Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 404–422.
DOI: 10.3934/math.2023019
Received: 01 July 2022
Revised: 16 September 2022
Accepted: 23 September 2022
Published: 08 October 2022

Research article

Existence of positive radial solutions for a problem involving the weighted Heisenberg $p(\cdot)$-Laplacian operator

Maria Alessandra Ragusa[1,*], **Abdolrahman Razani**[2] **and Farzaneh Safari**[2]

[1] Dipartimento di Matematica e Informatica, Università di Catania, Italy

[2] Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Postal code 3414896818, Qazvin, Iran

* **Correspondence:** Email: mariaalessandra.ragusa@unict.it; Tel:+390957383060; Fax: +39095330094.

Abstract: A variational principle is applied to examine a Muckenhoupt weighted $p(\cdot)$-Laplacian equation on the Heisenberg groups. The existence of at least one positive radial solution to the problem under the Dirichlet boundary condition belongs to the first order Heisenberg-Sobolev spaces is proved.

Keywords: Heisenberg $p(\cdot)$-Laplacian operator; variational principle; Muckenhoupt weight function; mountain pass geometry theorem
Mathematics Subject Classification: 35R03, 35B65, Secondary 35J60, 35J70

1. Introduction

Let Ω be the unit Korányi ball in the Heisenberg group $\mathbb{H}^n$ $(n \geq 1)$; the $p(\cdot)$-Laplacian problem

$$\begin{cases} \mathcal{L}(u) = \rho(\xi)|u|^{\theta(\xi)-2}u - \varrho(\xi)|u|^{\vartheta(\xi)-2}u & \xi \in \Omega, \\ u > 0 & \xi \in \Omega, \\ u = 0 & \xi \in \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

with Dirichlet boundary condition is studied. Assume

$$\rho(\xi) = \alpha(|\xi|_{\mathbb{H}^n}) \qquad \text{and} \qquad \varrho(\xi) = \beta(|\xi|_{\mathbb{H}^n}), \tag{1.1}$$

where $\alpha, \beta \in L^\infty(0,1)$ such that α is a positive non-constant radially non-decreasing function and β is a non-negative radially non-increasing function. $p, \theta, \vartheta \in C_+(\Omega)$ such that

$$p^+ < \theta^- < \theta(\xi) < \theta^+ < \mathfrak{p}^* \qquad \text{and} \qquad \vartheta^+ < \theta^- \text{ a.e.} \quad \text{in } \Omega$$

where

$$\mathfrak{p}^* = \frac{\mathfrak{p}Q}{Q-\mathfrak{p}} \qquad \text{and} \qquad \mathfrak{p} := p_s^- = \frac{sp^-}{s+1}$$

for s with

$$s \in [\frac{1}{p^- - 1}, +\infty) \cap (\frac{Q}{p^-}, +\infty).$$

The weighted operator $\mathcal{L}$ is defined by

$$\mathcal{L}(u(\cdot)) := div_{\mathbb{H}^n}(w(\cdot)|\nabla_{\mathbb{H}^n} u|^{p(\cdot)-2}\nabla_{\mathbb{H}^n} u) + w(\cdot)\mathcal{R}(\cdot)|u|^{p(\cdot)-2}u,$$

where w is a Muckenhoupt weight function of class A_s. And, finally, $\mathcal{R} : \Omega \to [0, +\infty)$ belongs to $L^\infty(\Omega)$ such that $\operatorname{ess\,inf}_\Omega \mathcal{R} > 0$.

The topicality of the theory of Sobolev spaces on Heisenberg groups is explained by numerous applications of it to the study of solutions for subelliptic differential equations, quasiconformal analysis, and many other related problems. The Heisenberg groups represent the best known and, in many respects, a model case of the Carnot-Carathéodory spaces.

Most papers in the literature on Muckenhoupt weighted problems focus their attention on the p-Laplacian operator (see [1, 3, 4, 6, 13–15, 18–21, 25, 26, 28–32, 35] and the references therein). A sharp distinction between this note and the mentioned works is that this paper discusses on the existence of a suitable interval for embedding of weighted Heisenberg Sobolev spaces with variable exponents into the Lebesgue spaces and using that for study of the existence of solutions for a weighted Heisenberg $p(\cdot)$-Laplacian problem.

Here, we are going to prove that the problem $(\mathcal{P})$, under the aforementioned assumptions, has at least one positive radial solution in $HW^{1,p(\cdot)}(\Omega, w) \cap L_\rho^{\theta(\cdot)}(\Omega) \cap L_\varrho^{\vartheta(\cdot)}(\Omega)$, where

$$L_\rho^{\theta(\cdot)}(\Omega) := \{u : \int_\Omega \rho(\xi)|u(\xi)|^{\theta(\xi)} d\xi < \infty\},$$

which has the norm

$$|u|_{\rho,\theta} = inf\{\lambda > 0 : \int_\Omega \rho(\xi)|\frac{u(\xi)}{\lambda}|^{\theta(\xi)} d\xi \leq 1\};$$

and, similarly,

$$L_\varrho^{\vartheta(\cdot)}(\Omega) := \{u : \int_\Omega \varrho(\xi)|u(\xi)|^{\vartheta(\xi)} d\xi < \infty\},$$

which has the norm

$$|u|_{\varrho,\vartheta} = inf\{\lambda > 0 : \int_\Omega \varrho(\xi)|\frac{u(\xi)}{\lambda}|^{\vartheta(\xi)} d\xi \leq 1\}$$

for $\rho, \varrho \in L^\infty(\Omega)$ as in (1.1), θ and ϑ are as above. We show that a radial weak solution to the problem $(\mathcal{P})$ is as follows.

Definition 1.1. *We say that*

$$u \in HW^{1,p(\cdot)}(\Omega, w) \cap L_\rho^{\theta(\cdot)}(\Omega) \cap L_\varrho^{\vartheta(\cdot)}(\Omega)$$

is a non-trivial radial weak solution of $(\mathcal{P})$ *if* $u > 0$ *in* Ω *is radial and the following equality is true:*

$$\mathcal{L}(u(\xi)) = \rho(\xi)|u|^{\theta(\xi)-2}u(\xi) - \varrho(\xi)|u|^{\vartheta(\xi)-2}u(\xi),$$

in the weak sense; that is

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^{p(\xi)-2}\nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} \phi w(\xi) d\xi + \int_\Omega \mathcal{R}(\xi)|u|^{p(\xi)-2} u\phi w(\xi) d\xi$$
$$= \int_\Omega \rho(\xi)|u|^{\theta(\xi)-2} u\phi d\xi - \int_\Omega \varrho(\xi)|u|^{\vartheta(\xi)-2} u\phi d\xi$$

for any $\phi \in C_0^\infty(\Omega)$.

The paper is arranged as follows. In Section 2 we present the notations and some facts related to the Heisenberg groups and Muckenhoupt weight functions. Additionally, we obtain a suitable interval for the embedding of weighted Heisenberg Sobolev spaces with variable exponents into the Lebesgue spaces. Plus that we bring some briefs from variational calculus and we introduce our main tool. Section 3 is devoted to the main result of the note and proof of the approach.

2. Notations and auxiliary remarks

In this note, $\mathbb{H}^n$ $(n \geq 1)$ is the Heisenberg Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold and is endowed with the following noncommutative law of product:

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(\langle y|x'\rangle - \langle x|y'\rangle)),$$

where $x, x', y, y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$ and $\langle\cdot|\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^n$. We denote by $|\cdot|_{\mathbb{H}^n}$ Korányi norm with respect to the parabolic dilation $\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$, i.e.,

$$|\xi|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$$

for $z = (x, y) \in \mathbb{R}^{2n}$ and $\xi = (z, t) \in \mathbb{H}^n$.

Definition 2.1. *(Radial Function) Let* $\Omega \subset \mathbb{H}^n$ *be a bounded open set. The function* $u : \Omega \to \mathbb{R}$ *is called a radial function if* $u(x, y, t) = \phi(\mathbf{r})$*, where* $\mathbf{r} = |(x, y, t)|_{\mathbb{H}^n}$ *and* $\phi : [0, +\infty) \to \mathbb{R}$.

A Korányi ball with the center ξ_0 and radius κ is defined by

$$B_{\mathbb{H}^n}(\xi_0, \kappa) := \{\xi : |\xi^{-1} \circ \xi_0|_{\mathbb{H}^n} \leq \kappa\},$$

and it satisfies the following equalities:

$$|B_{\mathbb{H}^n}(\xi_0, \kappa)| = |B_{\mathbb{H}^n}(0, \kappa)| = \kappa^Q |B_{\mathbb{H}^n}(0, 1)|,$$

where $|U|$ denotes the $(2n+1)$-dimensional Lebesgue measure of U and $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. The Heisenberg gradient is given by

$$\nabla_{\mathbb{H}^n} = (X_1, \cdots, X_n, Y_1, \cdots, Y_n),$$

where

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, 3, \cdots, n,$$

are vector fields that constitute a basis for the real Lie algebra of left-invariant vector fields on $\mathbb{H}^n$; More precisely, the family

$$\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, [X_1, Y_1]\}$$

satisfies the Hörmander's condition which means that it spans the whole tangent space $T\mathbb{R}^{2n+1}$. Let us recall that Hörmander's condition is a crucial condition for many problems consisting of hypoelliptic operators (see more details in [13] and the references therein).

For any horizontal vector field function $X = X(\xi), X = \{x_i X_i + x_i' Y_i\}_{i=1}^n$, of the class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the horizontal divergence of X by

$$div_{\mathbb{H}^n} X := \sum_{i=1}^{n} [X_i(x_i) + Y_i(x_i')].$$

Definition 2.2. *(Horizontal Curve) A piecewise smooth curve $y : [0,1] \to \mathbb{H}^n$ is called a horizontal curve if $\dot{y}(t)$ belongs to the span of $\{X_i, Y_i\}_{i=1}^n$ a.e. in $[0,1]$. The horizontal length of y is defined as follows*

$$L_{\mathbb{H}^n}(y) = \int_0^1 \sqrt{(\dot{y}(t), \dot{y}(t))_{\mathbb{H}^n}}\, dt = \int_0^1 |\dot{y}(t)|_{\mathbb{H}^n} dt,$$

where

$$(X, Y)_{\mathbb{H}^n} = \sum_{i=1}^{n} (x_i y_i + x_i' y_i'),$$

for each $X = \{x_i X_i + x_i' Y_i\}_{i=1}^n$ and $Y = \{y_i X_i + y_i' Y_i\}_{i=1}^n$.

The Carnot-Carathéodory distance of two points $\xi_1, \xi_2 \in \mathbb{H}^n$ is defined by

$$d_{cc}(\xi_1, \xi_2) = inf\{L_{\mathbb{H}^n}(y) : y \text{ is a horizontal curve joining } \xi_1, \xi_2 \text{ in } \mathbb{H}^n\}.$$

Notice that according to the Chow-Rashevsky theorem [5, 24], for any two arbitrary points $\xi_1, \xi_2 \in \mathbb{H}^n$, there is a horizontal curve between them in $\mathbb{H}^n$; then, the above definition is well-defined. d_{cc} is a left invariant metric on $\mathbb{H}^n$ and has a homogeneity of degree 1 with respect to dilations δ_λ, that is

$$d_{cc}(\delta_\lambda(\xi_1), \delta_\lambda(\xi_2)) = \lambda d_{cc}(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in \mathbb{H}^n$. In the case of the Heisenberg group, it is easy to check that the Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under left translations. Thus, from here on, we denote by $d\xi$ the Haar measure on $\mathbb{H}^n$ that coincides with the $(2n+1)$-Lebesgue measure, this is because the Haar measures on Lie groups are unique up to constant multipliers.

As usual, for any measurable set $\Omega \subset \mathbb{H}^n$ $(n \geq 1)$ and $m > 1$, we denote by $L^m(\Omega)$ the canonical Banach space, endowed with the norm

$$|u|_m = \Big(\int_\Omega |u|^m d\xi\Big)^{\frac{1}{m}}.$$

The first-order Heisenberg Sobolev space on Ω is defined as follows

$$HW^{1,m}(\Omega) := \{u \in L^m(\Omega) : \ |\nabla_{\mathbb{H}^n} u| \in L^m(\Omega)\},$$

endowed with the norm

$$||u||_{1,m} = |u|_m + |\nabla_{\mathbb{H}^n} u|_m.$$

Definition 2.3. *(Poincaré-Sobolev Domain) An open set* Ω *of* $\mathbb{H}^n$ *is said to be a Poincaré-Sobolev domain, if there exists a bounded open set* $U \subset \mathbb{H}^n$, *with* $\Omega \subset \bar{\Omega} \subset U$, *which is a covering* $\{B\}_{B \in \mathfrak{F}}$ *of* Ω *by Carnot-Carathéodory balls B and the numbers* $N > 0, \alpha \geq 1$ *and* $\nu \geq 1$ *such that*
(i) $\sum_{B \in \mathfrak{F}} \mathbf{1}_{(a+1)B} \leq N \mathbf{1}_{\Omega}$ *in U, where* $\mathbf{1}_D$ *is the characteristic function of a Lebesgue measurable subset D.*
(ii) *there exists a (central) ball* $B_0 \in \mathfrak{F}$ *such that, for all* $B \in \mathfrak{F}$ *there is a finite chain* $B_0, B_1, \cdots, B_{s(B)}$, *with* $B_i \cap B_{i+1} \neq \emptyset$ *and*

$$|B_i \cap B_{i+1}| \geq \frac{max\{|B_i|, |B_{i+1}|\}}{N}, \quad i = 0, 1, \cdots, s(B) - 1;$$

and moreover, $B \subset \nu B_i$ *for* $i = 0, 1, \cdots, s(B)$.

This definition is purely metric. There is a multiplicity of Poincaré-Sobolev domains in $\mathbb{H}^n$, as explained in details in [10]. The next result is a special case of [11, Theorem 1.3.1].

Theorem 2.1. (i) *Let* Ω *be a bounded Poincaré-Sobolev domain in* $\mathbb{H}^n$, *and let* $1 \leq m \leq Q$. *Then, the embedding*

$$HW^{1,m}(\Omega) \hookrightarrow\hookrightarrow L^{\sigma}(\Omega), \quad for \;\; 1 \leq \sigma < m^*$$

is compact for all σ, *where* $m^* = \frac{mQ}{Q-m}$ *is the critical Sobolev exponent related to m.*
(ii) *The Carnot-Carathéodory balls are Poincaré-Sobolev domains.*

Remark 2.1. *Combining Theorem 2.1, with the fact that the Carnot-Carathéodory distance and the Korányi distance are equivalent on* $\mathbb{H}^n$, *we get that the following embedding is compact*

$$HW^{1,m}(\Omega) \hookrightarrow\hookrightarrow L^{\sigma}(\Omega), \quad for \;\; 1 \leq \sigma < m^*,$$

when $1 \leq m \leq Q$ *and* Ω *is any Korányi ball centered at* $\xi_0 \in \mathbb{H}^n$ *with a radius* $R > 0$. *Furthermore, there exists a* $C_\sigma > 0$ *such that*

$$|u|_{\sigma} \leq C_{\sigma} \, \|u\|_{1,m}, \quad for \;\; 1 \leq \sigma \leq m^*$$

for all $u \in HW^{1,m}(\Omega)$.

From now on we denote by Ω the unit Korányi ball centered at the origin, and we set

$$q^- = \inf_{\xi \in \Omega} q(\xi) \qquad \text{and} \qquad q^+ = \sup_{\xi \in \Omega} q(\xi)$$

for $q \in C_+(\bar{\Omega}) = \{g \in C(\bar{\Omega}) : g^- > 1\}$. The generalized Lebesgue space $L^{q(\cdot)}(\Omega)$ is the collection of all measurable functions u on Ω for which there exists a $\lambda > 0$ such that

$$\int_{\Omega} (\frac{u(\xi)}{\lambda})^{q(\xi)} d\xi < \infty,$$

and it has the norm

$$|u|_{q(\cdot)} = inf\{\lambda > 0 : \int_{\Omega} |\frac{u(\xi)}{\lambda}|^{q(\xi)} d\xi \leq 1\}.$$

We know that for any $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, i.e., the conjugate space of $L^{q(\cdot)}(\Omega)$, the Hölder type inequality

$$|\int_\Omega uvd\xi| \leq (\frac{1}{q^-} + \frac{1}{q'^-})|u|_{q(\cdot)}|v|_{q'(\cdot)} \tag{2.1}$$

holds true. Following the authors of [23], for any $\kappa > 0$, we put

$$\kappa^{\check{r}} := \begin{cases} \kappa^{r^+} & \kappa < 1, \\ \kappa^{r^-} & \kappa \geq 1, \end{cases}$$

and

$$\kappa^{\hat{r}} := \begin{cases} \kappa^{r^-} & \kappa < 1, \\ \kappa^{r^+} & \kappa \geq 1 \end{cases}$$

for $r \in C_+(\Omega)$. Then the well-known proposition 2.7 of [12] will be rewritten as follows.

Proposition 2.1. *For each $u \in L^{q(\cdot)}(\Omega)$ and $q \in C_+(\Omega)$, we have*

$$|u|_{q(\cdot)}^{\check{q}} \leq \int_\Omega |u(\xi)|^{q(\xi)} d\xi \leq |u|_{q(\cdot)}^{\hat{q}}.$$

The next lemma was established in [9].

Lemma 2.1. *Assume that $q, r \in C_+(\bar{\Omega})$. If $q(\xi) \leq r(\xi)$ for all $\xi \in \bar{\Omega}$, then $L^{r(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.*

Remark 2.2. *Let $q \in C_+(\Omega)$ with $q(\xi) < q^+ < m^*$ a.a. in Ω. Thanks to Remark 2.1 and Lemma 2.1, we have the following compact embedding:*

$$HW^{1,m}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$$

as $1 \leq m \leq Q$. Thus, there exists a $C_{q(\cdot)} > 0$ such that

$$|u|_{q(\cdot)} \leq C_{q(\cdot)} \|u\|_{1,m},$$

and for every bounded sequence $\{u_n\}$ in $HW^{1,m}(\Omega)$, up to the subsequence, $\{u_n\}$ converges to some $\bar{u}$ in $L^{q(\cdot)}(\Omega)$.

We continue by defining the Muckenhoupt weight functions rewritten on the Heisenberg groups.

Definition 2.4. (Muckenhoupt Weight) *Let $w : \mathbb{H}^n \to (0, \infty)$ be a locally integrable function. Then, we say that w belongs to the Muckenhoupt class A_m if there exists a positive constant $c_{m,w}$ depending only on m and w such that, for all Korányi balls B in $\mathbb{H}^n$,*

$$(\frac{1}{|B|}\int_B w d\xi)(\frac{1}{|B|}\int_B w^{-\frac{1}{m-1}} d\xi)^{m-1} \leq c_{q,w}.$$

A subclass of A_m: Let us define a subclass of A_m by

$$A_s = \Big\{ w \in A_m : w^{-s} \in L^1(\Omega) \text{ for some } s \in [\frac{1}{m-1}, \infty) \cap (\frac{Q}{m}, \infty) \Big\}.$$

Example 2.1. $w(\xi) = |\xi|^\alpha \in A_s \subset A_m$*, for any $-\frac{Q}{s} < \alpha < \frac{Q}{s}$, provided $1 < m < Q$.*

For $1 < m < Q$ and $w \in A_s$ with $s \in [\frac{1}{m-1}, \infty) \cap (\frac{Q}{m}, \infty)$, we set

$$m_s = \frac{sm}{s+1} \qquad \& \qquad m_s^* = \frac{m_s Q}{Q - m_s}.$$

Notice that by simple calculations one can show that $1 \leq m_s < m < m_s^*$. Define

$$L_w^m(\Omega) := \{u : \Omega \to \mathbb{R}^n \text{ measurable} : \int_\Omega |u(\xi)|^m w(\xi) d\xi < \infty\},$$

which has the norm

$$|u|_{m,w} = (\int_\Omega |u(\xi)|^m w(\xi) d\xi)^{\frac{1}{m}}.$$

We define the weighted Heisenberg-Sobolev space $HW^{1,m}(\Omega, w)$ by

$$HW^{1,m}(\Omega, w) := \{u \in L_w^m(\Omega) : \ |\nabla_{\mathbb{H}^n} u| \in L_w^m(\Omega)\},$$

endowed with the norm

$$\|u\|_{1,m,w} = |u|_{m,w} + |\nabla_{\mathbb{H}^n} u|_{m,w}.$$

We need the following fact for embeddings.

Lemma 2.2. *We have*

$$HW^{1,m}(\Omega, w) \hookrightarrow HW^{1,m_s}(\Omega).$$

Proof. Let $u \in HW^{1,m}(\Omega, w)$. Since $\frac{m}{m_s} > 1$, using the Hölder inequality with the exponents $\frac{m}{m_s}$ and $(\frac{m}{m_s})' = s + 1$, we obtain

$$\begin{aligned} |u|_{m_s}^{m_s} &= \int_\Omega |u(\xi)|^{m_s} w^{\frac{m_s}{m}}(\xi) w^{-\frac{m_s}{m}}(\xi) d\xi \\ &\leq (\int_\Omega |u(\xi)|^m w(\xi) d\xi)^{\frac{m_s}{m}} (\int_\Omega w^{-s}(\xi) d\xi)^{\frac{1}{s+1}}, \end{aligned}$$

which implies that

$$|u|_{m_s} \leq (\int_\Omega w^{-s}(\xi) d\xi)^{\frac{1}{sm}} |u|_{m,w}. \tag{2.2}$$

Replacing u by $\nabla_{\mathbb{H}^n} u$, we gain

$$|\nabla_{\mathbb{H}^n} u|_{m_s} \leq (\int_\Omega w^{-s}(\xi) d\xi)^{\frac{1}{sm}} |\nabla_{\mathbb{H}^n} u|_{m,w}. \tag{2.3}$$

Adding (2.2) and (2.3), one has $\|u\|_{1,m_s} \leq |w^{-s}|_1^{\frac{1}{sm}} \|u\|_{1,m,w}$. Thus,

$$HW^{1,m}(\Omega, w) \hookrightarrow HW^{1,m_s}(\Omega).$$

□

Remark 2.3. *It is easy to see that, by standard embeddings in the Heisenberg-Sobolev spaces mentioned in Remark 2.1 and Lemma 2.2, one has*

$$HW^{1,m}(\Omega, w) \hookrightarrow L^{\sigma}(\Omega), \quad \text{for all } 1 \le \sigma \le m_s^*$$

as $1 \le m \le Q$*; so, there exists a constant* k_σ *such that*

$$|u|_\sigma \le k_\sigma \|u\|_{1,m,w}$$

for each $u \in HW^{1,m}(\Omega, w)$*.*

Notice that embedding $HW^{1,m}(\Omega, w) \hookrightarrow L^{\sigma}(\Omega)$ is compact if $1 \le \sigma < m_s^*$.

Remark 2.4. *Let* $q \in C_+(\Omega)$ *such that* $q(\xi) < q^+ < m_s^*$ *a.a. in* Ω*. Thanks to Remark 2.2 and Lemma 2.2, for* $1 \le m \le Q$*, we have the following compact embedding*

$$HW^{1,m}(\Omega, w) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$

Now, for $p \in C_+(\Omega)$, define the weighted Lebesgue space with a variable exponent as follows:

$$L_w^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R}^n \text{ measurable} : \int_\Omega |u(\xi)|^{p(\xi)} w(\xi) d\xi < \infty\},$$

which has the norm

$$|u|_{p(\cdot),w} = inf\{\lambda > 0 : \int_\Omega |\frac{u(\xi)}{\lambda}|^{p(\xi)} w(\xi) d\xi \le 1\}.$$

We denote the weighted Heisenberg-Sobolev space with a variable exponent by

$$HW^{1,p(\cdot)}(\Omega, w) := \{u \in L_w^{p(\cdot)}(\Omega) : \ |\nabla_{\mathbb{H}^n} u| \in L_w^{p(\cdot)}(\Omega)\},$$

equipped with the norm

$$\|u\| := |u|_{p(\cdot),w} + |\nabla_{\mathbb{H}^n} u|_{p(\cdot),w}.$$

Here-in-after, for $p \in C_+(\Omega)$ with $p(\xi) \le p^+ \le Q$ a.a. in Ω, we put

$$X := HW^{1,p(\cdot)}(\Omega, w)$$

with the norm $\|u\|$.

Remark 2.5. *Let* $q \in C_+(\Omega)$ *such that* $q(\xi) < q^+ < \mathfrak{p}^* := (p_s^-)^*$ *a.a. in* Ω*. Then,*

$$X \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$

Furthermore, we denote the Sobolev embedding constant of this compact embedding by $K_{q(\cdot)} > 0$*, i.e.,*

$$|u|_{q(\cdot)} \le K_{q(\cdot)} \|u\|$$

for each $u \in X$*.*

Proof. As a consequence of Lemma 2.1, for $p, q \in C_+(\bar{\Omega})$, one has

$$X = HW^{1,p(\cdot)}(\Omega, w) \hookrightarrow HW^{1,q(\cdot)}(\Omega, w)$$

if $q(\xi) \leq p(\xi)$ a.e. $\xi \in \Omega$. In a special case, we gain

$$X \hookrightarrow HW^{1,p^-}(\Omega, w).$$

On the other hand, from Remark 2.4, for $q \in C_+(\Omega)$ with $q(\xi) < q^+ < \mathfrak{p} = (p_s^-)^*$ a.a. in Ω, we have

$$HW^{1,p^-}(\Omega, w) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$

So, the proof is completed. □

Remark 2.6. *For $u \in X$, there exist $\mu, M > 0$ such that*

$$\mu\|u\|^{\check{p}} \leq \int_\Omega (|\nabla_{\mathbb{H}^n} u|^{p(\xi)} + \mathcal{R}(\xi)|u|^{p(\xi)})w(\xi)d\xi \leq M\|u\|^{\hat{p}}.$$

Proof. Since $\operatorname{ess\,inf}_\Omega \mathcal{R} > 0$, there exists $0 < \delta < 1$ such that $\delta < \mathcal{R}(\xi)$ a.e. in Ω. Using Proposition 2.1 and the hypothesis $\mathcal{R} \in L^\infty(\Omega)$, we gain

$$\delta|u|^{\check{p}}_{p(\cdot),w} \leq \int_\Omega \mathcal{R}(\xi)|u(\xi)|^{p(x)}w(\xi)d\xi \leq \|\mathcal{R}\|_\infty |u|^{\hat{p}}_{p(\cdot),w},$$

and

$$\delta|\nabla_{\mathbb{H}^n} u|^{\check{p}}_{p(\cdot),w} \leq |\nabla_{\mathbb{H}^n} u|^{\check{p}}_{p(\cdot),w} \leq \int_\Omega |\nabla_{\mathbb{H}^n} u(\xi)|^{p(\xi)}w(\xi)d\xi \leq |\nabla_{\mathbb{H}^n} u|^{\hat{p}}_{p(\cdot),w}.$$

Bearing in mind the following elementary inequality due to J.A. Clarkson: for all $\gamma > 0$, there exists a $C_\gamma > 0$ such that

$$|a + b|^\gamma \leq C_\gamma(|a|^\gamma + |b|^\gamma)$$

for all $a, b \in \mathbb{R}$. Then, we deduce

$$\frac{\delta}{C_{\check{p}}}\|u\|^{\check{p}} \leq \int_\Omega (|\nabla_{\mathbb{H}^n} u|^{p(\xi)} + \mathcal{R}(\xi)|u|^{p(\xi)})w(\xi)d\xi \leq (1 + \|\mathcal{R}\|_\infty)\|u\|^{\hat{p}}.$$

So the proof is complete; it is enough to put $\mu = \frac{\delta}{C_{\check{p}}}$, $M = 1 + \|\mathcal{R}\|_\infty$. □

We continue by providing some briefs from variational calculus; the interested reader can see more details in [22] and the references therein.

Let V be a real Banach space and V^* be its topological dual; and, also assume that the pairing between V and V^* denoted by $\langle \ , \ \rangle$.

Definition 2.5. *(Subdifferential) Let $\Psi : V \to (-\infty, +\infty]$ be a proper (i.e. Dom $\Psi \neq \emptyset$), convex function. The subdifferential (generalized gradient) of Ψ denoted by $\partial\Psi$, $\partial\Psi : V \to 2^{V^*}$, for $u \in$ Dom$(\Psi) = \{v \in V; \Psi(v) < \infty\}$, is defined as the following set-value operator*

$$\partial\Psi(u) = \{u^* \in V^* : \Psi(v) \geq \Psi(u) + \langle u^*, v - u\rangle \text{ for all } v \in V\},$$

and $\partial\Psi(u) = \emptyset$ if $u \notin$ Dom(Ψ).

Notice that, if Ψ is Gâteaux differentiable at u, which has a derivative that is denoted by $D\Psi(u)$, $\partial\Psi(u)$ is a singleton. In this case, $\partial\Psi(u) = \{D\Psi(u)\}$.

Definition 2.6. *(Critical Point) Let V be a real Banach space , $\Phi \in C^1(V, \mathbb{R})$ and $\Psi : V \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Let $K \subset V$ be a weakly closed convex set. Define the function $\Psi_K : V \to (-\infty, +\infty]$ by*

$$\Psi_K(u) := \begin{cases} \Psi(u) & u \in K, \\ +\infty & u \notin K. \end{cases} \tag{2.4}$$

Consider the functional

$$I := \Psi_K - \Phi; \tag{2.5}$$

a point $u \in V$ is called a critical point of I, if $D\Phi(u) \in \partial\Psi_K(u)$ or, equivalently, it satisfies the following inequality:

$$\langle D\Phi(u), u - v\rangle + \Psi_K(v) - \Psi_K(u) \geq 0, \quad \textit{for all } v \in V. \tag{2.6}$$

The following result has been proved in [2, Theorem 1.5.6].

Theorem 2.2. *Let V be a reflexive Banach space and $I : V \to \mathbb{R}$ be a continuous, convex and coercive functional. Then, I has a global minimum point.*

Notice that a global minimum point is a critical point.

Definition 2.7. *((PS) Condition) We say that I mentioned in (2.5) satisfies the Palais-Smale compactness condition (in short, (PS) condition) if, for every sequence $\{u_n\}$, the following states are satisfied:*

- *$I(u_n) \to c \in \mathbb{R}$;*
- *$\langle D\Phi(u_n), u_n - v\rangle + \Psi_K(v) - \Psi_K(u_n) \geq -\epsilon_n\|v - u_n\|$ for all $v \in V$ as $\epsilon_n \to 0$;*

then, $\{u_n\}$ possesses a convergent subsequence.

The following mountain pass geometry (MPG) theorem was proved in [34].

Theorem 2.3. *Suppose that $I : V \to (-\infty, +\infty]$ is of the form (2.5) and satisfies the (PS) condition and the following conditions:*

(i) *$I(0) = 0$;*
(ii) *there exists $e \in V$ such that $I(e) \leq 0$;*
(iii) *there exists a positive constant λ such that $I(u) > 0$, if $\|u\| = \lambda$;*

then, I has a critical value $c \geq \lambda$ which is characterized by

$$c = \inf_{g\in\Gamma} \sup_{t\in[0,1]} I(g(t)),$$

where $\Gamma = \{g \in C([0, 1], V) : g(0) = 0\ ,\ g(1) = e\}$.

Definition 2.8. *(Pointwise Invariance Condition) Let $\Phi, \Psi : V \to \mathbb{R}$ be defined as in Definition 2.6 and K be any subset of V. We say that the triple (Ψ, Φ, K) satisfies the pointwise invariance condition at a point $u \in V$ if there exist a convex Gâteaux-differentiable function $G : V \to \mathbb{R}$ and a point $v \in K$ such that*

$$D\Psi(v) + DG(v) = D\Phi(u) + DG(u).$$

Here, we recall a variational principle established in [16] which we apply to prove our main approach.

Theorem 2.4. *Let V be a reflexive Banach space and K be a convex and weakly closed subset of V. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function which is Gâteaux differentiable on K, and let $\Phi \in C^1(V, \mathbb{R})$. Assume that the following two assertions hold:*

(i) The functional $I : V \to \mathbb{R} \cup \{+\infty\}$ defined by $I(w) = \Psi_K(w) - \Phi(w)$ has a critical point $u \in V$ in the sense of Definition 2.6,

(ii) The triple (Ψ, Φ, K) satisfies the pointwise invariance condition at the point u.

Then, $u \in K$ is a (weak) solution of the equation

$$D\Psi(u) = D\Phi(u).$$

Remark 2.7. *Notice that if Ψ is Gâteaux differentiable on $Dom\Psi$, u is a critical point of $I(w) = \Psi(w) - \Phi(w)$ and there exists $v \in Dom\Psi$ such that*

$$D\Psi(v) + DG(v) = D\Phi(u) + DG(u);$$

then u is a solution of $D\Psi(u) = D\Phi(u)$, but it does not necessarily belongs to K.

The next is a fact mentioned in [17, problem 127, page 81] or in [23].

Theorem 2.5. *Assume that $\{u_n\}$ is a sequence of monotonic (continuous or discontinuous) real functions on $[c, d]$ which converge pointwise to a continuous function $u : [c, d] \to \mathbb{R}$; then, the convergence is uniform.*

Remark 2.8. *Let Ω be a bounded open domain. Consider the closed convex set K as follows:*

$$K = \{u : \Omega \to \mathbb{R} : \ u \geq 0, \ u \text{ is an increasing radial function}\}.$$

Suppose that $\{u_n\}$ is a sequence in K such that $u_n \to \bar{u}$ a.e. in Ω. Then, regardless of a set of zero measures, $\{|u_n - \bar{u}|\}_{n\in\mathbb{N}}$ converge to zero uniformly.

Proof. Clearly $\bar{u}$ is a positive radial function; moreover, $u \in K$, since K is closed. If $\bar{u}$ is a continuous function, then Theorem 2.5 deduces $u_k \to \bar{u}$ uniformly. Otherwise, imagine that E contains all of the discontinuous points of $\bar{u}$. According to [27, Theorem 4.30] every monotonic function is discontinuous at a countable set of points at most, so E is at most countable with a Lebesgue measure of zero. Thus $\bar{u}$ is continuous on $\Omega \backslash E$ and the convergence of $\{|u_k - \bar{u}|\}_{k\in\mathbb{N}}$ to zero is uniform. □

3. Existence result

Here, we state the main result of this paper.

Theorem 3.1. *Let Ω be the unit Korányi ball in the Heisenberg group $\mathbb{H}^n$ $(n \geq 1)$ and $p \in C_+(\Omega)$ with $p(\xi) \leq p^+ \leq Q$ a.a. in Ω. Let $\theta, \vartheta \in C_+(\Omega)$ such that*

$$p^+ < \theta^- < \theta(\xi) < \theta^+ < \mathfrak{p}^* \quad \text{and} \quad \vartheta^+ < \theta^- \text{ a.e. on } \Omega,$$

where

$$\mathfrak{p}^* = \frac{\mathfrak{p}Q}{Q - \mathfrak{p}} \qquad \textit{and} \qquad \mathfrak{p} := p_s^- = \frac{sp^-}{s+1}$$

for s with

$$s \in [\frac{1}{p^- - 1}, +\infty) \cap (\frac{Q}{p^-}, +\infty).$$

Let w be a Muckenhoupt weight function of the class A_s and $\mathcal{R} : \Omega \to [0, +\infty)$ belong to $L^\infty(\Omega)$ such that $\operatorname{ess\,inf}_\Omega \mathcal{R} > 0$. *Assume $\rho, \varrho \in L^\infty(\Omega)$ satisfy the condition* (1.1). *Then the Dirichlet problem $(\mathcal{P})$ admits at least one radially increasing (weak) solution.*

Set

$$V := X_{rad} \cap L_\rho^{\theta(\cdot)}(\Omega) \cap L_\varrho^{\vartheta(\cdot)}(\Omega),$$

equipped with the norm

$$\|u\|_V := \|u\| + |u|_{\rho,\theta} + |u|_{\varrho,\vartheta},$$

where

$$X_{rad} = \{u \in X : u \geq 0,\ u \text{ is a radial function}\}.$$

It is clear that V is a reflexive Banach space. Now, consider the Euler-Lagrange energy functional corresponding to the problem $(\mathcal{P})$, i.e.,

$$\begin{aligned} E(u) := & \int_\Omega \frac{1}{p(\xi)}(|\nabla_{\mathbb{H}^n} u|^{p(\xi)} + \mathcal{R}(\xi)|u|^{p(\xi)})w(\xi)d\xi \\ & + \int_\Omega \frac{1}{\vartheta(\xi)}\varrho(\xi)|u|^{\vartheta(\xi)}d\xi - \int_\Omega \frac{1}{\theta(\xi)}\rho(\xi)|u|^{\theta(\xi)}d\xi, \end{aligned}$$

as well as the closed convex set

$$K := \{u \in V : u \geq 0, u \text{ is increasing with respect to the radius } \mathbf{r} = |\xi|_{\mathbb{H}^n}\}.$$

To adapt Theorem 2.4 to our problem, we define $\psi, \varphi : V \to \mathbb{R}$ by

$$\psi(u) := \int_\Omega \frac{1}{p(\xi)}(|\nabla_{\mathbb{H}^n} u|^{p(\xi)} + \mathcal{R}(\xi)|u|^{p(\xi)})w(\xi)d\xi + \int_\Omega \frac{1}{\vartheta(\xi)}\varrho(\xi)|u|^{\vartheta(\xi)}d\xi,$$

and

$$\varphi(u) := \int_\Omega \frac{1}{\theta(\xi)}\rho(\xi)|u|^{\theta(\xi)}d\xi.$$

Notice that ψ is a proper, convex, lower semicontinuous function and $D\varphi(u) = \rho(\xi)|u|^{\theta(\xi)-2}u$; therefore, φ is a C^1- function on the space V. Let us introduce the functional $I : V \to (-\infty, +\infty]$ as follows:

$$I(u) = \psi_K(u) - \varphi(u), \tag{3.1}$$

where ψ_K is defined as (2.4).
We should be aware that I is indeed the Euler-Lagrange functional corresponding to our problem (denoted by $E(\cdot)$) as restricted to K, and it is clear that the critical points of I are exactly the radially increasing weak solutions of $(\mathcal{P})$.

We prove Theorem 3.1 in two steps:

Step1. We show that I has a critical point and, for this reason, we need the following lemma.

Lemma 3.1. *Let* $V = X_{rad} \cap L_{\rho}^{\theta(\cdot)}(\Omega) \cap L_{\varrho}^{\vartheta(\cdot)}(\Omega)$ *and consider the functional* $I : V \to \mathbb{R}$ *by applying*

$$I(u) := \psi_K(u) - \varphi(u)$$

as in (3.1). *Then, I has a nontrivial critical point in K.*

Proof. We apply the MPG theorem (Theorem 2.3) to prove this lemma.

First, we verify that I satisfies the following MPG conditions:

It is clear that $I(0) = 0$. Take $e \in K$. From Remark 2.6, we have the following estimate

$$I(te) \leq \frac{M}{p^+} t^{\hat{p}} \|e\|^{\hat{p}} + t^{\hat{\vartheta}} \int_{\Omega} \frac{1}{\vartheta(\xi)} \varrho(|\xi|_{\mathbb{H}^n}) |e|^{\vartheta(\xi)} d\xi - t^{\hat{\theta}} \int_{\Omega} \frac{1}{\theta(\xi)} \rho(|\xi|_{\mathbb{H}^n}) |e|^{\theta(\xi)} d\xi,$$

since $\theta^- > \vartheta^+$ and $\theta^- > p^+$, for t sufficiently large, $I(te)$ is negative. We now prove Condition (*iii*) of the MPG theorem. Take $u \in Dom(\psi)$ with $\|u\| = \lambda > 0$. Notice that from Lemmas 2.5 and 2.6, for $u \in K$, we have

$$\begin{aligned} \varphi(u) &= \int_{\Omega} \frac{1}{\theta(\xi)} \rho(|\xi|_{\mathbb{H}^n}) |u|^{\theta(\xi)} d\xi \\ &\leq \frac{1}{\check{\theta}} \|\rho\|_{\infty} |u|_{\theta(\cdot)}^{\hat{\theta}} \\ &\leq \frac{1}{\check{\theta}} \|\rho\|_{\infty} K_{\theta(\cdot)} \|u\|^{\hat{\theta}} \\ &\leq C_3 \lambda^{\hat{\theta}}. \end{aligned}$$

Thus

$$I(u) \geq \frac{\mu}{p^+} \lambda^{\check{p}} - C_1 \lambda^{\hat{\theta}} > 0,$$

provided $\lambda > 0$ is small enough as $2 \leq \check{p} < \hat{\theta}$ and C_1 is a positive constant. If $u \notin Dom(\psi)$, clearly, $I(u) > 0$. Therefore, the MPG holds for the functional I.

Second, we verify the following (PS) condition:

Suppose that $\{u_n\}$ is a sequence in K such that

$$I(u_n) \to c \in \mathbb{R}, \quad \text{as } \epsilon_n \to 0,$$

and let, for all $v \in V$,

$$\langle D\varphi(u_n), u_n - v \rangle + \psi_K(v) - \psi_K(u_n) \geq -\epsilon_n \|v - u_n\|. \tag{3.2}$$

We show that $\{u_n\}$ has a convergent subsequence in V. First notice that $u_n \in Dom(\psi)$; then,

$$I(u_n) = \psi_K(u_n) - \varphi(u_n) \to c, \quad as\ n \to \infty.$$

Thus, for large values of n we have

$$\psi_K(u_n) - \varphi(u_n) \leq 1 + c. \tag{3.3}$$

Now, consider the function $g(t) = t^{\vartheta^+} - \theta^-(t-1) - 1$ on the interval $(1, +\infty)$ and set $\hat{t} = (\frac{\theta^-}{\vartheta^+})^{\frac{1}{\vartheta^+-1}}$. It is easy to see that for every $t \in (1, \hat{t})$ we have $g(t) < 0$. We choose such a number t for which we have $t > 1$ and $t^{\vartheta^+} - 1 < \theta^-(t-1)$. In (3.2), set $v = tu_n$; then,

$$(1-t)\langle D\varphi(u_n), u_n\rangle + (t^{\vartheta^+} - 1)\psi_K(u_n) \geq -\epsilon_n(t-1)\|u_n\|. \tag{3.4}$$

Furthermore,

$$\langle D\varphi(u_n), u_n\rangle = \int_\Omega \rho(\xi)\ u_n(\xi)^{\theta(\xi)}\ d\xi \geq \theta^-\ \varphi(u_n). \tag{3.5}$$

Since $t^{\vartheta^+} - 1 < \theta^-(t-1)$, we can take $\gamma > 0$ such that

$$\frac{1}{\theta^-(t-1)} < \gamma < \frac{1}{t^{\vartheta^+} - 1}.$$

Multiplying (3.4) by γ and adding it to (3.3) we obtain

$$[1 - \gamma\theta^-(1-t)]\varphi(u_n) + [1 - \gamma(t^{\vartheta^+} - 1)]\psi_K(u_n) \leq 1 + c + \gamma C\|u_n\|.$$

So, using Remark 2.6 for some suitable constant $C' > 0$, we have

$$\frac{\mu}{p^+}\|u_n\|^{\check{p}} \leq \psi_K(u_n) \leq C'(1 + \|u_n\|).$$

Therefore, $\{u_n\}$ is a bounded sequence in the reflexive space X. Thanks to Remark 2.5, we gain that there exists $\bar{u} \in X$ such that, up to the subsequences, the following holds true

- $u_n \rightharpoonup \bar{u}$ in X;
- $u_n \to \bar{u}$ in $L^{q(\xi)}(\Omega)$, $q \in C_+(\Omega)$ and $q(\xi) < q^+ < \mathfrak{p}^*$;
- $u_n(\xi) \to \bar{u}(\xi)$ a.e in Ω;

On the one hand, $\{u_n\} \subset K$; so, according to Remark 2.8, regardless of a set of measure of zeros, $\{|u_n - \bar{u}|\}_{n\in\mathbb{N}}$ converges to zero uniformly. Then,

$$\begin{aligned}\psi(u_n - \bar{u}) = &\int_\Omega \frac{1}{p(\xi)}(|\nabla_{\mathbb{H}^n}(u_n - \bar{u})|^{p(\xi)} + \mathcal{R}(\xi)|u_n - \bar{u}|^{p(\xi)})w(\xi)d\xi \\ &+ \int_\Omega \frac{1}{\vartheta(\xi)}\varrho(\xi)|u_n - \bar{u}|^{\vartheta(\xi)}d\xi \to 0, \quad \text{as } n \to +\infty.\end{aligned} \tag{3.6}$$

On the other hand, Remark 2.6 gives

$$\frac{\mu}{\hat{p}}\|u_n - \bar{u}\|^{\check{p}} \leq \psi(u_n - \bar{u}). \tag{3.7}$$

From the inequalities (3.6) and (3.7), we deduce that $\|u_n - \bar{u}\| \to 0$. As we mentioned before, by applying the standard embeddings

$$u_n \to \bar{u} \quad \text{in} \quad L^{\vartheta(\cdot)}_{\varrho}(\Omega)\ \&\ L^{\theta(\cdot)}_{\rho}(\Omega),$$

$u_n \to \bar{u}$ strongly in V, as desired. □

Remark 3.1. *Notice that for each $n \in \mathbb{N}$, $u_n \in K$ is radial, so $\bar{u}$ is radial. K is a closed subset of V; then, $\bar{u} \in K$. Therefore, the MPG theorem guarantees that the existence of a critical point belongs to K, namely $\hat{u}$.*

Step2. We show that for any $u \in K$, in a special case $\hat{u}$, the triple $(\psi_K, \varphi, Dom\psi)$ satisfies the pointwise invariance condition at u when $G = 0$. To this end, we shall need following lemma.

Lemma 3.2. *Let $\mathcal{R} \in L^\infty(\Omega)$ be a nonnegative real functional. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and there exist constants $a, b > 0$ such that*

$$|f(t)| \leq a + b|t|^{\gamma-1} \qquad \text{for all } t \in \mathbb{R}, \tag{3.8}$$

where $\gamma \in (1, \mathfrak{p}^*)$; moreover,

$$f(t)t \leq 0 \qquad \text{for all } t \in \mathbb{R}.$$

Then for every $h \in L^{\frac{sp^-}{s(p^--1)-1}}(\Omega)$ the problem

$$\begin{cases} \mathcal{L}(u) = f(u) + h(\xi) & \xi \in \Omega, \\ u = 0 & \xi \in \partial\Omega, \end{cases} \tag{3.9}$$

where

$$\mathcal{L}(u(\xi)) = div_{\mathbb{H}^n}(w(\xi)|\nabla_{\mathbb{H}^n} u|^{p(\xi)-2}\nabla_{\mathbb{H}^n} u) + \mathcal{R}(\xi)w(\xi)||u|^{p(\xi)-2}u,$$

admits at least one solution.

Proof. First notice that by integration one can see that there exist $a_1, b_1 > 0$ such that

$$|F(t)| \leq a_1 + b_1|t|^\gamma \qquad \text{for all } t \in \mathbb{R},$$

and that $F(t) \leq 0$ for all $t \in \mathbb{R}$, where

$$F(t) = \begin{cases} \int_0^t f(\tau)d\tau & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Now, consider the following energy functional on X which is corresponding to Problem (3.9):

$$J(u) = \int_\Omega \frac{1}{p(\xi)}(|\nabla_{\mathbb{H}^n} u|^{p(\xi)} + \mathcal{R}(\xi)|u|^{p(\xi)})w(\xi)d\xi - \int_\Omega F(u)d\xi - \int_\Omega hud\xi.$$

Because of the growth condition (3.8) J is well-defined on X. According to the Hölder inequality, one has $\int_\Omega hud\xi \leq |u|_{\mathfrak{p}}|h|_{\mathfrak{p}'}$ where, by (2.2),

$$|u|_{\mathfrak{p}} \leq c|u|_{p^-,w} \leq c'||u||,$$

and by the assumption of the lemma $h \in L^{\mathfrak{p}'}(\Omega)$, $\mathfrak{p}' = \frac{sp^-}{s(p^--1)-1}$. Then thanks to Remark 2.2 we have

$$J(u) \geq \frac{\mu}{p^+}||u||^{\check{p}} - C||u||.$$

This is because $p^- > 1$, J is coercive. Thus according to Theorem 2.2, J has a global minimum point, meaning that Problem (3.9) admits at least one solution. $\square$

Lemma 3.3. *Let $u \in K$, $\mathcal{R} \in L^{\infty}(\Omega)$ and $\rho, \varrho \in L^{\infty}(\Omega)$ be defined as in (1.1). Let $p, \theta, \vartheta \in C_{+}(\Omega)$ with*

$$p^{+} < \theta^{-} < \theta(\xi) < \theta^{+} < \mathfrak{p}^{*} \quad \& \quad \vartheta^{+} < \theta^{-} a.e.\ in\ \ \Omega.$$

Then, there exists $v \in Dom\psi$ such that

$$\begin{cases} \mathcal{L}(v) = \rho(\xi)u^{\theta(\xi)-1} - \varrho(\xi)v^{\vartheta(\xi)-1} & in\ \Omega, \\ v = 0 & on\ \partial\Omega, \end{cases}$$

in the weak sense.

Proof. Let $u \in K$; so, $0 \leq u \in K \subset X$; also, set

$$f(v(\xi)) = -\varrho(\xi)v(\xi)^{\vartheta(\xi)-1}.$$

Then, thanks to Lemma 3.2, it is enough to show that

$$h(\xi) = \rho(\xi)\ u(\xi)^{p(\xi)-1} \in L^{\mathfrak{p}'}(\Omega).$$

But u is a radial function (i.e., $u(\xi) = \phi(\mathbf{r})$); so, by [7, equation (2.4)] we have $|\nabla_{\mathbb{H}^n} u| = \frac{\mathbf{r}}{r}|\phi'|$, where $\mathbf{r} = |\xi|_{\mathbb{H}^n} = |(z,t)|_{\mathbb{H}^n}$, $r = |z|$. Using the fundamental theorem of calculus and Hölder inequality, one has following estimate:

$$\begin{aligned} |u(\xi)| =& |\phi(\mathbf{r})| = |\int_0^{\mathbf{r}} \phi'(\tau)d\tau + \phi(0)| \\ \leq& \int_0^{\mathbf{r}} |\phi'(\tau)|d\tau + |\phi(0)| \\ \leq& (\int_0^{\mathbf{r}} |\phi'(\tau)|^{\mathfrak{p}}\tau^{Q-1}d\tau)^{\frac{1}{\mathfrak{p}}}(\int_0^{\mathbf{r}} \tau^{\frac{-(Q-1)\mathfrak{p}'}{\mathfrak{p}}}d\tau)^{\frac{1}{\mathfrak{p}'}} + |\phi(0)| \\ \leq& C(\omega_{Q-1}^{-1}\int_{\Omega} |\phi'(\xi)|^{\mathfrak{p}}d\xi)^{\frac{1}{\mathfrak{p}}}\mathbf{r}^{(1-\frac{(Q-1)\mathfrak{p}'}{\mathfrak{p}})\frac{1}{\mathfrak{p}'}} + |\phi(0)| \\ \leq& C'\Big(\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^{\mathfrak{p}}d\xi\Big)^{\frac{1}{\mathfrak{p}}}\mathbf{r}^{(1-\frac{(Q-1)\mathfrak{p}'}{\mathfrak{p}})\frac{1}{\mathfrak{p}'}} + |\phi(0)| \\ \leq& C_{*}\ \|u\| |\xi|_{\mathbb{H}^n}^{(1-\frac{(Q-1)\mathfrak{p}'}{\mathfrak{p}})\frac{1}{\mathfrak{p}'}}, \end{aligned}$$

where ω_{Q-1} is the measure of the unit ball in $\mathbb{H}^n$ and C, C' and C_{*} are positive constants. A computation shows that $u(\xi)^{\theta-1} \in L^{\mathfrak{p}'}(\Omega)$. So, the proof is complete. □

4. Conclusions

In this paper, first, we looked for a suitable interval embedding of weighted Heisenberg-Sobolev spaces with variable exponents into the Lebesgue spaces in a step by step manner. In Remark 2.3, using Lemma 2.2, we found the following embedding

$$HW^{1,m}(\Omega, w) \hookrightarrow L^{\sigma}(\Omega) \qquad \text{for all } 1 \leq \sigma \leq m_s^{*},$$

as $1 \leq m \leq Q$. Thanks to Remark 2.2 and Lemma 2.2, for $1 \leq m \leq Q$, we generalized the result as follows:

$$HW^{1,m}(\Omega, w) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$

Finally, in Remark 2.5, we proved that if $q \in C_+(\Omega)$ such that $q(\xi) < q^+ < \mathfrak{p}^* := (p_s^-)^*$ a.a. in Ω, then

$$HW^{1,p(\cdot)}(\Omega, w) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$

Employing the result and MPG theorem, we proved that $I = \psi_K - \varphi$ has a critical point in K, namely $\hat{u}$, which is radial but may not necessarily be a solution of $(\mathcal{P})$. In Lemma 3.3, we showed that, for any $u \in K$, particularly for $\hat{u}$, there exists $v \in Dom\psi$ satisfying the equation $D\psi(v) = D\varphi(u)$. Indeed, we showed that the triple $(\psi, \varphi, Dom\psi)$ satisfies the point wise invariance condition at any $u \in K$, especially at $\hat{u}$, given $G = 0$. Therefore, Theorem 2.4, Remark 2.7 and the maximum principle for the $p(\cdot)$-Laplacian operator ensure the trueness of Theorem 3.1.

Acknowledgments

M. A. Ragusa was partially supported by Piano della Ricerca 2016–2018-Linea di intervento 2: “Metodi variazionali ed equazioni differenziali”.

Conflict of interest

All authors declare no conflicts of interest regarding this study.

References

1. F. Anceschi, S. Muzzioli, S. Polidoro, Existence of a fundamental solution of partial differential equations associated to Asian options, *Nonlinear Anal.,* **62** (2021), 103373. https://doi.org/10.1016/j.nonrwa.2021.103373
2. M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners, Existence Results via the Variational Approach*, Springer science and Business Media, 2010.
3. I. Birindelli, F. Ferrari, E. Valdinoci, Semilinear PDEs in the Heisenberg group: The role of the right invariant vector fields, *Nonlinear Anal.,* **72** (2010), 987–997. https://doi.org/10.1016/j.na.2009.07.039
4. G. M. Bisci, P. Pucci, Critical Dirichlet problems on H domains of Carnot groups, *Two nonlinear days in Urbino 2017, Electron. J. Diff. Eqns., Special Volume dedicated to the memory of Anna Aloe, Conf.,* **25** (2018), 179–196.
5. W. L. Chow, Über systeme von liearren partiellen differentialgleichungen erster ordnung, *Math. Ann.,* **117** (1940), 98–105. https://doi.org/10.1007/BF01450011
6. G. Citti, A. Pascucci, S. Polidoro, On the regularity of solutions to a nonlinear ultraparabolic equation arising in mathematical finance, *Differ. Integral Equ.,* **14** (2001), 701–738. https://doi.org/10.1177/197140090101400616

7. L. D'Ambrosio, Some Hardy inequalities on the Heisenberg group, *J. Differ. Equ.,* **40** (2004), 552–564. https://doi.org/10.1023/B:DIEQ.0000035792.47401.2a

8. L. Evans, *Partial Differential Equations, Sobolev Spaces*, Academic Press, New York, 1998.

9. X. L. Fan, D. Zhao, On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$, *J. Gansu Educ. College,* **12** (1998), 1–6.

10. N. Garofalo, D. M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, *Commun Pure Appl. Math.,* **49** (1996), 1081–1144. https://doi.org/10.1002/(SICI)1097-0312(199610)49:10¡1081::AID-CPA3¿3.0.CO;2-A

11. S. P. Ivanov, D. N. Vassilev, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, World Scientific, Publishing Co. Pte. Ltd., Hackensack, N. J., 2011.

12. Y. Karagiorgos, N. Yannakaris, A Neumann problem involving the $p(x)$-Laplacian with $p = \infty$ in a subdomain, *Adv. Calc. Var.,* **9** (2016), 65–76.

13. A. E. Kogoj, Y. Pinchover, S. Polidoro, On Liouville-type theorems and the uniqueness of the positive Cauchy problem for a class of hypoelliptic operators, *J. Evol. Equ.,* **16** (2015), 905–943.

14. H. Y. Lan, J. J. Nieto, Solvability of second-order uniformly elliptic inequalities involving demicontinuous ψ-dissipative operators and applications to generalized population models, *Eur. Phys. J. Plus,* **136** (2021), 1–17.

15. Y. Li, J. Xie, L. Mao, Existence of solutions for the boundary value problem of non-instantaneous impulsive fractional differential equations with p-Laplacian operator, *AIMS Math.,* **7** (2022), 17592–17602. https://doi.org/10.3934/math.2022968

16. A. Moameni, L. Salimi, Existence results for a supercritical Neumann problem with a convex-concave non-linearity, *Ann. Mat. Pura Appl.,* **198** (2019), 1165–1184. https://doi.org/10.1007/s10231-018-0813-1

17. G. Pólya, G. Szegö, *Problems and Theorems in Analysis I: Series. Integral Calculus. Theory of Functions*, Springer-Verlag Berlin Heidelberg New York, 1978.

18. P. Pucci, Critical Schrödinger-Hardy systems in the Heisenberg group, *Discrete Contin. Dyn. Syst. Ser. S, Special Issue on the occasion of the 60th birthday of Professor Vicentiu D. Radulescu,* **12** (2019), 375–400. https://doi.org/10.14225/Joh1634

19. P. Pucci, Existence of entire solutions for quasilinear equations in the Heisenberg group, *Minimax Theory Appl.,* **4** (2019).

20. P. Pucci, Existence and multiplicity results for quasilinear equations in the Heisenberg group, *Opuscula Math.,* **39** (2019), 247–257. https://doi.org/10.7494/OpMath.2019.39.2.247

21. P. Pucci, L. Temperini, Existence for (p, q) critical systems in the Heisenberg group, *Adv. Nonlinear Anal.,* **9** (2020), 895–922.

22. P. H. Rabinowitz, Some critical point theorems and applications to semilinear elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **2** (1978), 215–223. https://doi.org/10.1080/07481187808253308

23. M. A. Ragusa, A. Razani, F. Safari, Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem, *Adv. Differ. Equ.,* **2021** (2021), 1–14.

24. P. K. Rashevsky, Any two points of a totally nonholonomic space may be connected by an admissible line, Uch. Zap. Ped. Inst. im. Liebknechta, *Ser. Phys. Math.,* **2** (1938), 83–94.

25. A. Razani, F. Safari, Existence of radial weak solutions to Steklov problem involving Leray-Lions type operator, *J. Nonlinear Math. Phys.*, (2022), 17. https://doi.org/10.1007/s44198-022-00078-1

26. D. Ricciotti, *p-Laplace Equation in the Heisenberg Group: Regularity of Solutions*, Springer, 2015. https://doi.org/10.1007/978-3-319-23790-9

27. W. Rudin, *Principles of mathematical analysis*, Vol. 3. New York: McGraw-hill, 1964.

28. F. Safari, A. Razani, Existence of positive radial solutions for Neumann problem on the Heisenberg group, *Bound. Value Probl.,* **1** (2020), 1–14. https://doi.org/10.1186/s13661-020-01386-5

29. F. Safari, A. Razani, Nonlinear nonhomogeneous Neumann problem on the Heisenberg group, *Appl. Anal.*, (2020), 1–14. https://doi.org/10.1080/00036811.2020.1807013

30. F. Safari, A. Razani, Existence of radial solutions of the Kohn-Laplacian problem, *Complex Var. Elliptic Equ.*, (2020), 1–15. https://doi.org/10.1080/17476933.2020.1818733

31. F. Safari, A. Razani, Positive weak solutions of a generalized supercritical Neumann problem, *Iran J. Sci. Technol. Trans. Sci.,* **44** (2020), 1891–1898. https://doi.org/10.1007/s40995-020-00996-z

32. F. Safari, A. Razani, Radial solutions for a general form of a p-Laplace equation involving nonlinearity terms, *Complex Var. Elliptic Equ.*, (2021), 1–11. https://doi.org/10.1080/17476933.2021.1991331

33. F. Safari, A. Razani, Existence of radial solutions for a weighted p-biharmonic problem with Navier boundary condition on the Heisenberg group, *Math. Slovaca,* **72** (2022), 677–692. https://doi.org/10.1515/ms-2022-0046

34. A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **3** (1986), 77–109. https://doi.org/10.1016/S0022-3476(86)80576-5

35. F. J. Torres, Existence of positive solutions for boundary value problems with p-Laplacian operator, *Math. Slovaca,* **72** (2022), 1091–1109. https://doi.org/10.1515/ms-2022-0075