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Distorted expectiles risk measure and LP formulation

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ABSTRACT

Given a concave distortion function, we provide a dual representation of the expectiles based on rank-dependent expected utility theory. With possible application to portfolio management in mind, we also derive an LP formulation of the related optimization problem.

1. Introduction

Distortion risk measures can be tracked back to the dual utility theory of Yaari (1987) according to which a decision maker should use distorted tail probabilities to evaluate their prospects. Now, they are firmly established as coherent, law invariant and comonotone additive mappings $\rho : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X}$ is a linear set of loss random variables, containing the constants, defined over an atomless probability space $(\Omega, \mathscr{F}, \mathsf{P})$. In insurance theory the premium principle is a well established risk measure, as well in finance theory a risk measure is needed to determine a solvency capital requirement. Starting from distortion risk measures based on premium principles (see Wang, 1995) and passing through the modern theory of financial risk (see Föllmer and Schied, 2016) some risk measures can be recovered by minimizing an appropriate objective function. For example, generalized quantile risk measures $\rho(X)$ come as solutions to the minimization problem

$$\min_{x\in\mathbb{R}} \left\{ \tau \mathsf{E}_{\mathsf{P}} \left(u_1((X-x)_+) \right) + (1-\tau)\mathsf{E}_{\mathsf{P}} \left(u_2((X-x)_-) \right) \right\}, \tag{1}$$

where $\tau \in (0,1)$ is a probability level, $x_+ = \max\{x, 0\}$, $x_- = \min\{-x, 0\}$ and u_1, u_2 are increasing convex functions.

Our aim in this paper is twofold. First, we propose a dual representation of special generalized quantiles based on rank-dependent expected utility theory (RDEU, see Mao and Cai, 2018 for a recent development) defined as the minimizer in problem (1), assuming $u_1(x) = u_2(x) = x^2$ and replacing the expectation E_{P} taken with respect to (w.r.t.) the original probability measure P, or the corresponding CDF,[1] with the distorted expectation E_{ψ}, see Föllmer and Schied (2016, Sec 4.6) and Definition 2.1 below. We recall that the RDEU is the mapping $H_{u,\psi} : \mathcal{X} \to \mathbb{R}$ defined by

$$H_{u,\psi}(X) = \int_{-\infty}^{\infty} u(x)\mathrm{d}\psi(F_X(x)), \tag{2}$$

where $u : \mathbb{R} \to \mathbb{R}$ is the decision maker's utility function (viz. increasing and continuous) and ψ is the distortion function. Under this hypothesis, a distorted expectile should be defined as a solution to (1) rewritten as

$$\min_{x\in\mathbb{R}} \left\{ \tau H_{u,\psi}((X-x)_+) + (1-\tau)H_{u,\psi}((X-x)_-) \right\}. \tag{3}$$

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[1] In this case E_{P} is represented as a Stieltjes integral.

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Remark 1.1. It is well known that the Savage representation of the stochastic utility of $X \in \mathcal{X}$ is in the form

$$U(X) = \inf_{\mathsf{Q} \in \mathcal{Q}} \mathsf{E}_{\mathsf{Q}}(u(X)),$$

where u is some numerical function and $\mathcal{Q}$ is a convex set of probability measures, see Föllmer and Schied (2016, Ch 2). In fact, the utility functional U can be related to a loss functional via $L(X) = -U(X)$, which establishes a one-to-one correspondence between risk measures and utility:

$$L(X) = \sup_{\mathsf{Q} \in \mathcal{Q}} \mathsf{E}_{\mathsf{Q}}(-u(-X)),$$

and turning back to the utility functional one has $U(X) = \rho(-u(X))$. Thus we can talk about risk measures/expected losses or expected utility from a unifying point of view (see Delbaen, 2012). The two representations above are usually called dual representations, and can be obtained by working out the first order conditions of a minimization problem like that in (1).

The reason why we propose E_ψ is that the shortfall $(X-x)_+$ and the over-required capital $(X-x)_-$ in (1) are evaluated under P as in classical expected utility, instead we use RDEU which enable us to account for empirical perception of extreme events.

Our second goal is to provide a computational counterpart of distorted expectile risk measures. In this respect, we propose an LP formulation of the corresponding dual characterization aimed at possible application in asset allocation problems, where X is a portfolio of losses and one has to consider also minimization w.r.t. portfolio weights.

2. Preliminaries

In defining a distortion risk measure the following is a crucial ingredient in the sense of Yaari (1987). From now on we work with $\mathcal{X} = L^1$, the space of all equivalence classes of loss random variables, defined over $(\Omega, \mathcal{F}, \mathsf{P})$, having finite expectation taken w.r.t. the original probability measure P.

Definition 2.1. Let $X \in \mathcal{X}$ and $\psi : [0,1] \to [0,1]$ be a right-continuous increasing function with $\psi(0) = 0$ and $\psi(1) = 1$, called distortion function. The distorted expectation is defined as

$$\mathsf{E}_\psi(X) = \int_{-\infty}^{0} \left(\psi(1 - F_X(x)) - 1\right) \mathrm{d}x + \int_0^{+\infty} \psi(1 - F_X(x))\mathrm{d}x = \int_{-\infty}^{+\infty} x \, \mathrm{d}\tilde{\psi} \circ F_X(x), \tag{4}$$

where $\tilde{\psi} = 1 - \psi(1-x)$ is the convex *dual* distortion function and $\tilde{\psi} \circ F_X(x) = \tilde{\psi}(F_X(x))$.

Clearly, if $\psi(x) = x$ then $\mathsf{E}_\psi(X) = \mathsf{E}(X)$. An analytical approximation of distorted expectation is given in Xianming et al. (2015). Observe that if $\psi(x) \geqslant x$ for all $x \in \mathbb{R}$ then $\mathsf{E}_\psi(X) \geqslant \mathsf{E}(X)$. It is well known (see Schmeidler, 1986; Föllmer and Schied, 2016) that if ψ is a concave distortion then for any $A \in \mathcal{F}$ the distorted probability $\psi(P(A))$ is sub-modular i.e. a sub-additive set function and the distorted expectation can be given by the maximization

$$\mathsf{E}_\psi(X) = \max_{Q \in \mathcal{Q}_\psi} \mathsf{E}_Q(X), \tag{5}$$

where $\mathcal{Q}_\psi$ is the convex-compact set of all finitely additive normalized set functions $Q : \mathcal{F} \to [0,1]$ with $Q(A) \leqslant \psi(P(A))$ for all $A \in \mathcal{F}$, usually referred as the *core* of ψ. The next one is our proposed definition of distorted expectile.

Definition 2.2. Let $X \in \mathcal{X}$ and $\psi : [0,1] \to [0,1]$ be a distortion function, as in Definition 2.1. The distorted expectile is the minimizer

$$e_{\tau,\psi}(X) = \operatorname*{argmin}_{x \in \mathbb{R}} \{\tau \mathsf{E}_\psi((X-x)_+)^2 + (1-\tau)\mathsf{E}_\psi((X-x)_-)^2\}. \tag{6}$$

Definition 2.2 is based on the distorted expectation, and, by replacing E with E_ψ in the proof of Bellini et al. (2014, Prop 7(b)) and by using (Bellini et al., 2014, Prop7(c)) one gets a coherent risk measure for $\tau \geqslant \frac{1}{2}$. By using the same arguments, it is possible to verify the elicitability in the sense of Bellini and Bignozzi (2015). Indeed, the objective function in (6) is the distorted expectation of the scoring function $S(a,x) = \tau((a-x)_+)^2 + (1-\tau)((a-x)_-)^2$ which for given $a \in \mathbb{R}$ is strictly convex. This, in addition, guarantees the uniqueness of the solution to the minimization problem.

Before giving the dual representation of $e_{\tau,\psi}$, we recall the definition of the classical expectile:

$$e_\tau(X) = \max_{\varphi \in \mathcal{M}_\tau} \mathsf{E}(\varphi X), \qquad \text{for } \tau \geqslant \tfrac{1}{2}, \tag{7}$$

where $\mathcal{M}_\tau = \left\{ \varphi \in L^\infty \,\middle|\, \varphi > 0 \text{ a.s.}, \ \mathsf{E}_\mathsf{P}(\varphi) = 1, \ \frac{\operatorname{ess\,sup} \varphi}{\operatorname{ess\,inf} \varphi} \leqslant \frac{\tau}{1-\tau} \right\}$ is the scenario set, see Bellini et al. (2014, Prop 8) and L^∞ is the set of all equivalence classes of essentially bounded random variables. As noted by Delbaen (2013), the scenario set $\mathcal{M}_\tau$ is different from $\mathcal{Q}_\psi$ and the distorted expectile is different from the classical expectile. Moreover, the expectile order does not imply the same order among distorted expectiles, i.e. $X \preccurlyeq_{\mathrm{e}} Y \Longleftrightarrow e_\tau(X) \leqslant e_\tau(Y) \quad \not\Leftrightarrow \quad e_{\tau,\psi}(X) \leqslant e_{\tau,\psi}(Y)$ as shown in Example 3.1. The distorted expectile as defined in (6) is not a distorted risk measure because, in general, it lacks of comonotonic additivity. Here, $\psi(\mathsf{P}(A))$ represents a distorted perception of probability by decision makers, since they overstate the likelihood of adverse events $A \in \mathcal{F}$. In terms of the dual distortion, $\tilde{\psi}(x) \leqslant x$ and $\tilde{\psi} \circ F_X$ puts more weight on high values (losses) of X, giving a more pessimistic model.

3. Dual representation and discrete computation

The following result provides the dual representation of distorted expectiles.

Proposition 3.1. *Let $\tau \geqslant \frac{1}{2}$ and $e_{\tau,\psi}(X)$ be the distorted expectile of $X \in L^1$ as in (6) with a concave distortion ψ as in Definition 2.1. Then*

$$e_{\tau,\psi}(X) = \max_{Q\in\mathscr{Q}_\psi} \max_{\varphi\in\mathscr{M}_\tau} \mathsf{E}_Q(\varphi X). \tag{8}$$

where $\mathscr{Q}_\psi$ is the core of ψ and $\mathscr{M}_\tau$ is the convex and weakly compact (in L^1) scenario set.

Notice that, by the Tychonoff theorem, $\mathscr{Q}_\psi \times \mathcal{M}_\tau$ is convex too.

Proof. Let us define $\pi_{\tau,\psi}(x, X) := \tau\mathsf{E}_\psi\left(((X-x)_+)^2\right) + (1-\tau)\mathsf{E}_\psi\left(((X-x)_-)^2\right)$, which is strictly convex in $x \in \mathbb{R}$. The f.o.c. for minimizing $\pi_{\tau,\psi}(X)$ is:

$$\frac{\partial\, \pi_{\tau,\psi}(x,X)}{\partial x} = \tau\frac{\partial\, \mathsf{E}_\psi\left(((X-x)_+)^2\right)}{\partial x} + (1-\tau)\frac{\partial\, \mathsf{E}_\psi\left(((X-x)_-)^2\right)}{\partial x} = 0.$$

Since the distortion function ψ is monotone and continuous from below, according to the Monotone Convergence Theorem 8.1 in Denneberg (1994) for monotone set functions, we have that the first partial on the right-hand side above is $-\mathsf{E}_\psi(2(X-x)_+)$ while the second is $\mathsf{E}_\psi(2(X-x)_-)$, then by the Continuous Mapping Theorem and the composition rule of differentiation we end up with

$$\frac{\partial\, \pi_{\tau,\psi}(x,X)}{\partial x} = -2\tau\mathsf{E}_\psi(X-x)_+ + 2(1-\tau)\mathsf{E}_\psi(X-x)_- = 0. \tag{9}$$

Equation (9) together with the proof of Proposition 8 in Bellini et al. (2014) give $e_{\tau,\psi}(X) = \max_{\varphi\in\mathscr{M}_\tau} \mathsf{E}_\psi(\varphi X)$ with scenario set $\mathscr{M}_\tau$. The representation of distorted expectation in Schmeidler (1986) completes the proof. □

Notice that, if $\tau = \frac{1}{2}$, then from the f.o.c. (9) we have $e_{\frac{1}{2},\psi}(X) = \mathsf{E}_\psi(X)$. Similarly to the classical expectile, we also have $\dfrac{\mathsf{E}_\psi(X-e_{\tau,\psi}(X))_+}{\mathsf{E}_\psi(X-e_{\tau,\psi}(X))_-} = \dfrac{1-\tau}{\tau}$, which leads to the following definition of *distorted* Omega ratio: $\Omega_{X,\psi}(m) = \dfrac{\int_m^\infty \psi(1-F_X(x))\mathrm{d}x}{\int_{-\infty}^m \psi(F_X(x))\mathrm{d}x}$. So, when $m = e_{\tau,\psi}(X)$ the distorted expectiles can be seen as threshold providing a profit/loss ratio of $\frac{1-\tau}{\tau}$ but with different conservatism in assessing the risk of X.

3.1. Discrete case

Assuming a finite sample space $\Omega = \{\omega_1, \dots, \omega_n\}$, for $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, the loss X is now a discrete random variable with $P(X = x_i) = p_i$. Given $\tau \geqslant \frac{1}{2}$, the distorted expectile in (8) can be formulated as the nonlinear programming problem in (10) below.

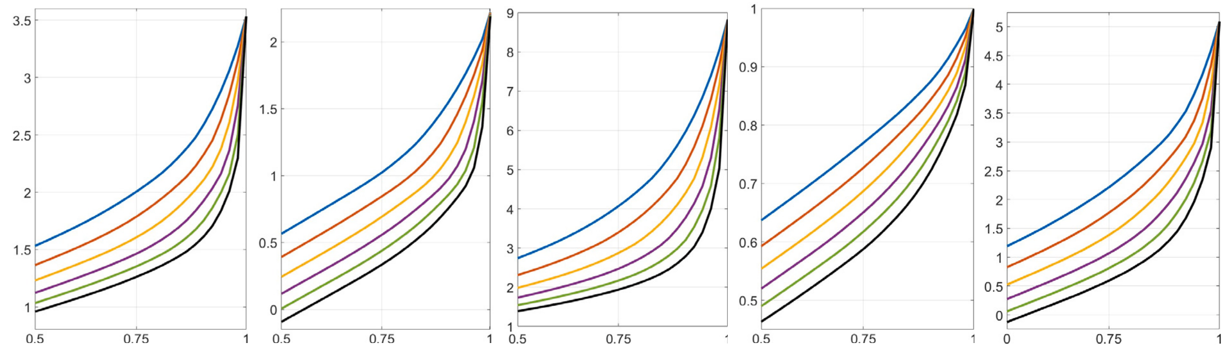
$$\begin{aligned}
\text{maximize} \quad & e_{\tau,\psi}(X) = \max \sum_{i=1}^n x_i\varphi_i q_i, \\
\text{subject to} \quad & \sum_{\omega_i\in A} q_i \leqslant \psi\left(\sum_{\omega_i\in A} p_i\right), \quad \forall A \in 2^n; \\
& \sum_{i=1}^n q_i\varphi_i = 1, \\
& \sum_{i=1}^n q_i = 1, \\
& \varphi_i \leqslant \tau m, \qquad\qquad i = 1,\dots,n, \\
& \varphi_i \geqslant (1-\tau)m, \qquad i = 1,\dots,n, \\
& 0 \leqslant q_i \leqslant 1, \qquad\qquad i = 1,\dots,n, \\
& m \geqslant 0.
\end{aligned} \tag{10}$$

It is worth noting that problem (10) serves to the calibration of the distorted expectile risk measure, from the point of view of a conservative decision maker assessing the risk of X. Therefore, if we in addition consider the values x_i as portfolio losses to be minimized with weights w_i (usually with no-leverage, $\sum_{i=1}^n w_i = 1$), then we have an asset allocation problem of the type min–max-max with intrinsic calibration of all relevant risk parameters.

The maximization in (10) is a bilinear programming problem performed with respect to the q_i and φ_i variables, with a number $(2^n + 3n + 3)$ of constraints and a fixed probability level τ. The constraint $m \geqslant 0$ together with $\tau \geqslant \frac{1}{2}$ forces each φ_i to be non-negative. When n is not too large, problem (10) can be efficiently solved by an optimization software. For example, by using the MatLab *fmincon* function with $n = 15$, $\tau = 0.8$ and an exponential distortion $\psi(p) = p^{0.8}$, the average running time is 45 s. For greater values of n it can be useful to have a procedure which is accurate and efficient as well. Now, according with (Wendell and Hurter J., 1976) a global solution of problem (10) can be found by using an enumeration approach, where $[0, 1]^n$ is compact and the objective

Table 1
In the first iteration ($t = 1$) one solves the LP I w.r.t. the variables $q_{t,i}$ by setting $\varphi_{t,i} = 1$ for $i = 1, \dots, n$, then we solves LP II w.r.t. the variables $\varphi_{t,i}$ with $q_{t,i}^{opt}$. In the second iteration ($t = 2$) one solves LP I w.r.t. the variables $q_{t,i}$ and $\varphi_{t,i}^{opt}$.

LP I	solution	LP II	solution
maximize $\sum_{i=1}^{n} x_i \varphi_{t,i} q_{t,i}$, s.t. $\sum_{\omega_i \in A} q_{t,i} \leqslant \psi\left(\sum_{\omega_i \in A} p_i\right)$, $A \in 2^n$, $\sum_{i=1}^{n} q_{t,i} = 1$, $q_{t,i} \geqslant 0$.	$\mathbf{q}_{t,i}^{opt} = \mathbf{q}_{t+1,i}$	maximize $\sum_{i=1}^{n} x_i q_{t,i} \varphi_{t,i}$, s.t. $\sum_{i=1}^{n} q_{t,i}^{opt} \varphi_{t,i} = 1$, $\varphi_{t,i} \leqslant \tau m$, $i = 1, \dots, n$, $\varphi_{t,i} \geqslant (1-\tau) m$, $i = 1, \dots, n$, $\varphi_{t,i} \geqslant 0$, $m \geqslant 0$.	$\boldsymbol{\varphi}_{t,i}^{opt} = \boldsymbol{\varphi}_{t+1,i}$



(a) Exponential with $\lambda = 1$ (b) Standard Normal (c) Log-Standard Normal (d) Uniform (e) Student t

Fig. 1. τ-distorted expectiles for 1(a) Exponential with ($\lambda = 1$), 1(b) Standard Normal, 1(c) Log-Standard Normal, 1(d) Uniform distributions, 1(e) Student's t ($\nu = 2$), distortion $\psi(p) = p^\alpha$ and parameter $\alpha = \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ colored: blue, red, yellow, violet, green and black, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

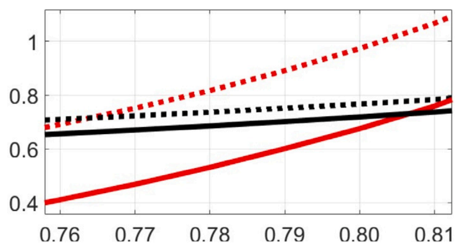


Fig. 2. Expectiles and distorted expectiles discordance w.r.t. the same values of τ.

function is bilinear. So, we propose to solve problem (10) by applying the iterative procedure described in Table 1 based on two LP problems.

Our proposed algorithm exhibits two advantages typical of linear programming: it provides an efficient and exact solution, then it handles large scale problems. Indeed, instead of LP I in Table 1 we could consider its dual which can be solved by a column generation procedure . The average running time with $n = 150$, $\tau = 0.8$ and an exponential distortion $\psi(p) = p^{0.8}$ is approximately 5.400 s.

To appreciate the differences between distorted and classical expectiles, Fig. 1 displays τ-distorted expectiles with respect to five parametric distributions: Exponential, Standard Normal, Log-Standard Normal, Uniform and Student's t. We used $n = 50 \times 5$ realizations and distortion $\psi(p) = p^\alpha$ with parameter $\alpha = \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$. Notice that for all concave distortion one has $e_{\tau,\psi}(X) \geqslant e_\tau(X)$ with equality for $\alpha = 1$.

Example 3.1. Let assume $X \sim \mathcal{N}(0,1)$ and $Y \sim U([0,1])$. Fig. 2 displays $e_\tau(X)$ and $e_\tau(Y)$ for $\tau \geqslant \frac{1}{2}$ (red and black solid lines respectively) as well as $e_{\tau,\psi}(X)$ and $e_{\tau,\psi}(Y)$ with distortion $\psi(p) = p^{0.8}$ (red and black dashed lines respectively). There is a whole interval of values of τ for which $e_\tau(X) \leqslant e_\tau(Y)$ and $e_{\tau,\psi}(X) \geqslant e_{\tau,\psi}(Y)$.

4. Conclusion

Expectile risk measures have been gaining increasing popularity in the finance industry, since they are coherent alternative to VaR and elicitable substitute of expected shortfall. We propose a distorted version of expectile risk measures, providing a dual representation in the spirit of RDEU to account for a decision maker's attitudes toward risk and wealth. We also give an LP formulation to provide the computational framework for working out practical applications of distorted expectile risk measures such as optimal asset allocation.

CRediT authorship contribution statement

Sally Giuseppe Arcidiacono: Methodology, Software, Writing – original draft, Writing – reviewing & editing. **Damiano Rossello:** Conceptualisation, Methodology, Software, Writing – original draft, Writing – reviewing & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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