathrm{r}_{\max}$, let m_k be the nonnegative number of modal preference structures in $\mathfrak{M}$ having rank k.[19] A *layout* for $\mathfrak{M}$ is any rooted tree T of height $\mathrm{h}(T) = \mathrm{r}_{\max} - \mathrm{r}_{\min} + 1$ with the property that there are exactly m_k leaves of height k for each $k = \mathrm{r}_{\min}, \dots, \mathrm{r}_{\max}$.

A family of MPSs may have several layouts, as the next example shows.

[18] Thus, by definition, any node of height 0 is a leaf, but there may be leaves with strictly positive height.

[19] Note that $\sum_{k=\mathrm{r}_{\min}}^{\mathrm{r}_{\max}} m_k = |I|$.

Example 10. Let $\mathfrak{M} = \{\mathscr{M}_1, \dots, \mathscr{M}_{12}\}$ be a family of MPSs on a set X such that $\mathrm{r}(\mathscr{M}_i) = 0$ for $1 \leqslant i \leqslant 6$, $\mathrm{r}(\mathscr{M}_i) = 1$ for $7 \leqslant i \leqslant 10$, and $r(\mathscr{M}_i) = 2$ for $11 \leqslant i \leqslant 12$. The rooted trees T_1 and T_2 depicted in Fig. 3 are both layouts for $\mathfrak{M}$.

Finally, we define the so-called 'T-tensor product' of an arbitrary family of modal preference structures on X, where T is a layout for that family. Definition 9 is a special case of this notion, obtained in the case that the layout T is a rooted tree of height 1.

Definition 11. Let $\mathfrak{M} = \{\mathscr{M}_i : i \in I\}$ be a finite family of MPSs on X, and T a layout for $\mathfrak{M}$. We recursively assign a modal preference structure on X to each node of the tree T as follows. For the base step, use a bijection $\alpha : \mathrm{Leaf}(T) \to \mathfrak{M}$ to assign all elements of $\mathfrak{M}$ to the leaves of T, in a way that $\mathrm{r}(\alpha(t)) = \mathrm{h}(t) + \mathrm{r}_{\min}$ for every $t \in \mathrm{Leaf}(T)$. (This assignment is sound, because T is a layout for $\mathfrak{M}$.)

Recursively, to any node $t \in T \setminus \mathrm{Leaf}(T)$ such that all $s \in \mathrm{Pred}(t)$ have been assigned a modal preference structure $\mathscr{M}_s$ on X, assign the tensor product $\otimes_{s \in \mathrm{Pred}(t)} \mathscr{M}_s$. (By Definition 9, this assignment is sound.) We proceed by rank, that is, before defining a certain modal preference structure of a certain rank, all MPSs of lower rank must have been defined (or they belong to $\mathfrak{M}$). At any rank, different tensor products can be performed in any order.

The last step of this process assigns an MPS of rank $\mathrm{r}_{\max} + 1$ to the root of T: we call this modal preference structure the *T-tensor product* of $\mathfrak{M}$, and denote it by $\otimes^T \mathfrak{M}$.

It is simple to show that Definition 11 is sound (proof is omitted):

Lemma 11. *The T-tensor product of a family of modal preference structures having maximum rank n is a modal preference structure of rank $n + 1$.*

We conclude this section with an example that illustrates in detail how the operation of tensor product is performed in a concrete scenario. Here as the input of the decision process we have twelve MPSs of several distinct ranks. We shall provide two slightly different outputs, obtained by applying two distinct layouts for the amalgamation of the given modal preferences structures into one of higher rank.

Example 11. We elaborate on the setting described in Example 10. Let $X = \{a, b, c\}$ be a set of three perspective investments, from which a multinational corporation has to select one or two. The agents involved in the decision process are of several types (officers, sub-departments, departments, branches, etc.). Suppose the pieces of preferential information provided by these agents are given by a family $\mathfrak{M} = \{\mathscr{M}_1, \ldots, \mathscr{M}_{12}\}$ of modal preference structures on X as in Example 10: six total preorders $\precsim_i$ for $1 \leqslant i \leqslant 6$, four NaP-preferences $\left(\succsim_i^{\forall}, \succsim_i^{\exists}\right)$ for $7 \leqslant i \leqslant 10$, and two GNaP-preferences $\left(\succsim_i^{\forall\forall}, \succsim_i^{\forall\exists}, \succsim_i^{\exists\forall}, \succsim_i^{\exists\exists}\right)$ for $11 \leqslant i \leqslant 12$, all defined on X.[20]

The input $\mathfrak{M}$ of the decision process is provided at different levels of the hierarchical organization of the corporation. This happens when some elementary pieces of information coming from below remain hidden, being only given in an aggregated form. For instance, imagine a situation in which the officers of the R&D sub-departments of a national unit give a technical evaluations of the three investments in the form of total preorders, and each sub-department forms a NaP-preference from them without disclosing the process that guided their construction. Then the pieces of informations provided by all sub-departments are amalgamated into a GNaP-preference synthesizing the overall evaluation provided by the R&D department into one MPS of rank 2.

Now suppose the twelve modal preference structures $\mathscr{M}_i$ in $\mathfrak{M}$ are defined as follows:

($\mathscr{M}_1$) $a \succ_1 b \succ_1 c$;
($\mathscr{M}_2$) $a \sim_2 b \succ_2 c$;
($\mathscr{M}_3$) $a \succ_3 b \sim_3 c$;
($\mathscr{M}_4$) $a \sim_4 c \sim_4 b$;
($\mathscr{M}_5$) $b \succ_5 c \succ_5 a$;
($\mathscr{M}_6$) $b \succ_6 c \sim_6 a$;
($\mathscr{M}_7$) $b \succ_7^{\forall} c \succ_7^{\forall} a$, $b \succ_7^{\forall} a$, and $b \sim_7^{\exists} c \succ_7^{\exists} a$, $b \succ_7^{\exists} a$;
($\mathscr{M}_8$) $a \succ_8^{\forall} b \succ_8^{\forall} c$, $a \succ_8^{\forall} c$, and $a \succ_8^{\exists} b \succ_8^{\exists} c$, $a \succ_8^{\exists} c$;
($\mathscr{M}_9$) $a \succ_9^{\forall} b$, and $a \succ_9^{\exists} b \sim_9^{\exists} c$, $a \sim_9^{\exists} c$;
($\mathscr{M}_{10}$) $b \succ_{10}^{\forall} c$, and $a \sim_{10}^{\exists} b \succ_{10}^{\exists} c$, $a \sim_{10}^{\exists} c$;
($\mathscr{M}_{11}$) $a \succ_{11}^{\forall\forall} b \succ_{11}^{\forall\forall} c$, $a \succ_{11}^{\forall\forall} c$, $\succsim_{11}^{\forall\exists} := X^2$, $a \sim_{11}^{\exists\forall} b \succ_{11}^{\exists\forall} c$, $a \succ_{11}^{\exists\forall} c$, and $\succsim_{11}^{\exists\exists} := X^2$;
($\mathscr{M}_{12}$) $b \succ_{12}^{\forall\forall} c$, $a \succ_{12}^{\forall\exists} b \succ_{12}^{\forall\exists} c$, $a \succ_{12}^{\forall\exists} c$, $b \succ_{12}^{\exists\forall} a \sim_{12}^{\exists\forall} c$, $b \succ_{12}^{\exists\forall} c$, and $\succsim_{12}^{\exists\exists} := X^2$.

In what follows, we separately apply the layouts T_1 and T_2 given in Example 10, and generate two (slightly) different MPSs of rank 3, namely $\otimes^{T_1}\mathfrak{M}$ and $\otimes^{T_2}\mathfrak{M}$.

We begin with the T_1-tensor product of $\mathfrak{M}$. The first step is to take the tensor product of the given family of total preorders $\{\precsim_1, \ldots, \precsim_6\}$ to generate a NaP-preference $\otimes_{i=1}^{6}\mathscr{M}_i := \left(\succsim_1^{\forall}, \succsim_1^{\exists}\right)$. Intersection and union of these total preorders yield that $\succsim_1^{\forall}$ and $\succsim_1^{\exists}$ are respectively given by $b \succ_1^{\forall} c$ and $\succsim_1^{\exists} = X^2$. Since there are no other modal preference structures of rank 1 to be built, next we define those of rank 2. To start, we take the rank-2-lifting of $\left(\succsim_1^{\forall}, \succsim_1^{\exists}\right)$, which is the GNaP-preference $\otimes\left(\succsim_1^{\forall}, \succsim_1^{\exists}\right) = \left(\succsim_1^{\forall}, \succsim_1^{\exists}, \succsim_1^{\forall}, \succsim_1^{\exists}\right)$. According to the shape of the layout T_1, the next step is to take the tensor product of the four NaP-preferences $\left(\succsim_i^{\forall}, \succsim_i^{\exists}\right)$ in $\mathfrak{M}$, where $7 \leqslant i \leqslant 10$. This is the GNaP-preference $\left(\succsim_2^{\forall\forall}, \succsim_2^{\forall\exists}, \succsim_2^{\exists\forall}, \succsim_2^{\exists\exists}\right)$ such that

$$\succsim_2^{\forall\forall} = \{(a,a),(b,b),(c,c)\},\, b \succ_2^{\forall\exists} c,\, b \sim_2^{\exists\forall} a \sim_2^{\exists\forall} c,\, b \succ_2^{\exists\forall} c,\, \succsim_2^{\exists\exists} = X^2.$$

Since there are no other modal preference structures of rank 2 to be built, we can finally take the tensor product of the four GNaP-preferences $\left(\succsim_i^{\forall\forall}, \succsim_i^{\forall\exists}, \succsim_i^{\exists\forall}, \succsim_i^{\exists\exists}\right)$, with $i \in \{1, 2, 11, 12\}$, which is, by definition, the T_1-tensor product of the original family $\mathfrak{M}$. This is the modal preference structure

$\left(\succsim_{T_1}^{\forall\forall\forall}, \succsim_{T_1}^{\forall\forall\exists}, \succsim_{T_1}^{\forall\exists\forall}, \succsim_{T_1}^{\exists\forall\forall}, \succsim_{T_1}^{\forall\exists\exists}, \succsim_{T_1}^{\exists\forall\exists}, \succsim_{T_1}^{\exists\exists\forall}, \succsim_{T_1}^{\exists\exists\exists}\right)$

of rank 3 whose components are defined as follows:

$\succsim_{T_1}^{\forall\forall\forall} = \{(a,a),(b,b),(c,c)\}$;
$b \succ_{T_1}^{\forall\forall\exists} c$;
$b \succ_{T_1}^{\forall\exists\forall} c$;
$a \succ_{T_1}^{\exists\forall\forall} b \succ_{T_1}^{\exists\forall\forall} c$, $a \succ_{T_1}^{\exists\forall\forall} c$;
$\succsim_{T_1}^{\forall\exists\exists} = X^2$;
$\succsim_{T_1}^{\exists\forall\exists} = X^2$;
$\succsim_{T_1}^{\exists\exists\forall} = X^2 \setminus \{(c,b)\}$;
$\succsim_{T_1}^{\exists\exists\exists} = X^2$.

Now we take the T_2-tensor product of $\mathfrak{M}$. Computations similar to those above yield the modal preference structure

$\left(\succsim_{T_2}^{\forall\forall\forall}, \succsim_{T_2}^{\forall\forall\exists}, \succsim_{T_2}^{\forall\exists\forall}, \succsim_{T_2}^{\exists\forall\forall}, \succsim_{T_2}^{\forall\exists\exists}, \succsim_{T_2}^{\exists\forall\exists}, \succsim_{T_2}^{\exists\exists\forall}, \succsim_{T_2}^{\exists\exists\exists}\right)$

defined as follows:

$\succsim_{T_2}^{\forall\forall\forall} = \{(a,a),(b,b),(c,c)\}$;
$b \succ_{T_2}^{\forall\forall\exists} c$, $a \succ_{T_2}^{\forall\forall\exists} c$;
$b \succ_{T_2}^{\forall\exists\forall} c$;
$a \succ_{T_2}^{\exists\forall\forall} b \succ_{T_2}^{\exists\forall\forall} c$, $a \succ_{T_2}^{\exists\forall\forall} c$;
$\succsim_{T_2}^{\forall\exists\exists} = X^2$;
$\succsim_{T_2}^{\exists\forall\exists} = X^2 \setminus \{(c,b)\}$;
$\succsim_{T_2}^{\exists\exists\forall} = X^2$;
$\succsim_{T_2}^{\exists\exists\exists} = X^2$.

Finally, we compare the outputs of the two aggregation processes. The first thing that jumps into attention is that the differences between them are really few. However, this is due to the small number of alternatives, which prevents the layouts from producing distinctively different aggregated results. Both outputs point in the direction that investment c should be dismissed. This result is re-enforced by the second tensor product, which displays a preference $a \succsim_{T_2}^{\forall\forall\exists} c$ not present in the first tensor product. Furthermore, if a single investment has to be selected, then both outputs suggest to choose a, due to the strict preference $a \succ_{T_i}^{\exists\forall\forall} b$ for $i = 1, 2$, whereas b is never strictly preferred to a. The two preferences $a \succ_{T_i}^{\exists\forall\forall} b$ are indeed quite strong, because they can be interpreted, for instance, as follows: there is a department in the corporation such that for all its sub-departments and for all officers of these sub-departments, investment a is always ranked better than investment b. Therefore, to take a final decision, the Board of Directors may want to look at the reliability of the department that is providing this type of strong input.

8. Conclusions and future directions of research

In this paper we have introduced modal preference structures, which are the natural extension of an approach to preference modeling that employs multiple binary relations for a representation. Among several possible interpretations, these structures

[20] The setting is purely didactical, because our unique goal is to explicitly illustrate how the T-tensor product works in a concrete case.

can be regarded as a sound way to aggregate rational preferences, aiming to describe how the two basic tenets of economic rationality – transitivity and completeness – collectively act on a family of binary relations. In fact, modal preference structures are characterized by the satisfaction of harmonic versions of the two tenets of rationality, namely transitive coherence and mixed completeness. The semantics of MPSs is connected to group decision making, because they offer a meaningful synthesis of the preference structures of all agents involved in the decision process.

One of the main directions of future research consists of proving further extensions of a famous theorem due to Schmeidler (1971), which establishes an intriguing connection among the fundamental properties of continuity, transitivity, and completeness in a connected topological space.[21] Schmeidler's theorem has already been (reformulated and) extended by Giarlotta and Watson (2020) to NaP-preferences. This extension is based on the notion of *order-section topology* – a bi-relation refinement of the standard order topology.[22] Any further generalization of Schmeidler's theorem to GNaP-preferences or MPSs requires the definition of a suitable variation of the order topology induced by all preference relations. A possible approach to this problem consists of defining a topology induced by the family of preorders that represent a given MPS, and then showing that such a topology is (i) invariant of the representation of the MPS, and (ii) the coarsest topology such that certain continuity properties hold.

A second possible stream of work is studying representability of a given MPS through utility functions. This could be achieved in many ways, for instance considering utility representations for the involved total preorders, and then representing an MPS by an indexed family of utility functions. The approach employed for this task should extends the *modal utility representation* of NaP-preferences, as suggested by Giarlotta and Greco (2013, Section 4).

A third possible direction of research concerns the number and the properties of the binary relations involved in an MPS.[23] Specifically, since all sequences of quantifiers of length n are considered, the number of binary relations in a modal preference structure is bounded to be exactly 2^n. Ideally, our theory should be generalized by considering any subset of these sequences, and get modal preference structures with any number of relations. A different but related query concerns a possible variation of the building blocks of modal preference structures, which at present are total preorders. It may be of some interest – although nontrivial – to extend the theory of modal preference structures to other basic types of binary relations, e.g., linear orders, incomplete preorders, semiorders, interval orders, and, more generally, (m, n)-Ferrers relations.

Still an additional direction of future research is related to the algebraic operation of tensor product. From this perspective, several queries may be addressed. For instance, are there nonhomogeneous families of modal preference structures such that their tensor product is invariant of all possible layouts (that is, they produce the same structure regardless of the layout used for the aggregation procedure)? For another example, given a family $\mathfrak{M}$ of modal preference structures, define two layouts S, T for $\mathfrak{M}$ to be equivalent if $\otimes^S \mathfrak{M} = \otimes^T \mathfrak{M}$. What are the common features of the layouts that belong to the same equivalence class? Furthermore, we can define a 'composition' operator for layouts, and consider all the possible layouts that, when composed, give a certain layout. Is it possible to characterize the family of all layouts such that their composition gives the same layout? And, finally, what about a stochastic setting in which probability distributions are associated to both layouts and sets of layouts? Answering all these queries may be important in applications, once that all pieces of information provided at a very low organizational level need be *consistently* aggregated for the consideration of the CEO.

A final, purely abstract, direction of research is related to set theory and category theory. Specifically, we have another proof of the characterization of modal preference structures, which is more technical and involved. However, the generality of this alternative proof allows us to apply a similar reasoning to the modelization of arbitrary structures (and not necessarily binary relations) indexed by strings of quantifiers. This proof involves a recursive construction that describes the amalgamation process of a composite structure from simpler ones, 'remembering' when a certain index has been introduced, and fetching it whenever needed.

21 David Schmeidler recently passed away. We believe that keep working on topics pioneered by him is a good way to honor his memory.

22 For details, see Giarlotta and Watson (2020, Section 3).

23 We thank one of the referees for pointing out these possible developments of our theory.

Acknowledgments

The authors wish to thank two anonymous referees for several useful suggestions, which improved the clarity of the paper. Alfio Giarlotta gratefully acknowledges the support of "Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR), Italy - PRIN 2017", project *Multiple Criteria Decision Analysis and Multiple Criteria Decision Theory*, grant 2017CY2NCA.

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