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Bounded rationality is rare ☆

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Abstract

Most bounded rationality properties in the literature are inherited by subchoices, but are not satisfied by at least one subchoice of most choice functions. Therefore the fraction of choice functions that can be explained by these models goes to zero as the number of items tends to infinity. Numerical estimates confirm the rarity of bounded rationality even for small sets of alternatives.

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1. Models of bounded rationality

According to the theory of revealed preferences pioneered by Samuelson (1938), choices are observed and preferences are revealed. In this approach, rationality coincides with *rationalizability*, when the choice behavior of a decision maker (DM) is justified by maximizing the binary relation of revealed preference. However, rationalizability fails to explain many observed phenomena. Following the inspiring analysis of Simon (1955), rationalizability has been weakened by forms of *bounded rationality*, which aim to explain a larger portion of choice behaviors by means of more flexible paradigms. Without claiming to be exhaustive, below we mention several models of bounded rationality introduced in the choice literature in the last twenty years.

Manzini and Mariotti (2007) propose an approach in which the DM selects from each menu the unique item that survives after the sequential application of distinct criteria (asymmetric relations). Xu and Zhou (2007) characterize a rationalization method which justifies the selection from any menu as the subgame perfect Nash equilibrium outcome of an associated extensive game. Rubinstein and Salant (2008) investigate a post-dominance rationality choice rule: the DM first discards any dominated alternative in the menu, and then chooses the best item from the remaining ones. The choice procedure proposed by Manzini and Mariotti (2012a) uses semiorders as rationales, always applied in a fixed order. In Manzini and Mariotti (2012b), the DM only considers those alternatives that belong to some salient categories. Masatlioglu et al. (2012) argue that the DM is typically endowed with a limited attention, and is unable to take into account all the alternatives in a menu. Apesteguia and Ballester (2013), elaborating on the work of Masatlioglu and Ok (2005), describe a DM who restricts her attention to alternatives that are superior to her *status quo*. In the theory of rationalization of Cherepanov et al. (2013), the DM preliminarily discards items not satisfying some psychological constraint. Apesteguia and Ballester (2013) describe a choice guided by routes. Yildiz (2016) discusses a choice rule based on a pairwise comparison of items according to an ordered list. Lleras et al. (2017) consider overwhelming choices, in which the DM maximizes a fixed preference over subsets of menus determined by a competition filter.

In relation to all these models, the following query arises:

Question. What is the fraction of choice functions that are rationalizable by these models? In other words, what is the explanatory power of the existing models of bounded rationality?

This note answers this query for all mentioned models (and for others). We show that as the number of items goes to infinity, the fraction of choice functions explained by them becomes negligible.[1] We also provide some numerical estimates, which confirm the rarity of bounded rationalizability for small sets of alternatives. Our results strengthen the case for the testability of existing theories, because the small fraction of choices justified by them can be regarded as truly representative of a coherent choice behavior.

The paper is organized as follows. In Section 2 we define the notion of hereditary property, and show that it applies to all mentioned models of bounded rationality. In Section 3 we prove that the fraction of choice functions satisfying any hereditary property tends to zero as the size of the ground set of alternatives goes to infinity. In Section 4 we obtain several estimates on small

[1] This assertion is by no means obvious. In fact, it could be the case that a model allows for a non-negligible fraction of choice functions even when the size of the set of alternatives grows very large.

sets of alternatives for all the presented models, which confirm the rarity of bounded rationality. All proofs are contained in the main body of the paper, except Lemma 8, for which the long computational proof is available online.

2. Hereditary properties

In what follows, X is the *ground set*, a finite nonempty set of alternatives. (Note that X is not fixed once and for all; in fact, its (finite) cardinality will vary.) Any nonempty $A \subseteq X$ is a *menu*, and $\mathscr{X} = 2^X \setminus \{\varnothing\}$ is the family of all menus. A *choice function* on X is a map $c\colon \mathscr{X} \to X$ such that $c(A) \in A$ for any $A \in \mathscr{X}$; for brevity, we refer to a choice function as a *choice*.

A binary relation $\succ$ on X is *asymmetric* if $x \succ y$ implies $\neg(y \succ x)$, *transitive* if $x \succ y \succ z$ implies $x \succ z$, and *complete* if $x \neq y$ implies $x \succ y$ or $y \succ x$ (here x, y, z are arbitrary elements of X). A *linear order* is an asymmetric, transitive, and complete relation. For any $A \in \mathscr{X}$, the symbol $\succ_{\restriction A}$ denotes the restriction of $\succ$ to $A \times A$. Note that if $\succ$ is a linear order, then so is $\succ_{\restriction A}$ for any $A \in \mathscr{X}$.

Given an asymmetric relation $\succ$ on X and a menu $A \in \mathscr{X}$, the set of *maximal* elements of A is $\max(A, \succ) = \{x \in X : y \succ x \text{ for no } y \in A\}$. A choice $c\colon \mathscr{X} \to X$ is *rationalizable* if there is an asymmetric relation $\succ$ on X (in fact, a linear order) such that, for any $A \in \mathscr{X}$, $c(A)$ is the unique element of the set $\max(A, \succ)$; in this case, we write $c(A) = \max(A, \succ)$.

Definition 1. Let $c\colon \mathscr{X} \to X$ be a choice. For any $A \in \mathscr{X}$, denote by $\mathscr{A}$ the family of nonempty subsets of A. The choice *induced by c on A* is $c_{\restriction A}\colon \mathscr{A} \to A$, defined by $c_{\restriction A}(B) = c(B)$ for any $B \in \mathscr{A}$. (Note that $c = c_{\restriction X}$.) We call $c_{\restriction A}$ a *subchoice* of c.

Definition 2. Two choices $c\colon \mathscr{X} \to X$ and $c'\colon \mathscr{X}' \to X'$ are *isomorphic* if there is a bijection (*isomorphism*) $\sigma : X \to X'$ such that $\sigma(c(A)) = c'(\sigma(A))$ for any $A \in \mathscr{X}$.

Definition 3. A *property $\mathscr{P}$ of choices* is a proper nonempty subset of the collection of all choices on finite ground sets, which is closed under isomorphism. (Thus, by definition, $\mathscr{P}$ holds for at least one choice on a finite set, and fails for at least one choice on a finite set.) A property $\mathscr{P}$ is *hereditary* when $\mathscr{P}$ holds for a choice implies that it also holds for all of its subchoices.[2]

In what follows, we shall view a model of choice as a property.

Example 1. Most properties of choices considered in the literature (often called *axioms of choice consistency*) are hereditary, e.g., α,[3] β, γ, ρ, *path independence*, *WARP*, etc.[4] But some properties fail to be hereditary: for example, *moody*, in the sense of Giarlotta et al. (2022b, Definition 2).[5]

2 More formally, $\mathscr{P}$ is hereditary if for all finite sets X and choices c on X, $c \in \mathscr{P} \implies (\forall A \in \mathscr{X})\ c_{\restriction A} \in \mathscr{P}$.

3 A choice $c\colon \mathscr{X} \to X$ satisfies *property α* when for any $x \in X$ and $A, B \in \mathscr{X}$, if $x \in A \subseteq B$ and $c(B) = x$, then $c(A) = x$. This property was introduced by Chernoff (1954).

4 See Cantone et al. (2021, Section 3.2), and references therein.

5 An example of a non-hereditary property of choice *correspondences* (where $c\colon \mathscr{X} \to \mathscr{X}$ and $\varnothing \neq c(A) \subseteq A$ for all $A \in \mathscr{X}$) is *choosing without dominated elements (CWDE)*, which says that for any $A \in \mathscr{X}$ and $x, y \in A$, if y is never chosen in any menu containing both x and y, then $c(A) = c(A \setminus \{y\})$. CWDE is used by García-Sanz and Alcantud (2015) to partially extend the characterization of the *rational shortlist method* of Manzini and Mariotti (2007) from choice functions to choice correspondences. See also Cantone et al. (2021, Section 3.5).

Next, we recall – in chronological order – thirteen models of choice that capture some notion of rationality. To focus, we omit their formal description, but mention behavioral characterizations. Let $c\colon \mathscr{X} \to X$ be a choice. Then:

(i) c is *rationalizable* (Samuelson, 1938) iff property α holds;
(ii) c is *sequentially rationalizable (SR)* (Manzini and Mariotti, 2007) iff weak reducibility (WR) holds[6];
(iii) c is a *rational shortlist method (RSM)* (Manzini and Mariotti, 2007) iff property γ and weak WARP (WWARP) hold[7];
(iv) c is *rationalizable by game trees (RGT)* (Xu and Zhou, 2007) iff weak separability (WS) and divergence consistency (DC) hold[8];
(v) c is *rationalizable by a post-dominance rationality procedure* (Rubinstein and Salant, 2008) iff exclusion consistency (EC) holds iff c is a RSM[9];
(vi) c is *a choice by lexicographic semiorders (CLS)* (Manzini and Mariotti, 2012a) iff reducibility (Re) holds[10];
(vii) c is *categorize-then-choose* (Manzini and Mariotti, 2012b) iff WWARP holds;
(viii) c is *with limited attention (CLA)* (Masatlioglu et al., 2012) iff WARP with limited attention (WARP(LA)) holds[11];
(ix) c is *consistent with basic rationalization theory* (Cherepanov et al., 2013) iff WWARP holds;
(x) c is *a sequential procedure guided by a set of routes* (Apesteguia and Ballester, 2013) iff Re holds[12];
(xi) c is an *(endogenous) status quo bias choice (SQB)* (Apesteguia and Ballester, 2013) iff it is either an extreme status quo biased choice or a weak status quo biased choice[13];
(xii) c is *list-rational (LR)* (Yildiz, 2016) iff the relation of revealed-to-follow is acyclic[14];

[6] *WR*: for any $\varnothing \neq \mathscr{S} \subseteq \mathscr{X}$, there is $S \in \mathscr{S}$ and a collection of pairs $\{x_i, y_i\}_{i\in I}$, with $x_i, y_i \in S$ for all $i \in I$, such that, for all $T \in \mathscr{S}$, if $T \setminus \bigcup\{y_i : x_i \in T\} \in \mathscr{S}$, then $c(T) = c\left(T \setminus \bigcup\{y_i : x_i \in T\}\right)$. The characterization of sequential rationalizability by WR is due to Manzini and Mariotti (2012a, Theorem 2).

[7] γ: for any $A, B \in \mathscr{X}$ and $x \in X$, if $c(A) = c(B) = x$, then $c(A \cup B) = x$. *WWARP*: for any $A, B \in \mathscr{X}$ and $x, y \in X$ with $x, y \in A \subseteq B$, if $c(B) = c(\{x, y\}) = x$, then $c(A) \neq y$. Model (iii) is a special case of (ii).

[8] *WS*: for any $A \in \mathscr{X}$ of size at least two, there is a partition $\{B, D\} \subseteq \mathscr{A}$ of A such that $c(S \cup T) = c(\{c(S), c(T)\})$ for any $S \subseteq B$ and $T \subseteq D$. For each $x, y, z \in X$, let $x \circlearrowleft \{y, z\}$ stand for $c(\{x, y, z\}) = x$ and x, y, z give rise to a *cyclic binary selection*, that is, either (i) $c(\{x, y\}) = x$, $c(\{y, z\}) = y$, and $c(\{x, z\}) = z$, or (ii) $c(\{x, y\}) = y$, $c(\{y, z\}) = z$, and $c(\{x, z\}) = x$. Then *DC* is: for any $x_1, x_2, y_1, y_2 \in X$, if $x_1 \circlearrowleft \{y_1, y_2\}$ and $y_1 \circlearrowleft \{x_1, x_2\}$, then $c(\{x_1, y_1\}) = x_1$ if and only if $c(\{x_2, y_2\}) = y_2$.

[9] *EC*: for any $A \in \mathscr{X}$ and $x \in X \setminus A$, if $c(A \cup \{x\}) \notin \{c(A), x\}$, then there is no $A' \in \mathscr{X}$ such that $x \in A'$ and $c(A') = c(A)$.

[10] *Re:* for any $\varnothing \neq \mathscr{S} \subseteq \mathscr{X}$, there is $S \in \mathscr{S}$ and $x, y \in S$ such that, for all $T \in \mathscr{S}$, if $T \setminus \{y\} \in \mathscr{S}$, then $c(T) = c(T \setminus \{y\})$. Note that this property implies weak reducibility (WR), as defined in Footnote 6.

[11] *WARP(LA)*: for any $A \in \mathscr{X}$, there is $x \in A$ such that for any B containing x, if $c(B) \in A$ and $c(B) \neq c(B \setminus \{x\})$, then $c(B) = x$.

[12] This characterization is obtained by combining Theorem 4 in Apesteguia and Ballester (2013) and Corollary 1 in Manzini and Mariotti (2012a).

[13] The notions of *extreme endogenous status quo biased choice* and *weak endogenous status quo biased choice* are given at page 92 of the mentioned paper.

[14] Formally, x is *revealed-to-follow* y if for some $A \in \mathscr{X}$, either (1) $x = c(A \cup y)$ and $\left[y = c(\{x, y\}) \text{ or } x \neq c(A)\right]$, or (2) $x \neq c(A \cup y)$ and $\left[x = c(\{x, y\}) \text{ or } x = c(A)\right]$.

(xiii) c is *overwhelming* (Lleras et al., 2017) iff WARP under choice overload (WARP-CO) holds iff WWARP holds.[15]

As announced, we have:

Lemma 1. *Models (i)–(xiii) are hereditary.*

Proof. (Sketch) Let $c\colon \mathscr{X} \to X$ be a choice function.

(i) Suppose c is rationalizable by a linear order $\rhd$ on X, and let $A \in X$. Since $c_{\restriction A}(B) = \max\left(B, \rhd_{\restriction A}\right)$ for any $B \in \mathscr{A}$, it follows that $c_{\restriction A}$ is rationalizable.

(ii) Suppose c is sequentially rationalized by an ordered list $(\succ^1, \ldots, \succ^n)$ of asymmetric relations on X, that is, for each $A \in \mathscr{X}$, defining recursively $M_0(A) := A$ and $M_i(A) := \max(M_{i-1}(A), \succ^i)$ for $i = 1, ..., n$, the equality $c(A) = M_n(A)$ holds. Let $A \in \mathscr{X}$. For each $i = 1, \ldots, n$ and $B \in \mathscr{A}$, we have $M_i(B) = \max\left(M_{i-1}(B), \succ^i_{\restriction A}\right)$. Thus, $\left(\succ^1_{\restriction A}, \ldots, \succ^n_{\restriction A}\right)$ sequentially rationalizes $c_{\restriction A}$.

(iii)-(v) Recall that c is a RSM if there is an ordered pair $\mathscr{L} = (\succ^1, \succ^2)$ of acyclic relations on X which sequentially rationalizes c. Thus, this proof is similar to that of (ii).

(iv) It is not difficult to check that both WS and DC are hereditary.

(vi) Suppose c satisfies Re. For any $\varnothing \neq \mathscr{S} \subseteq \mathscr{X}$, there is $S \in \mathscr{S}$ and $x, y \in S$ such that, for all $T \in \mathscr{S}$, if $T \setminus \{y\} \in \mathscr{S}$, then $c(T) = c(T \setminus \{y\})$. Let $A \in \mathscr{X}$. Since $\mathscr{A} \subseteq \mathscr{X}$, for any $\varnothing \neq \mathscr{S} \subseteq \mathscr{A}$, there is $S \in \mathscr{S}$ and $x, y \in S$ such that, for all $T \in \mathscr{S}$, if $T \setminus \{y\} \in \mathscr{S}$, then $c(T) = c_{\restriction A} = c(T \setminus \{y\}) = c_{\restriction A}(T \setminus \{y\})$. Thus $c_{\restriction A}$ satisfies Re.

(vii)-(ix)-(xiii) WWARP is hereditary: see Cantone et al. (2019).

(viii) By definition, if c is a CLA, then $c(A) = \max(\Gamma(A), \rhd)$, where $\Gamma\colon \mathscr{X} \to \mathscr{X}$ is a choice correspondence such that $x \notin \Gamma(B)$ implies $\Gamma(B) = \Gamma(B \setminus \{x\})$ for all $B \in \mathscr{X}$ and $x \in X$, and $\rhd$ is linear order on X. For any $A \in \mathscr{X}$, it is easy to check that $c_{\restriction A}(B) = \max(\Gamma_{\restriction A}(B), \rhd_{\restriction A})$ for any $B \in \mathscr{A}$, where $\Gamma_{\restriction A}$ satisfies the required property.

(xi) By definition, c is SQB if there is a triple $(\rhd, d, Q)$, with $\rhd$ linear order on X, $d \in X$, and $Q \subseteq \{x \in X : x \rhd d\}$, such that for any $S \in \mathscr{X}$, either properties (1)-(2)-(3) or properties (1)-(2)-(3′), given below, hold:
(1) if $d \notin S$, then $c(S) = \max(S, \rhd)$;
(2) if $d \in S$ and $Q \cap S = \varnothing$, then $c(S) = d$;
(3) if $d \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(Q \cap S, \rhd)$;
(3′) if $d \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(S \setminus \{d\}, \rhd)$.
Fix $A \in \mathscr{X}$. Select $a_0 \in A$, and set

$$d_A := \begin{cases} d & \text{if } d \in A \\ a_0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad Q_A := \begin{cases} Q \cap A & \text{if } d \in A \\ \{a \in A\colon a \rhd d_A\} & \text{otherwise.} \end{cases}$$

[15] *WARP-CO*: for any $A \in \mathscr{X}$, there is $x \in A$ such that for any B containing x, if $c(B) \in A$ and $c(B') = x$ for some $B' \supsetneq B$, then $c(B) = x$.

One can check that the triple $(\rhd_A, d_A, Q_A)$ witnesses that $c_{\restriction A}$ is SQB.[16]

(xii) By definition, c is LR if there is a linear order $\rhd$ on X such that $c(A) = c(A \setminus \{x\}, x)$ for any $A \in \mathscr{X}$, where $x = \min(A, \rhd)$.[17] Fix $A \in \mathscr{X}$. For any $B \in \mathscr{A}$, $c_{\restriction A}(B) = c(B) = c(\{c(B \setminus \{x\}), x\})$, where $x = \max(B, \rhd) = \max\left(B, \rhd_{\restriction A}\right)$. Thus, $c_{\restriction A}$ is LR. □

3. Asymptotic rarity of bounded rationality

If $\mathscr{P}$ is a property, then let $T(n)$ and $T(n, \mathscr{P})$ denote, respectively, the total number of choices on a ground set of size $n \geqslant 2$, and the total number of those choices satisfying $\mathscr{P}$. Furthermore, let $F(n, \mathscr{P}) = \frac{T(n,\mathscr{P})}{T(n)}$ be the fraction of choices on X satisfying $\mathscr{P}$. Note that, by Definition 3, we have $0 < F(n, \mathscr{P}) < 1$. Here we prove:

Theorem 1. *If $\mathscr{P}$ is a hereditary property of choices, then* $\lim_{n\to\infty} F(n, \mathscr{P}) = 0$.

Lemma 1 and Theorem 1 readily yield:

Corollary 1. *The fraction of choices explained by models (i)–(xiii) tends to zero as the size of the ground set tends to infinity.*

Corollary 1 formally states what is sometimes informally assumed, but has never been proved, to the best of our knowledge, for all known models of bounded rationality: when the cardinality of the ground set increases, these models become extremely selective.[18]

We shall derive Theorem 1 as a consequence of the more general Theorem 2 below.

Definition 4. Let $\mathscr{P}$ be a property of choices. A choice $c \colon \mathscr{X} \to X$ *hereditarily satisfies* $\mathscr{P}$ if every subchoice of c (including c itself) satisfies $\mathscr{P}$, i.e., $\left\{c_{\restriction A} : A \in \mathscr{X}\right\} \subseteq \mathscr{P}$. $H(n, \mathscr{P})$ denotes the total number of choices on a ground set of size n that hereditarily satisfy $\mathscr{P}$.

Definitions 3 and 4 may look similar at first sight. Indeed, if $\mathscr{P}$ is a hereditary property (according to Definition 3) and c satisfies $\mathscr{P}$, then obviously c hereditarily satisfies $\mathscr{P}$ (according to Definition 4). However, the converse is false, because a choice may hereditarily satisfy a non-hereditary property. So $H(n, \mathscr{P}) \leqslant T(n, \mathscr{P})$, and equality holds if $\mathscr{P}$ is hereditary.

Theorem 2. *For any property $\mathscr{P}$ of choices,* $\lim_{n\to\infty} \dfrac{H(n, \mathscr{P})}{T(n)} = 0$.

In words, the fraction of choices that hereditarily satisfy *any* property of choices tends to zero as the size of the ground set diverges. Theorem 1 is a special case of Theorem 2, because the hereditariness of $\mathscr{P}$ implies $H(n, \mathscr{P}) = T(n, \mathscr{P})$. However, Theorem 2 is more general than Theorem 1, because the former also applies to non-hereditary properties.

[16] Three cases must be considered: (a) $d \notin A$; (b) $d \in A$ and $Q \cap A = \varnothing$; (c) $d \in A$ and $Q \cap A \neq \varnothing$.

[17] Given an asymmetric relation $\succ$ on X and a menu $A \in \mathscr{X}$, the set of *minimal* elements of A is $\min(A, \succ) = \{x \in X : x \succ y \text{ for no } y \in A\}$.

[18] On this point, see also Remark 3 at the end of the paper.

The remainder of this section is devoted to the illuminating proof of Theorem 2. Hereafter, $\mathscr{P}$ is a property of choices, and X is any set of fixed size $n \geqslant 2$.

Lemma 2. $T(n) = \prod\{|A| : A \subseteq X, |A| > 1\} = \prod_{k=2}^{n} k^{\binom{n}{k}}$.

Proof. For $A \in \mathscr{X}$, there are $|A|$ possible choices for $c(A)$. We can omit menus of size 1. □

Note that $T(n)$ grows very fast, e.g., $T(4) = 20\,736$, and $T(5) = 309\,586\,821\,120$.

Lemma 3. *Let $1 \leqslant m \leqslant n$. For any list of menus $(X_j)_{j=1}^{p}$ in $\mathscr{X}$, all having the same size m and pairwise intersecting in at most one item, we have*

$$\frac{H(n,\mathscr{P})}{T(n)} \leqslant \left(\frac{H(m,\mathscr{P})}{T(m)}\right)^{p}.$$

Proof. By Lemma 2, $T(m) = \prod\big\{|A| : A \subseteq X_j, |A| > 1\big\}$, for any j in $\{1,\dots,p\}$. Let

$$M := \prod\big\{|B| : B \subseteq X, |B| > 1, B \nsubseteq X_j \text{ for all } j\text{'s}\big\}.$$

By regrouping the product in Lemma 2, we get $T(n) = T(m)^p M$. Furthermore, we have $H(n,\mathscr{P}) \leqslant H(m,\mathscr{P})^p M$. It follows that

$$\frac{H(n,\mathscr{P})}{T(n)} \leqslant \frac{H(m,\mathscr{P})^p\, M}{T(m)^p\, M} = \left(\frac{H(m,\mathscr{P})}{T(m)}\right)^{p}. \quad \square$$

Definition 5. Let $c\colon \mathscr{X} \to X$ be a choice. For any permutation π of X, let $c_\pi : \mathscr{X} \to X$ be the *permuted choice* of c defined by $c_\pi(A) := \pi^{-1}(c(\pi(A)))$ for all $A \in \mathscr{X}$.

Clearly, c is isomorphic to c_π for any permutation π of X, and all choices that are isomorphic to c are of the type c_π for some π. By the next result, all c_π's are distinct.

Lemma 4. *For any choice c on X, $\big|\{c_\pi : \pi \text{ is a permutation of } X\}\big| = n!$.*

Proof. Let $c\colon \mathscr{X} \to X$ be a choice. We show that distinct permutations of X generate distinct permuted choices on X. Toward a contradiction, suppose π and σ are two distinct permutations of X such that $c_\pi = c_\sigma$. It follows that $c_{\pi\sigma^{-1}} = c$, with $\pi\sigma^{-1} \neq \mathrm{id}_X$. Thus, we can assume without loss of generality that $c_\pi = c$, with $\pi \neq \mathrm{id}_X$. Let $A \in \mathscr{X}$ be the menu $A = \{x \in X : \pi(x) \neq x\}$, hence $\pi(A) = A$. Then $c(A)$ is a fixed point of π, because

$$c(A) = c_\pi(A) = \pi^{-1}(c(\pi(A))) \implies \pi(c(A)) = \pi(\pi^{-1}(c(\pi(A)))) = c(\pi(A)) = c(A)\,.$$

Now the definition of A yields $c(A) \notin A$, which is impossible. □

By Lemmas 2 and 4, we get

Corollary 2. *There are exactly $\frac{T(4)}{4!} = 864$ non-isomorphic choices on 4 items.*

The fraction $F(n,\mathscr{P})$ can be computed by only considering pairwise non-isomorphic choices, since all equivalence classes of isomorphism have the same size ($= n!$).

Lemma 5. *If $\mathscr{P}$ holds hereditarily for exactly q non-isomorphic choices on m items, then*

$$\frac{H(m,\mathscr{P})}{T(m)} = \frac{q \cdot m!}{T(m)}.$$

Proof. $\mathscr{P}$ holds hereditarily for c iff $\mathscr{P}$ holds hereditarily for all c_π, where π is a permutation of the ground set. By Lemma 4, there are $m!$ permuted choices associated to c. □

Lemmas 3 and 5 readily yield the key upper bound:

Corollary 3. *Let $1 \leqslant m \leqslant n$. Suppose X has size n, and $\mathscr{P}$ is a property that holds hereditarily for at most $q < T(m)/m!$ non-isomorphic choices on a set of size m. If $(X_j)_{j=1}^p$ is a list of menus in $\mathscr{X}$ having size m and pairwise intersecting in at most one item, then*

$$\frac{H(n,\mathscr{P})}{T(n)} \leqslant \left(\frac{q \cdot m!}{T(m)}\right)^p.$$

In particular, if $\mathscr{P}$ is hereditary, then $F(n,\mathscr{P}) \leqslant \left(\frac{q \cdot m!}{T(m)}\right)^p$.

We are ready to prove the main result of this section.

Proof of Theorem 2. By hypothesis $\mathscr{P}$ fails for a choice c on a set of size m (and holds for $q < \frac{T(m)}{m!}$ non-isomorphic choices on this set). Set $\zeta := \frac{q \cdot m!}{T(m)} < 1$. Let $\epsilon > 0$. Select $p \in \mathbb{N}$ such that $\zeta^p < \epsilon$. For any integer $n \geqslant mp$ and any set X of size n, there is a list of pairwise disjoint menus $(X_j)_{j=1}^p$ in $\mathscr{X}$ all having size m. Corollary 3 yields $\frac{H(n,\mathscr{P})}{T(n)} \leqslant \zeta^p < \epsilon$. □

The proof of Theorem 2 does not use the full power of Corollary 3, because we are taking pairwise disjoint menus. In the next section we shall use Corollary 3 in its full extent.

4. Rarity of bounded rationality on small sets

Our goal is to design a user-friendly algorithm to assess the testability of behavioral choice models on small sets of items. For rationalizability (i.e., property α), we get exact numbers.

Lemma 6. *The fraction $F(n,\alpha)$ of rationalizable choices on n items is (rounding decimals)*

n	3	4	5	6
$F(n,\alpha)$	0.25	0.0012	4×10^{-10}	6×10^{-26}

Proof. Since α is hereditary, $F(n,\alpha) = \frac{T(n,\alpha)}{T(n)} = \frac{H(n,\alpha)}{T(n)}$. Up to isomorphism, there is exactly one choice on n items satisfying α, hence Lemmas 2 and 5 yield the claim. □

By Lemma 6, the fraction of rationalizable choices is trifling even on tiny ground sets. This further validates the necessity to switch to models of bounded rationality. To get good estimates for properties weaker than α, we need a more refined combinatorial approach.

Definition 6. For any integers n, m such that $1 \leqslant m \leqslant n$, denote by $P(n, m)$ the maximum size p of a list $(X_j)_{j=1}^{p}$ of subsets of $\{1, \ldots, n\}$ such that $|X_j| = m$ for all $j = 1, \ldots, p$, and $|X_i \cap X_j| \leqslant 1$ for all distinct $i, j = 1, \ldots, p$.

Using Definition 6, we can rewrite Corollary 3 as a simple formula:

Corollary 4. *For any integers $1 \leqslant m \leqslant n$, and any property $\mathscr{P}$ that holds hereditarily for at most q non-isomorphic choices on a set of size m, we have*

$$\frac{H(n, \mathscr{P})}{T(n)} \leqslant \left(\frac{q \cdot m!}{T(m)}\right)^{P(n,m)} .$$

Corollary 4 gives an upper bound to the fraction of choices that hereditarily satisfy a property $\mathscr{P}$. In particular, this bound holds for the fraction of choices satisfying any hereditary property, and so we can apply Corollary 4 to all models (ii)-(xiii). Clearly, the sharper the lower bounds to $P(n, m)$ are, the finer the upper bounds to $H(n, \mathscr{P})/T(n)$ become. The following recursive estimate of $P(n, m)$ comes handy:

Lemma 7. *For any $3 \leqslant k \leqslant n$, where n is a power of a prime, $P(kn, k) \geqslant n^2 + kP(n, k)$.*

Proof. Since n is a power of a prime number, there are operations $+$, $-$, $\cdot$, and $/$ on the set $\{0, \ldots, n-1\}$, which make it into a field. For each $i, j \in \mathbb{N}$ such that $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-1$, define sets A_{ij} by

$$A_{ij} := \left\{\overline{i + mj} + mn : 0 \leqslant m \leqslant k-1\right\} \subseteq \{0, \ldots, kn-1\},$$

where $\overline{i + mj} \in \{0, \ldots, n-1\}$ is computed by using the field operations on $\{0, \ldots, n-1\}$. All sets A_{ij} are well-defined and have size k. Furthermore, there are n^2 such sets.

Claim: $|A_{ij} \cap A_{i'j'}| \leqslant 1$. Suppose A_{ij} and $A_{i'j'}$ overlap in $\overline{i + mj} + mn = \overline{i' + m'j'} + m'n$. Since both $\overline{i + mj}$ and $\overline{i' + m'j'}$ are less than n, we get $m = m'$ and $\overline{i + mj} = \overline{i' + mj'}$. Working in the field on $\{0 \ldots n-1\}$, we have $i + mj = i' + mj'$, and so $i - i' = m(j' - j)$.
- Case 1: If $j' - j = 0$ (in the field), then $i - i' = 0$, hence $i = i'$ and $m(j' - j) = 0$. If $j = j'$, we are done. Otherwise, $m = 0$, and A_{ij} and $A_{i'j'}$ overlap on $i = i'$.
- Case 2: If $j' - j \neq 0$ (in the field), then $m = (i - i')/(j' - j)$, and so $A_{ij}, A_{i'j'}$ intersect only on $\{mn, \ldots, mn + n - 1\}$ and at one point in that set.

The Claim gives us n^2 sets, each of which intersects each $\{mn, \ldots, mn+n-1\}$ in one point, with $0 \leqslant m \leqslant k-1$. We can also find $P(n, k)$ additional sets that are subsets of each $\{mn, \ldots, mn + n-1\}$, and we can do this for each m. $\square$

Corollary 5. *The following lower bounds to $P(n, 4)$ hold:*

n	16	20	28	32	36	44
$P(n, 4)$	$\geqslant 20$	$\geqslant 29$	$\geqslant 57$	$\geqslant 72$	$\geqslant 93$	$\geqslant 141$

Proof. Apply Lemma 7.[19] For instance, $P(9,4)=3$ implies $P(36,4) \geqslant 9^2 + 4 \cdot 3 = 93$. □

Applying Corollaries 4 and 5 for $m=4$ and $n = 16, 20, 28, 32, 36, 44$, we get:

Corollary 6. *If $\mathscr{P}$ holds hereditarily for at most q non-isomorphic choices on a set of size 4, then on a ground set of size n the following upper bounds to $H(n,\mathscr{P})/T(n)$ hold:*

n	16	20	28	32	36	44
$H(n,\mathscr{P})/T(n)$	$\leqslant (q/864)^{20}$	$\leqslant (q/864)^{29}$	$\leqslant (q/864)^{57}$	$\leqslant (q/864)^{72}$	$\leqslant (q/864)^{93}$	$\leqslant (q/864)^{141}$

Now we compute the number q for all models (ii)-(xiii); calculations are available online (Giarlotta et al., 2022a).

Lemma 8. *Let $\mathscr{P}$ be any of the properties (models) SQB, LR, RGT, RSM, CLS, SR, WWARP, and CLA. The number q of non-isomorphic choices on 4 items satisfying $\mathscr{P}$ is*

$\mathscr{P}$	SQB	LR	RGT	RSM	CLS	SR	WWARP	CLA
q	6	10	11	11	15	15	304	324

Our last result justifies the title of this section:

Theorem 3. *Let $\mathscr{P}$ be any of the properties (models) listed below. The fractions $F(n,\mathscr{P})$ of choices satisfying $\mathscr{P}$ on $n = 16, 20, 28, 32, 36, 44$ items are, respectively:*[20]

$\mathscr{P}$	$F(16,\mathscr{P})$	$F(20,\mathscr{P})$	$F(28,\mathscr{P})$	$F(32,\mathscr{P})$	$F(36,\mathscr{P})$	$F(44,\mathscr{P})$
Status Quo Bias Choice	$\leqslant 10^{-43}$	$\leqslant 10^{-62}$	$\leqslant 10^{-123}$	$\leqslant 10^{-155}$	$\leqslant 10^{-200}$	$\leqslant 10^{-304}$
List-Rational Choice	$\leqslant 10^{-38}$	$\leqslant 10^{-56}$	$\leqslant 10^{-110}$	$\leqslant 10^{-139}$	$\leqslant 10^{-180}$	$\leqslant 10^{-273}$
Rationalization by Game Tree	$\leqslant 10^{-37}$	$\leqslant 10^{-54}$	$\leqslant 10^{-108}$	$\leqslant 10^{-136}$	$\leqslant 10^{-176}$	$\leqslant 10^{-267}$
Rational Shortlist Method	$\leqslant 10^{-37}$	$\leqslant 10^{-54}$	$\leqslant 10^{-108}$	$\leqslant 10^{-136}$	$\leqslant 10^{-176}$	$\leqslant 10^{-267}$
Choice by Lexicographic Semiorders	$\leqslant 10^{-35}$	$\leqslant 10^{-51}$	$\leqslant 10^{-100}$	$\leqslant 10^{-126}$	$\leqslant 10^{-163}$	$\leqslant 10^{-248}$
Sequentially Rationalizable Choice	$\leqslant 10^{-35}$	$\leqslant 10^{-51}$	$\leqslant 10^{-100}$	$\leqslant 10^{-126}$	$\leqslant 10^{-163}$	$\leqslant 10^{-248}$
Weak WARP	$\leqslant 10^{-9}$	$\leqslant 10^{-13}$	$\leqslant 10^{-25}$	$\leqslant 10^{-32}$	$\leqslant 10^{-42}$	$\leqslant 10^{-63}$
Choice with Limited Attention	$\leqslant 10^{-8}$	$\leqslant 10^{-12}$	$\leqslant 10^{-24}$	$\leqslant 10^{-30}$	$\leqslant 10^{-39}$	$\leqslant 10^{-60}$

Proof. Apply Corollary 6 and Lemma 8. □

The numerical estimates given by Theorem 3 complete the analysis of models (i)-(xiii). In fact, the bounds for RSM also apply to (v) post-dominance rationality procedure, those for CLS also apply to (x) sequential procedure guided by a set of routes, and those for Weak WARP apply to three models, namely (vii) categorize-then-choose, (ix) consistency with basic rationalization theory, and (xiii) overwhelming choice.

[19] To prove $P(16,4) \geqslant 20$, we can also use a simple geometric approach. Display the 16 items on a 4×4 matrix, and take the 4 rows, the 4 columns, and the $\frac{4!}{2}$ products of even class obtained by computing the determinant of the matrix. These 20 sets pairwise intersect in at most one item.

[20] The number of atoms in the observable universe is between 10^{78} and 10^{82}.

Remark 1. A natural direction of research is to estimate the ratios between the number of choices satisfying α (equivalently, WARP) and the number of choices satisfying weaker axioms.[21] The rationale of this investigation is that WARP is often considered excessively demanding, whereas the testability of weaker axioms of consistency is sometimes debated. We believe that most of these ratios asymptotically tend to zero, and they may be close to zero even for a relatively small number of items. Computations support this conjecture: for instance, by Lemma 6 the fraction of rationalizable choices on 5 items is 4×10^{-10}, whereas for choices satisfying Weak WARP a similar upper bound holds on 16 items.

Remark 2. The fraction $F(n, \mathscr{P})$ of choices on n items satisfying $\mathscr{P}$ is an *ex-ante* approximation of the *hit rate*, as defined by Selten (1991, p. 194). This score, which gives the relative frequency of correct predictions, is a component of a global measure of predictive success of a theory. Starting from Afriat (1974), several attempts have been made to identify a *measure of rationality*, which may take into account deviations of individual behavior from the maximization principle. In this respect, Apesteguia and Ballester (2017) define the *swap index*, which is the sum, across all the observed menus, of the number of alternatives that must be swapped with the chosen one to obtain a choice rationalizable by the linear order(s) maximizing this sum. Our numerical estimates may be an additional tool to investigate performances of rationality indices.

Remark 3. We define choice functions on the full domain $\mathscr{X} = 2^X \setminus \{\varnothing\}$, implicitly assuming that the DM's behavior is observable for all possible menus. However, this hardly happens in practice. Additional work is needed to obtain estimates when using a different definition of choice function, which allows for a *limited dataset* (de Clippel and Rozen, 2021). In this perspective, instead of computing upper bounds to the fraction $F(n, \mathscr{P}) = \frac{T(n,\mathscr{P})}{T(n)}$, one should estimate the fraction $F^*(n, \mathscr{P}) = \frac{E(n,\mathscr{P})}{E(n)}$, where $E(n)$ is the number of partial choices on n elements that arise from experimental/empirical settings.[22]

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