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HOMOLOGICAL INVARIANTS OF SOME SPECIAL VARIETIES

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To Elena and Giuseppe

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Introduction

Given a homogeneous ideal I in a polynomial ring, it has a graded minimal free resolution which contains interesting numerical data. When we know more information about I, e.g., when I defines a special variety, we can try to explicitly compute those numbers. In this Ph.D. thesis, we discuss several different results about some homological invariants (e.g., graded Betti numbers, Hilbert function, regularity) of some special varieties. In particular, we focus on the codimension two ACM varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the edge ideals of bicyclic graphs.

Given a variety $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, an interesting problem is the description of the homological invariants of the coordinate ring of X. This problem has been primarily studied for points, although there is not a general answer in this direction. One difficulty comes from the fact that a set of distinct points $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ is not necessarily arithmetically Cohen-Macaulay (ACM). See, for instance [43, 55, 57, 58, 59, 60] for some results on this topic, and [45, 46] for a recent characterization of the ACM property in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ and, under certain conditions, in $\mathbb{P}^1 \times \mathbb{P}^m$.

The motivation to study multiprojective spaces and their subvarieties has recently increased, since they arise in many applications. For example, the value of the Hilbert function in multidegree $(1, \ldots, 1)$ of a collection of (2-fat) points in a multiprojective space is related to a classical problem of algebraic geometry concerning the dimensions of certain secant varieties of Segre varieties (see, for instance [8, 21, 22]) which parameterize decomposible tensors, while the values of the Hilbert function at other multidegrees are related to the dimensions of Segre-Veronese varieties, which parameterize partially symmetric tensors. Another example appears in [9, 26] where the authors deduce new results about tensors, and in [34], the author focus on the implicitization problem for tensor product surfaces.

Chapter 2 of this thesis is concerned with finite arrangements in multiprojective space. In combinatorial algebraic geometry, there has been interest in studying finite arrangements of lines (see [28, 73] for recent developments in $\mathbb{P}^2$). A line arrangement over an algebraically closed field K is a finite collection $L_1, \ldots, L_d \subseteq \mathbb{P}^2$, $d > 1$, of distinct lines in the projective plane and their crossing points (i.e., the points of intersections of the lines).

In this thesis, we investigate special arrangements of lines in multiprojective spaces by focusing on ACM codimension two varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, called *varieties of lines*, since we want to generalize the codimension two ACM property of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Specifically, we study the Hilbert function of Ferrers varieties of lines, a special case of ACM variety of lines, and we describe the trigraded minimal free resolution of the defining ideal of a variety of lines arising from a complete intersection of points. We also compute the Castelnuovo-Mumford regularity of the defining ideals of some special cases of varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, that is grids of lines and complete intersections of lines. The study of the regularity of other more general classes of varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a future research topic we want to explore more deeply, thus extending this work.

In Chapter 3 of this thesis we study the regularity of the edge ideal of bicyclic graphs and their powers. Castelnuovo-Mumford regularity is one of the most fundamental invariants in commutative algebra and algebraic geometry. One of its first hidden appearances may be found in Castelnuovo's work on linear systems on smooth projective space curves [20]. Castelnuovo's result gives a sharp upper bound on the largest degree r such that the complete linear system of the r-fold plane sections on the given curve is not cut out by surfaces of degree r. Another early invisible occurrence of Castelnuovo-Mumford regularity is found in the work of Hermann [68]. The results of Hermann show that the minimal free resolution of an ideal generated by finitely many homogeneous polynomials can be computed in a finite number of steps which depends on the number of indeterminates of the ambient ring and the maximal degree of the given polynomials.

In 1966, Mumford gave a first formal definition of Castelnuovo-Mumford regularity [86], defining the notion of being m-regular in the sense of Castelnuovo for a coherent sheaf of ideals over a projective space and a given integer m. Although Castelnuovo-Mumford regularity was originally defined in terms of sheaf cohomology, it may be expressed in terms of the degrees of the syzygies. Thus, it is of basic significance in classical projective algebraic geometry. In the classical case (standard graded algebra) the Castelnuovo-Mumford regularity measures the maximum degree of the syzygies and provides a quantitative version of Serre's vanishing theorem for the associated sheaf. In particular, it bounds the largest degree of the minimal generators and the smallest twist for which the sheaf is generated by its global sections.

The regularity has also been used as a measure of the complexity of computational problems in algebraic geometry and it also found much interest in commutative algebra (see for example [12] and [36]). In 1984, Eisenbud and Goto [36] made explicit the link between this algebraic Castelnuovo-Mumford regularity of a graded module over a polynomial ring and its minimal free resolution.

The Castelnuovo-Mumford regularity of an ideal I in a standard polynomial ring $R = K[x_0, \ldots, x_n]$, denoted by $\operatorname{reg}(I)$, is then an important invariant that can be associated to a projective variety $X \subseteq \mathbb{P}^n$ having defining ideal I. It has been the object of many authors to estimate $\operatorname{reg}(I)$

since, as mentioned above, not only it bounds the degrees of a minimal set of defining equations for X, but it also gives a uniform bound on the degrees of syzygies of I (we recall that there are two equivalent definitions of Castelnuovo-Mumford regularity, the first is in terms of graded Betti numbers and the second is in terms of local cohomology).

The most fundamental situation is when X is a set of points (see for example [23], [41], [49], [95]). Many authors have been also interested in extending our understanding on regularity for sets of points in $\mathbb{P}^n$ to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ (cf. [24], [50], [53], [55], [97]). In the context of $\mathbb{N}^2$-graded rings, Aramova, Crona and De Negri [5] have introduced a finer notion of regularity that places bounds on each coordinate of the degree of a multi-graded syzygy. Extending the definition of regularity to multi-graded rings is also considered in [81]. For sets of points in multi-projective spaces, Hà and Van Tuyl [65] compute the regularity of an ideal defining a set of distinct points in generic position and bound the regularity for a set of fat points with generic support. Their strategy was based on viewing the ideal defining a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ both as $\mathbb{N}^h$-homogeneous ideal and as a homogeneous ideal in the normal sense, i.e., a homogeneous ideal in a $\mathbb{N}^1$-graded ring.

Using the same approach, i.e., by considering their defining ideals as homogeneous ideals, we compute the Castelnuovo-Mumford regularity of some particular varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, that is grids of lines and complete intersections of lines.

In this thesis we also study the regularity of another special variety, i.e., the edge ideal of a bicyclic graph and its powers. If I is a homogeneous ideal of the polynomial ring $R = K[x_1, \ldots, x_r]$, the Castelnuovo-Mumford regularity of I and its powers has been an interesting and active research topic for the past decades. An important result of the vast literature on the study of regularity of I and its powers was given in 1999 by Cutkosky, Herzog, and Trung (see [32]). In 2000, Kodiyalam used a different method to prove the same result (see [79]). In both papers, it is proved that for all positive integers $q \geq q_0$, the regularity of powers of I is a linear function of the form $\operatorname{reg}(I^q) = dq + b$, where q_0 is the so called stabilizing index, and b is the so-called constant. The value of d in the above formula is well understood. For example, d is equal to the degree of the generators of I when I is equigenerated. The method used by Cutkoswky, Herzog and Trung does not give precise information on q_0 and b. Since then, many researchers have tried to compute q_0 and b for special families of ideals.

The most simple case, yet interesting, is when I is the edge ideal of a finite simple graph. Recall that a graph $G = (V(G), E(G))$ consists of a vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and a collection $E(G)$ of non empty subsets of $V(G)$ of cardinality 2 which are called edges of G. The graph G is simple if it has no multiple edges or loops. Let $G = (V(G), E(G))$ be a simple graph, and let R be the polynomial ring $R = K[x_i \mid x_i \in V(G)]$

where K is any field. We recall that the edge ideal $I(G)$ of G is the ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)) \subseteq R.$$

Several authors have settled the problem of determining the stabilizing index and the constant for special families of graphs. Banerjee proved that $\operatorname{reg} I(G)^q = 2q$, for all $q \geq 2$, when G is a gap-free and cricket-free graph (see [10]). Moghimian, Fakhari and Yassemi answered the question for the family of whiskered graphs (see [84]). Beyarslan, Hà and Trung settled the problem for the family of forests and cycles (see [14]). Their results were expanded to the family of unicyclic graphs by Alilooee, Beyarslan and Selvaraja (see [3]). Moreover, Alilooee and Banerjee determined the stabilizing index and the constant for the family of bipartite graphs with regularity equal to three (see [2]). Jayanthan and Selvaraja settled the problem for the family of very well-covered graphs (see [76]). Recently, Erey proved that if G is a gap-free and diamond-free graph, then $\operatorname{reg} I(G)^q = 2q$ for all $q \geq 2$ (see [40]).

The above-mentioned results have given rise to the following problem: characterize graphs G for which the edge ideals $I = I(G)$ satisfy the equality $\operatorname{reg} I^q = 2q + \nu(G) - 1$ for all $q \gg 0$ or $\operatorname{reg} I^q = 2q + \operatorname{reg} I - 2$ for all $q \gg 0$, where $\nu(G)$ denotes the induced matching number of G.

The simplest situation for an edge ideal is when its powers have linear resolutions. We recall that an ideal I has a d-linear resolution if I is generated by homogeneous elements of degree d and $\operatorname{reg}(I) = d$. Among all the interesting problems in Castelnuovo-Mumford regularity, classification of ideals with linear resolution is of great importance. Proving that a class of ideals has a d-linear resolution is difficult in general. However, some classes of ideals with linear resolution may be found in [6], [27], [71], [104]. It was proved (see [48], [101]) that the edge ideal of a graph G has a linear resolution if and only if G^c is chordal. It also follows from [71] that if $I(G)$ has a linear resolution, then so does $I(G)^q$ for all $q \geq 1$ and from [88] that if a power of $I(G)$ has a linear resolution then G^c is chordal. It is, thus, of interest to characterize graphs whose (sufficiently large) powers have linear resolutions.

Our approach in this thesis is focused on the relations between the combinatorics of graphs and algebraic properties of edge ideals. We refer the reader to [4],[15], [66], [72], [78],[89] and [103] for more information on this topic. The purpose of Chapter 3 of this thesis is to extend the results of [3] to the family of bicyclic graphs, i.e., a graph with exactly two cycles (see Figure 1). The base case of the family of bicyclic graphs is that of dumbbell graphs. A dumbbell graph $C_n \cdot P_l \cdot C_m$ is a graph consisting of two cycles C_n and C_m connected with a path P_l, whose vertices are $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ and $\{z_1, \ldots, z_l\}$, respectively (see Figure 2).

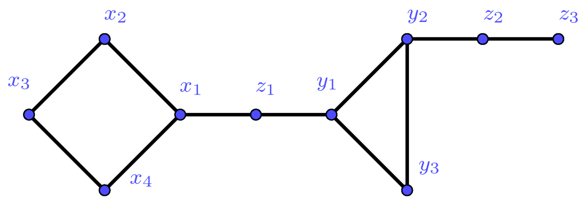


Figure 1: An example of bicyclic graph.

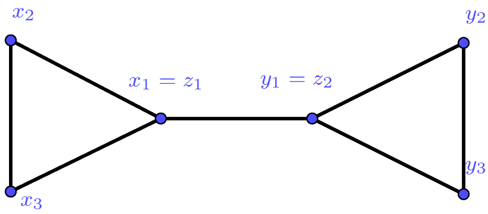


Figure 2: The dumbbell graph $C_3 \cdot P_2 \cdot C_3$.

We first compute the regularity of the edge ideal of a dumbbell graph, and then we give a combinatorial characterization of the regularity of the edge ideal of an arbitrary bicyclic graph in terms of its induced matching number. Finally we study the regularity of powers of edge ideals of some specific bicyclic graphs, i.e., dumbbell graphs with path having at most two vertices. Our approach takes advantage of the notion of even-connectedness and the relations between the induced matching number of graphs and the regularity of the edge ideal. Our strategy is to show $2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m)$ is actually an upper bound and a lower bound for $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ for all $q \geq 1$ where $l \leq 2$. To obtain the upper bound, we follow the argument of Banerjee from [10, Theorem 5.2]. To compute the lower bound, we proceed by looking at some specific induced subgraphs of $C_n \cdot P_l \cdot C_m$.

As a side result, we answer an interesting question on the behavior of the constant term of the asymptotycally linear regularity function. Let I be an arbitrary ideal generated in degree d and let $\operatorname{reg}(I^q) = dq + b_q$ for $q \geq q_0$. An interesting question is the study the sequence $\{b_i\}_{i\geq 1}$. In [38] Eisenbud and Harris proved that if $\dim(R/I) = 0$, then $\{b_i\}_{i\geq 1}$ is a weakly decreasing sequence of non-negative integers. In [11] Banerjee, Beyarslan and Hà conjectured that for any edge ideal, $\{b_i\}_{i\geq 1}$ is a weakly decreasing sequence (see [11, Conjecture 7.11]). For the edge ideal of any dumbbell graph with $l \leq 2$, we prove $b_i = b_1$ for all $i \geq 1$. However, we expect $b_i \leq b_1$ for all $i \geq 1$ for any graph.

We now describe the structure of this thesis.

In Chapter 1, we fix some notation, and we present a survey of crucial known results. The notions and remarks in this chapter are essential for the remaining chapters of this thesis.

In Chapter 2, we describe varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Definitions 2.1.1 and 2.1.2) and we make some observations that make clear the connection between a set of lines having the same direction and a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (see Remark 2.1.9). In this chapter we study special arrangements of lines, i.e., the codimension two ACM varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We first describe a connection between ideals of varieties of lines and some squarefree monomial ideals (see Lemma 2.2.2). In particular, we introduce the $Hyp_n(\star)$-property (see Definition 2.2.9) to give a combinatorial characterization of ACM varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ using a well-known property of chordal graphs:

Theorem A. *(Theorem 2.2.15) Let X be a variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X is ACM if and only if X has the $Hyp_n(\star)$-property for $n = 4, 5, 6$.*

Then we introduce a numerical way to check the ACM property for any varieties of lines (see Propositions 2.3.6, 2.3.7 and 2.3.8). We also describe the Hilbert function of Ferrers varieties of lines (see Theorem 2.4.3 and Corollary 2.4.4), a special case of ACM variety of lines (see Definition 2.4.1). After that, we initiate an investigation on varieties of lines whose set of crossing points is a complete intersection of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Definition 2.5.1). Additionally we describe a set of minimal generators and the trigraded minimal free resolution of its defining ideal:

Theorem B. *(Lemma 2.5.5 and Theorem 2.5.8) Let*

$$\mathcal{C} := \{\mathcal{L}(A_i) \cap \mathcal{L}(B_j) \cap \mathcal{L}(C_k) \mid i \in [a], j \in [b], k \in [c]\}$$

be a complete intersection of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of type (a, b, c) and let $X := X_{\mathcal{C}}$ be the grid of lines arising from $\mathcal{C}$. Then a set of minimal generators of the ideal defining X is

$$I_X = \Big(\prod_{i\in[a]} A_i \cdot \prod_{j\in[b]} B_j \ , \ \prod_{i\in[a]} A_i \cdot \prod_{k\in[c]} C_k \ , \ \prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k \Big).$$

Furthermore, $X_{\mathcal{C}}$ is ACM and a trigraded minimal free resolution of $I_{X_{\mathcal{C}}}$ is

$$0 \to R^2(-a, -b, -c) \to R(-a, -b, 0) \oplus R(-a, 0, -c) \oplus R(0, -b, -c) \to I_{X_{\mathcal{C}}} \to 0.$$

We also characterize varieties of lines defined by a complete intersection ideal in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

Theorem C. *(Theorem 2.6.2) Let X be a variety of lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then the ideal I_X is a complete intersection if and only if $I_X = (F_1, F_2)$, with $\deg F_1 = a\boldsymbol{e}_i$ and $\deg F_2 = b\boldsymbol{e}_j + c\boldsymbol{e}_k$ with $j, k \neq i$, for some $a, b, c \in \mathbb{N}$.*

We also start a preliminary investigation on the Castelnuovo-Mumford regularity of defining ideals of varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In particular, we compute the regularity of the defining ideals of grids of lines (see Corollary 2.5.11) and complete intersections of lines (see Corollary 2.6.4).

We end this chapter by proposing three possible research topics to explore: (1) the connection between our varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and special configurations of lines in $\mathbb{P}^3$; (2) the Hilbert function of any ACM variety of lines; (3) the regularity of other more general classes of varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

In Chapter 3, we study the regularity of bicyclic graphs and their powers. First, we use combinatorial techniques to compute the induced matching number of a dumbbell graph:

Theorem D. *(Theorem 3.1.4) Let $n, m \geq 3$ and $l \geq 1$. Then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Then, applying inductive methods and basing our approach on the Lozin transformation (see [16] and [80]), we study the regularity of the edge ideals of dumbbell graphs:

Theorem E. *(Theorem 3.1.6) Let $m, n \geq 3$ and $l \geq 1$.*

(i) If $l \equiv 0, 1$ (mod 3), then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \ (mod\ 3), \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

(ii) If $l \equiv 2$ (mod 3), then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n \equiv 0, 1 \ (mod\ 3), \\ & \quad m \equiv 2 \quad (mod\ 3) \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Then, for an arbitrary bicyclic graph G, we give a combinatorial characterization of $\operatorname{reg} I(G)$ in terms of the induced matching number $\nu(G)$:

Theorem F. *(Theorem 3.2.2) Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold.*

(I) If $n, m \equiv 0, 1$ (mod 3), then

$$\operatorname{reg} I(G) = \nu(G) + 1.$$

(II) If $n \equiv 0, 1$ (mod 3) and $m \equiv 2$ (mod 3), then

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2,$$

and $\operatorname{reg} I(G) = \nu(G) + 2$ if and only if $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

(III) If $n, m \equiv 2$ (mod 3) and $l \geq 3$, then

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 3.$$

Moreover:

(i) $\operatorname{reg} I(G) = \nu(G) + 3$ *if and only if* $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$;

(ii) $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if the following conditions hold:*

(a) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;

(b) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;

(c) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

(IV) If $n, m \equiv 2$ (mod 3) and $l \leq 2$, then

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2.$$

Moreover, if x is a vertex on P_l and if $\mathcal{L}_x(G)$ is the Lozin transformation of G with respect to x, then $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if the following conditions are satisfied:*

(a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;

(b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;

(c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

We then investigate the asymptotic behavior of regularity of powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leq 2$. We prove that

Theorem G. *(Theorem 3.3.6) Let $C_n \cdot P_l \cdot C_m$ with $l \leq 2$. Then for any $q \geq 1$ we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2.$$

We also give examples that show the above theorem does not hold for given n, m, l and q (see Remark 3.3.8).

Chapter 1

Definitions, Notation and Preliminaries

1.1 ACM varieties in multiprojective spaces

In this section we present the relevant background on Hilbert function, resolutions and Cohen-Macaulay ideals within the context of multigraded rings (see for instance [96], [97], [98]) by extending to the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ the definitions and results given in [59] in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$.

1.1.1 Multi-graded rings and Hilbert function

Throughout this thesis $\mathbf{k}$ will denote an algebraically closed field of characteristic zero.

Now we extend the theory of graded rings to the theory of multi-graded rings S with a special emphasis on the case that S is the quotient of a polynomial ring, or more generally, a finitely generated $\mathbf{k}$-algebra. We also extend the definition of the Hilbert function to this context.

Let $\mathbb{N} := \{0, 1, 2, \dots\}$ denote the set of non-negative integers. We let $\mathbb{N}^h := \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{h\ times}$ and we denote $(i_1, \dots, i_h) \in \mathbb{N}^h$ by $\underline{i}$. We set $|\underline{i}| := \sum_k i_k$.

Let $\preceq$ denote the natural partial ordering on the elements of $\mathbb{N}^h$ defined by $\underline{i} \preceq \underline{j}$ in $\mathbb{N}^h$ if and only if $i_k \leq j_k$ for every $k = 1, \dots, h$. We also observe that $\mathbb{N}^h$ is a semi-group generated by $\{e_1, \dots, e_h\}$ where e_i is the i^{th} standard basis vector of $\mathbb{N}^h$, that is, $e_i := (0, \dots, 1, \dots, 0)$ with 1 being in the i^{th} position.

Definition 1.1.1. An $\mathbb{N}^h$-*graded ring* (or simply, a *multi-graded ring*) is a ring R that has a direct sum decomposition

$$R = \bigoplus_{\underline{i} \in \mathbb{N}^h} R_{\underline{i}} \quad \text{such that} \quad R_{\underline{i}} R_{\underline{j}} \subseteq R_{\underline{i}+\underline{j}} \quad \text{for all} \quad \underline{i}, \underline{j} \in \mathbb{N}^h.$$

An element $r \in R$ is said to be $\mathbb{N}^h$-*homogeneous* (or simply, *homogeneous*) if $r \in R_{\underline{i}}$ for some $\underline{i} \in \mathbb{N}^h$. If r is homogeneous, then we let $\deg r := \underline{i}$.

Now we will assume that $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, x_{2,0}, \ldots, x_{2,n_2}, \ldots, x_{h,0}, \ldots, x_{h,n_h}]$. We induce an $\mathbb{N}^h$-grading on R by setting $\deg x_{i,j} = e_i$, so that R is a multi-graded ring. If $m \in R$ is a monomial, then

$$m = x_{1,0}^{a_{1,0}} \cdots x_{1,n_1}^{a_{1,n_1}} x_{2,0}^{a_{2,0}} \cdots x_{2,n_2}^{a_{2,n_2}} \cdots x_{h,0}^{a_{h,0}} \cdots x_{h,n_h}^{a_{h,n_h}}.$$

We denote m by $X_1^{\underline{a}_1} X_2^{\underline{a}_2} \cdots X_h^{\underline{a}_h}$ where $\underline{a}_i \in \mathbb{N}^{n_i+1}$. It follows that $\deg m = (|\underline{a}_1|, |\underline{a}_2|, \ldots, |\underline{a}_h|)$. If $F \in R$, then we can write $F = F_1 + \cdots + F_t$ where each F_i is homogeneous. The F_i's are called *homogeneous terms* of F.

For each $\underline{i} \in \mathbb{N}^h$, let $R_{\underline{i}}$ denote the finite dimensional vector space over k spanned by all the monomials $m = X_1^{\underline{a}_1} X_2^{\underline{a}_2} \cdots X_h^{\underline{a}_h} \in R$ of degree $\deg m = (|\underline{a}_1|, |\underline{a}_2|, \ldots, |\underline{a}_h|) = \underline{i}$. It follows that

$$\dim_{\mathbf{k}} R_{\underline{i}} = \binom{n_1 + i_1}{i_1} \binom{n_2 + i_2}{i_2} \cdots \binom{n_h + i_h}{i_h}.$$

Definition 1.1.2. Let $I = (F_1, \ldots, F_r) \subseteq R$ be an ideal. If each generator F_j is $\mathbb{N}^h$-homogeneous, then we say that I is an $\mathbb{N}^h$-*homogeneous ideal* (or simply, a *homogeneous ideal*).

It can be shown that I is homogeneous if and only if for every $F \in I$, all of the homogeneous terms of F also belong to I.

If $I \subseteq R$ is any ideal, then we set $I_{\underline{i}} := I \cap R_{\underline{i}}$ for all $\underline{i} \in \mathbb{N}^h$. Each $I_{\underline{i}}$ is a subvector space of $R_{\underline{i}}$, and furthermore, $I \supseteq \bigoplus_{\underline{i} \in \mathbb{N}^h} I_{\underline{i}}$. If I is homogeneous, then we have an equality, i.e., $I = \bigoplus_{\underline{i} \in \mathbb{N}^h} I_{\underline{i}}$, because the homogeneous terms of F belong to I if $F \in I$.

When I is a homogeneous ideal of R, then the quotient ring R/I also inherits an $\mathbb{N}^h$-graded ring structure:

$$R/I = \bigoplus_{\underline{i} \in \mathbb{N}^h} (R/I)_{\underline{i}} = \bigoplus_{\underline{i} \in \mathbb{N}^h} R_{\underline{i}}/I_{\underline{i}}.$$

Remark 1.1.3. The multi-graded ring $R = \bigoplus_{\underline{i} \in \mathbb{N}^h} R_{\underline{i}}$ can also be viewed as a standard graded ring if we set $\deg x_{i,j} = 1$ and for each $t \in \mathbb{N}$, we define

$$R_t := \bigoplus_{\{\underline{j} \in \mathbb{N}^h \mid \ |\underline{j}| = t\}} R_{\underline{j}}.$$

Similarly, a homogeneous ideal I of R and the multi-graded quotient ring R/I are also $\mathbb{N}^1$-graded.

Now we introduce the multi-graded analog of the Hilbert function.

Definition 1.1.4. Let I be a homogeneous ideal of R with the $\mathbb{N}^h$-grading. The *Hilbert function of* R/I is the numerical function $H_{R/I} : \mathbb{N}^h \to \mathbb{N}$ defined by

$$H_{R/I}(\underline{i}) := \dim_{\mathbf{k}}(R/I)_{\underline{i}} = \dim_{\mathbf{k}} R_{\underline{i}} - \dim_{\mathbf{k}} I_{\underline{i}}.$$

Definition 1.1.5. Let $H : \mathbb{N}^h \to \mathbb{N}$ be a numerical function. We call $\Delta H : \mathbb{N}^h \to \mathbb{N}$ the *first difference function of* H where

$$\Delta H(\underline{i}) := \sum_{\underline{0} \leq \underline{l} = (l_1,\dots,l_h) \leq (1,\dots,1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \dots, i_h - l_h),$$

where $H(\underline{j}) = 0$ if $\underline{j} \not\geq \underline{0}$.

For example, if I is a homogeneous ideal of $R = k[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}]$ with the $\mathbb{N}^3$-grading, then the first difference function of H is the numerical function $\Delta H : \mathbb{N}^3 \to \mathbb{N}$ defined by

$$\Delta H(i,j,k) := \sum_{(0,0,0) \leq (l,m,n) \leq (1,1,1)} (-1)^{l+m+n} H(i-l, j-m, k-n).$$

1.1.2 Multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$

We now generalize the classical definition of a projective space and its subvarieties to a multi-projective setting.

We define the *multi-projective space* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ to be the set of equivalence classes

$$\left. \left\{ \begin{matrix} ((a_{1,0}, \dots, a_{1,n_1}), \dots, (a_{h,0}, \dots, a_{h,n_h})) \in \mathbf{k}^{n_1+1} \times \cdots \times \mathbf{k}^{n_h+1} \\ \text{with no } \underline{a}_i = (a_{i,0}, \dots, a_{i,n_i}) = \underline{0} \ \text{ for all } i = 1, \dots, h \end{matrix} \right\} \middle/ \sim \right.$$

where $\sim$ is the following equivalence relation

$$(\underline{a}_1, \dots, \underline{a}_h) \sim (\underline{b}_1, \dots, \underline{b}_h) \iff \exists \lambda_1, \dots, \lambda_h \in \mathbf{k} \setminus \{0\} \text{ such that}$$

$$\underline{a}_i = (a_{i,0}, \dots, a_{i,n_i}) = (\lambda_i b_{i,0}, \dots, \lambda_i b_{i,n_i}) = \lambda_i \underline{b}_i \quad , \quad i = 1, \dots, h.$$

An element of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ is called a *point*. We sometimes denote the equivalence class of $((a_{1,0}, \dots, a_{1,n_1}), \dots, (a_{h,0}, \dots, a_{h,n_h}))$ by $[a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{h,0} : \cdots : a_{h,n_h}]$. It follows that $[a_{i,0} : \cdots : a_{i,n_i}]$ is a point of $\mathbb{P}^{n_i}$ for every i.

We now consider the polynomial ring $R = \mathbf{k}[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{h,0}, \dots, x_{h,n_h}]$ with the $\mathbb{N}^h$-grading. If $F \in R$ is an $\mathbb{N}^h$-homogeneous element of degree $(d_1, \dots, d_h)$ and $P = [a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{h,0} : \cdots : a_{h,n_h}]$ is a point of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, then for all non-zero $\lambda_1, \lambda_2, \dots, \lambda_h \in \mathbf{k}$ we have

$$F(\lambda_1 a_{1,0}, \dots, \lambda_2 a_{2,0}, \dots, \lambda_h a_{h,0}, \dots) =$$
$$\lambda_1^{d_1} \lambda_2^{d_2} \cdots \lambda_h^{d_h} F(a_{1,0}, \dots, a_{2,0}, \dots, a_{h,0}, \dots).$$

To say that F vanishes at a point of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ is, therefore, a well-defined notion.

If T is any set of homogeneous elements of R, then we define

$$V(T) := \{P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h} \mid F(P) = 0 \text{ for all } F \in T\}.$$

If I is a homogeneous ideal of R, then $V(I) := V(T)$ where T is the set of all homogeneous elements of I. If $I = (F_1, \dots, F_r)$, then $V(I) = V(F_1, \dots, F_r)$. The multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ can be endowed with a topology by defining the *closed sets* to be all the subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ of the form $V(T)$ where T is a collection of $\mathbb{N}^h$-homogeneous elements of R. If Y is a subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ that is closed and irreducible with respect to this topology, then we say Y is a *multi-projective variety*, or simply, a *variety*.

If Y is any subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, then we set

$$I(Y) := \{F \in R \mid F(P) = 0 \text{ for all } P \in Y\}.$$

The set $I(Y)$ is an $\mathbb{N}^h$-homogeneous ideal of R and is called the *ideal associated to* Y. If $Y \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, then we set $I_Y := I(Y)$, and we call R/I_Y the $\mathbb{N}^h$*-homogeneous coordinate ring of* Y, or simply, the *coordinate ring of* Y. If H_{R/I_Y} is the Hilbert function of R/I_Y, then we sometimes write H_Y for H_{R/I_Y}, and we say H_Y is the *Hilbert function of* Y.

By adopting the proofs (see for example [30]) of the well known homogeneous case, it can be shown that

Proposition 1.1.6. *If R is an $\mathbb{N}^h$-graded polynomial ring, then*

(i) If $I_1 \subseteq I_2$ are $\mathbb{N}^h$-homogeneous ideals of R, then $V(I_1) \supseteq V(I_2)$.

(ii) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, then $I(Y_1) \supseteq I(Y_2)$.

(iii) For any two subsets Y_1, Y_2 of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

As in the graded case, the Nullstellensatz holds in the $\mathbb{N}^h$-graded context. Again, the proof follows as in the graded case (see for example [30]).

Theorem 1.1.7 ($\mathbb{N}^h$-graded Nullstellensatz)**.** *If $I \subseteq R$ is an $\mathbb{N}^h$-homogeneous ideal and $F \in R$ is an $\mathbb{N}^h$-homogeneous polynomial with $\deg F > \underline{0}$ such that $F(P) = 0$ for all $P \in V(I) \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, then $F^t \in I$ for some $t > 0$.*

This $\mathbb{N}^h$-graded version of the Nullstellensatz allows us to establish a correspondence between $\mathbb{N}^h$-homogeneous ideals of R and subvarieties of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, similar to the standard correspondence between graded ideals and varieties in $\mathbb{P}^n$. One difference between the standard graded case and the $\mathbb{N}^h$-graded case is the notion of irrelevant ideals.

Definition 1.1.8. Set $\mathbf{m}_i := (x_{i,0}, x_{i,1}, \dots, x_{i,n_i})$ for $i = 1, \dots, h$. An $\mathbb{N}^h$-homogeneous ideal I of R is called *projectively irrelevant* if $\mathbf{m}_i^a \subseteq I$ for some $i \in \{1, \dots, h\}$ and some positive integer a. An ideal $I \subseteq R$ is *projectively relevant* if it is not projectively irrelevant.

Remark 1.1.9. By adapting the proof of the graded case and using the $\mathbb{N}^h$-graded Nullstellensatz, it can be proved that there is a one-to-one correspondence between the nonempty closed subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ and the $\mathbb{N}^h$-homogeneous ideals of R that are radical ($I = \sqrt{I}$) and projectively relevant. The correspondence is given by

$$Y \mapsto I(Y) \text{ and } I \mapsto V(I).$$

This construction of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ and its subsets follows the classical definition of the projective space $\mathbb{P}^n$. While multiprojective spaces can be constructed via the modern methods of schemes, it will suffice for our purposes to only consider the classical definition because we wish to focus on sets of distinct points and sets of distinct lines. In the language of schemes, a set of distinct points (or lines) is a reduced scheme, and hence, the classical approach is equivalent to the schematic approach.

We recall two known results about the ideal associated to a point or a set of distinct points in a multi-projective space [96, Proposition 2.2.7].

Proposition 1.1.10. *For any point $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$, let I_P the ideal associated to the point P. Then*

(i) I_P is a prime ideal;

(ii) $I_P = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{h,1}, \ldots, L_{h,n_h})$ where deg $L_{i,j} = e_i$.

Proposition 1.1.11. *Let $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ be a set of s distinct points and suppose that I_{P_i} is the ideal associated to the point P_i. Then*

(i) $I_X = I_{P_1} \cap I_{P_2} \cap \cdots \cap I_{P_s}$;

(ii) K-dim $R/I_X = h$, where K-dim denotes the Krull dimension that is defined in the next section.

1.1.3 Resolutions and Projective Dimension

Now we recall the necessary results and definitions about the resolution and projective dimension of an R-module, where $R = \mathbf{k}[x_0, \ldots, x_n]$ is an $\mathbb{N}^1$-graded ring, although these definitions and results hold more generally.

Definition 1.1.12. An R-module M is a *graded R-module* if

(i) the module M has a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ where each M_i is an additive abelian group;

(ii) $R_i M_j \subseteq M_{i+j}$ for all $i \in \mathbb{N}$ and all $j \in \mathbb{Z}$.

If I is a homogeneous ideal of R, then I can be viewed as a graded R-module if we take $I_i = 0$ for $i < 0$. Similarly, for any homogeneous ideal $I \subseteq R$, the quotient ring R/I is a graded R-module.

If M is any R-graded module, and d is any integer, we let $M(d)$ denote the direct sum $M(d) = \bigoplus_{i\in\mathbb{Z}} M_{d+i}$. Then $M(d)$ is also a graded R-module, and it is sometimes referred to as the *twisted graded module*.

Definition 1.1.13. Let M and N be graded R-modules. A homomorphism of graded R-modules $\varphi : M \to N$ is said to be a *graded homomorphism of degree d* if $\varphi(M_i) \subseteq N_{i+d}$ for all $i \in \mathbb{Z}$.

In particular, an R-module graded homomorphism $\varphi : M \to N$ has degree 0 if $\varphi(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$.

Definition 1.1.14. Let M be a graded R-module. The *minimal graded free resolution of M* is an exact sequence of the form

$$\cdots \longrightarrow \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{2,j}(M)} \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{1,j}(M)} \xrightarrow{\varphi_1}$$

$$\xrightarrow{\varphi_1} \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{0,j}(M)} \xrightarrow{\varphi_0} M \longrightarrow 0$$

where each $\mathcal{F}_i := \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{i,j}(M)}$ is a graded free R-module and each map $\varphi_i : \mathcal{F}_i \to \mathcal{F}_{i-1}$, with $\mathcal{F}_{-1} := M$, is a graded homomorphism of degree zero such that $\varphi_{i+1}(\mathcal{F}_{i+1}) \subseteq \mathbf{m}\mathcal{F}_i$ for all $i \geq 0$.

The numbers $\beta_{i,j}(M)$ are the *graded Betti numbers of M*. In particular, the number $\beta_i = \sum_{j\in\mathbb{N}} \beta_{i,j}(M)$ is called *i-th Betti number of M* and $\beta_{i,j}(M)$ is the *i-th Betti number of M of degree j*.

Definition 1.1.15. Let M be a graded R-module. The *projective dimension of M*, denoted $\text{proj-dim}_R M$, is the length of a minimal graded free resolution of M.

1.1.4 Cohen-Macaulay rings

In this section we define Cohen-Macaulay rings and collect the facts we need in the later chapters. Our primary references for the material of this section on theory of Cohen-Macaulay rings is developed in [17] and [82].

We assume that $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1},\ x_{2,0}, \ldots, x_{2,n_2}, \ldots, x_{h,0}, \ldots, x_{h,n_h}]$ and we induce an $\mathbb{N}^h$-grading on R by setting $\deg x_{i,j} = e_i$, where e_i is the i-th standard basis vector in $\mathbb{N}^h$. We define $\mathbf{m}$ to be the ideal $\mathbf{m} := \bigoplus_{0\neq \underline{j}\in\mathbb{N}^h} R_{\underline{j}}$.

We recall the following definitions.

Definition 1.1.16. Let $I \subseteq R$ be an homogeneous ideal of R and let $\mathfrak{p}$ be a prime ideal of R/I. The *height of* $\mathfrak{p}$ is the largest integer t such that there exist prime ideals $\mathfrak{p}_i$ of R/I such that $\mathfrak{p} = \mathfrak{p}_t \supsetneq \mathfrak{p}_{t-1} \supsetneq \cdots \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0$. We write $\text{ht}_{R/I}(\mathfrak{p}) = t$.

Definition 1.1.17. For any ideal I of R, the *Krull dimension of R/I*, denoted by K-dim(R/I), is the number

$$\text{K-dim}(R/I) := \sup\{\text{ht}_{R/I}(\mathfrak{p}) \mid \mathfrak{p} \text{ is a prime ideal of } R/I\}.$$

Definition 1.1.18. Let $F_1, \ldots, F_r$ be a sequence of non-constant elements of R and let I be an $\mathbb{N}^h$-homogeneous ideal. Then we say $F_1, \ldots, F_r$ is a *regular sequence modulo I* or *give rise to a regular sequence in R/I* if and only if

(i) $(I, F_1, \ldots, F_r) \subseteq \mathfrak{m}$,

(ii) $\overline{F_1}$ is not a zero-divisor in R/I,

(iii) $\overline{F_i}$ is not a zero-divisor in $R/(I, F_1, \ldots, F_{i-1})$ for $1 < i \leq r$.

The sequence $F_1, \ldots, F_r$ is called a *maximal regular sequence modulo I* if $F_1, \ldots, F_r$ is a regular sequence which cannot be made longer.

Observe that for an arbitrary ring it is not true that all maximal regular sequence have the same length. However, since we shall only consider $\mathbb{N}^h$-homogeneous ideals of R, the following theorem applies.

Theorem 1.1.19. *[17, Theorem 1.2.5] Suppose that $I \subseteq \boldsymbol{m}$ is an $\mathbb{N}^h$-homogeneous ideal of the Noetherian ring R. Then all maximal regular sequence modulo I have the same length.*

By this theorem arises in a natural way the following definition:

Definition 1.1.20. Let $I \subseteq \mathbf{m}$ be an $\mathbb{N}^h$-homogeneous ideal of R. The *depth of R/I*, written depth(R/I), is the length of a maximal regular sequence modulo I.

One can show, using *Krull's Principal Ideal Theorem* [92, Theorem 15.2], that depth$(R/I) \leq$ K-dim(R/I) always holds. If equality occurs, then we give the ring R/I a special name.

Definition 1.1.21. Let $I \subseteq \mathbf{m}$ be an $\mathbb{N}^h$-homogeneous ideal of R. Then the ring R/I is called *Cohen-Macaulay* (or CM for short) if depth$(R/I) =$ K-dim(R/I). In this case we say that the ideal I is *Cohen-Macaulay.*

Definition 1.1.22. Let M be a module over the commutative Noetherian ring A, and let $\mathfrak{p}$ be a prime ideal of A. Then we say $\mathfrak{p}$ is an *associated prime ideal of M* when there exists an element $m \in M$ such that $(0\colon m) = \mathfrak{p}$. The set of associated prime ideals of M is denoted by $\text{Ass}_A(M)$.

Remark 1.1.23. Suppose that I is a proper ideal of a commutative Noetherian ring A and suppose that $I = Q_1 \cap \cdots \cap Q_r$ is the primary decomposition of I. Set $\mathfrak{p}_i = \sqrt{Q_i}$. Then we have $\text{Ass}_A(A/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Definition 1.1.24. Let A be a commutative Noetherian ring and I be an ideal of A. Suppose that $\mathrm{Ass}_A(A/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. We say that I is *unmixed* if $\mathrm{ht}_A(\mathfrak{p}_i) = \mathrm{ht}_A(I)$ for all $i = 1, \ldots, r$.

Now we recall a series of results about Cohen-Macaulay local rings that allow us to prove that if I is a homogeneous ideal of a polynomial ring R with the property that R/I is Cohen-Macaulay, then the associated prime ideals of I all have the same height (for the proofs of these following results, see [7] and [82]).

Lemma 1.1.25. *If (A, m) is a local CM ring, then for any ideal $I \subsetneq A$, we have $ht_A(I) + K\text{-}dim(A/I) = K\text{-}dimA$.*

Lemma 1.1.26. *If A is a CM ring, and if S is any multiplicatively closed subset, then $S^{-1}A$ is also CM (i.e. the CM property is preserved under localization).*

Lemma 1.1.27. *Let S be a multiplicative subset of A, and let M be a finitely generated A-module. Put $A' = S^{-1}A$ and $M' = S^{-1}M$. Then there exists a $1-1$ correspondence between the sets*

$$Ass(M) \cap \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime, } \mathfrak{p} \cap S = \emptyset\} \stackrel{1-1}{\longleftrightarrow} Ass_{A'}(M')$$

via the map $\mathfrak{p} \mapsto \mathfrak{p}S^{-1}A$.

Proposition 1.1.28. *Let (A, m) be a Noetherian local ring, and let M be a finitely generated CM A-module. Then depth $M = K\text{-}dim(A/\mathfrak{p})$ for every $\mathfrak{p} \in Ass_A(M)$.*

By these results it follows the next well known theorem, that also applies in the case of a graded polynomial ring.

Theorem 1.1.29. *Let I be a homogeneous ideal of $R = k[x_0, \ldots, x_n]$ and suppose that $I \subseteq \boldsymbol{m} := (x_0, \ldots, x_n)$. Then*

(i) $ht_R(I) + K\text{-}dim(R/I) = K\text{-}dim\ R$;

(ii) if R/I is a CM ring, then the ideal I is unmixed.

Definition 1.1.30. A variety $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ is *arithmetically Cohen-Macaulay* (ACM for short) if the multi-graded coordinate ring $R/I_{\mathbb{X}}$ is CM.

Remark 1.1.31. Suppose $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ is a variety. The multi-homogeneous ideal $I_{\mathbb{X}}$ in the $\mathbb{N}^h$-graded polynomial ring R corresponding to $\mathbb{X}$ is also a homogeneous ideal in the normal sense. We let $\tilde{\mathbb{X}}$ denote the variety in $\mathbb{P}^N$, where $N = \left(\sum_{i=1}^{h}(n_i + 1)\right) - 1$, defined by $I_{\mathbb{X}}$. The condition of being CM is a condition on the depth of $R/I_{\mathbb{X}}$. Because the grading of a ring does not influence the depth of a ring,

$$\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h} \text{ is ACM } \iff \tilde{\mathbb{X}} \subseteq \mathbb{P}^N \text{ is ACM.}$$

Note that the dimension of the variety $\tilde{\mathbb{X}}$ is bigger than the dimension of $\mathbb{X}$. Specifically, $\dim \tilde{\mathbb{X}} = \dim \mathbb{X} + h$.

The following results about CM rings will be required in the later chapters.

Lemma 1.1.32. *If A is a CM ring and x is a non-zero divisor in A, then the ring $A/(x)$ is also CM. Moreover, K-dim $A/(x) =$ K-dim $A - 1$.*

Lemma 1.1.33. *Let $J = (F_1, \ldots, F_r) \subseteq \boldsymbol{m} \subseteq R$ be an $\mathbb{N}^h$-homogeneous ideal. Suppose that $F_1, \ldots, F_r$ give rise a regular sequence in R. Then R/J is CM.*

Definition 1.1.34. Suppose that $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ is a variety. If the $\mathbb{N}^h$-homogeneous ideal $I_{\mathbb{X}}$ is generated by a regular sequence in R, then we say $\mathbb{X}$ is a *complete intersection*.

Remark 1.1.35. By Lemma 1.1.33, a complete intersection is always ACM.

The following result from homological algebra, that is a special case of the Auslander-Buchsbaum formula, allows us to link the depth of the quotient ring R/I to its projective dimension.

Theorem 1.1.36 (Auslander-Buchsbaum Formula). *Let I be a homogeneous ideal in the standard graded ring $R = k[x_0, \ldots, x_n]$, then*

$$\textit{proj-dim}_R(R/I) + \textit{depth}(R/I) = \textit{K-dim}(R).$$

This theorem allows us to characterize Cohen-Macaulay rings via the projective dimension of the ring, rather than in terms of the depth of the ring as in the definition of the CM property.

Theorem 1.1.37. *Let I be a homogeneous ideal in the standard graded ring $R = k[x_0, \ldots, x_n]$, then*

$$R/I \textit{ is CM if and only if } \textit{proj-dim}(R/I) = n + 1 - \textit{K-dim}(R/I).$$

Proof. The ring R/I is CM if and only if $\text{depth}(R/I) = \text{K-dim}(R/I)$. Hence, by the Auslander-Buchsbaum formula we have

$$\text{proj-dim}(R/I) + \text{K-dim}(R/I) = \text{proj-dim}(R/I) + \text{depth}(R/I) = n + 1.$$

□

1.1.5 General remarks on points in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we refer to [59] to recall the basic properties of a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$ and describe its associated homogeneous ideal and some algebraic invariants of the coordinate ring $R_X := R/I(X)$.

A point $P \in \mathbb{P}^1 \times \mathbb{P}^1$ has the form $P = A \times B$, with $A, B \in \mathbb{P}^1$ not necessarily distinct. Given a point $P = A \times B$ of $\mathbb{P}^1 \times \mathbb{P}^1$, its associated homogeneous ideal $I(P)$ in the bigraded ring $R = k[x_0, x_1, y_0, y_1]$ has the following properties [59, Theorem 3.1]:

(i) $I(P)$ is a prime ideal of R;

(ii) $I(P) = (H, V)$ where $\deg H = (1, 0)$ and $\deg G = (0, 1)$;

(iii) Let $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s distinct points and suppose that $I(P_i)$ is the ideal associated to the point P_i. Then $I(X) = I(P_1) \cap \dots \cap I(P_s)$.

Remark 1.1.38. By the proof of (ii) property it follows that if $P = A \times B \in \mathbb{P}^1 \times \mathbb{P}^1$, where $A = [a_0 : a_1] \in \mathbb{P}^1$ and $B = [b_0 : b_1] \in \mathbb{P}^1$, then $I(P) = (a_1x_0 - a_0x_1, b_1y_0 - b_0y_1)$.

Remark 1.1.39. The bihomogeneous ideal $I(P)$ associated to a point $P \in \mathbb{P}^1 \times \mathbb{P}^1$ corresponds geometrically to a line in $\mathbb{P}^3$. We now explain this point of view. Consider the two skew lines L_1 and L_2 in $\mathbb{P}^3$ defined by $I(L_1) = (x_0, x_1)$ and $I(L_2) = (y_0, y_1)$. Given a point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$, by the previous Remark we have that $I(P) = (a_1x_0 - a_0x_1, b_1y_0 - b_0y_1)$. Since $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[\mathbb{P}^3]$ as rings, we can regard $I(P)$ as a $\mathbb{N}^1$-homogeneous ideal which defines a line L in $\mathbb{P}^3$ through the points $B = [0 : 0 : b_0 : b_1]$ on L_1 and $A = [a_0 : a_1 : 0 : 0]$ on L_2. So, as a graded ideal, $I(P)$ defines a line in $\mathbb{P}^3$ that intersects both L_1 and L_2, and furthermore, the coordinates of P describe where the line defined by $I(P)$ intersects these two skew lines.

Now we recall a combinatorial description of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ [59, Section 3.2], that gives us some information related to some of the algebraic invariants of the associated bigraded coordinate ring.

On $\mathbb{P}^1 \times \mathbb{P}^1$ there exist two families of lines $\{H_C\}$ and $\{V_C\}$, each parametrized by $C \in \mathbb{P}^1$, with the property that if $A \neq B \in \mathbb{P}^1$, then $H_A \cap H_B = \emptyset$ and $V_A \cap V_B = \emptyset$, and for all $A, B \in \mathbb{P}^1$, $H_A \cap V_B = A \times B$ is a point on $\mathbb{P}^1 \times \mathbb{P}^1$. We can thus view $\mathbb{P}^1 \times \mathbb{P}^1$ as a grid with horizontal and vertical rulings. A point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ can be viewed as the intersection of the horizontal ruling defined by the degree $(1, 0)$ line $H = a_1x_0 - a_0x_1$ and the vertical ruling defined by the degree $(0, 1)$ line $V = b_1y_0 - b_0y_1$. Hence, to any nonempty finite set $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ of points we associate a set L_X of integer lattice points indicating which points lie on the same horizontal or vertical ruling. The idea is to enumerate the horizontal and vertical rulings whose intersection with X is nonempty. We thus obtain, say $H_1, \dots, H_h$ and

$V_1, \ldots, V_\nu$ where $X \subset \bigcup_{i=1}^{h} H_i$ and $X \subset \bigcup_{j=1}^{\nu} V_j$, and L_X consists of all pairs (i, j) such that $X \cap H_i \cap V_j \neq \emptyset$.

If $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denotes the natural projection morphism onto the first coordinate, then note that $h = |\pi_1(X)|$, the number of distinct first coordinates that appear in X. Similarly, if $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection morphism onto the second coordinate, then $\nu = |\pi_2(X)|$ is the number of distinct second coordinates.

Example 1.1.40. Let X be the following set of points in $\mathbb{P}^1 \times \mathbb{P}^1$

$$X = \{A_1 \times B_2, A_1 \times B_4, A_1 \times B_5, A_2 \times B_2, A_2 \times B_3, A_2 \times B_4, A_2 \times B_5,$$
$$A_3 \times B_1, A_3 \times B_2, A_3 \times B_3, A_3 \times B_4, A_3 \times B_5\}.$$

Using the above mentioned construction, the set of points X can be represented as in Figure 1.1.

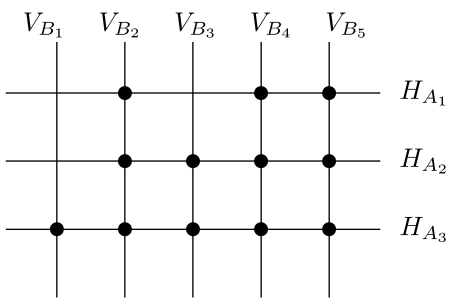


Figure 1.1: The set of points X.

We now make two conventions by which we will introduce a combinatorial description of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Convention 1.1.41. We shall abuse notation to let H_A, respectively V_B, denote both the horizontal ruling, respectively the vertical ruling, and the degree $(1, 0)$ form, respectively the degree $(0, 1)$ form, that defines the ruling. Thus, given a point $P = A \times B \in \mathbb{P}^1 \times \mathbb{P}^1$, its defining ideal is given by $I(P) = (H_A, V_B)$, and geometrically, $P = H_A \cap V_B$.

Convention 1.1.42. By relabeling the horizontal and vertical rulings, we can always assume that $|X \cap H_{A_1}| \geq |X \cap H_{A_2}| \geq \cdots$ and $|X \cap V_{B_1}| \geq |X \cap V_{B_2}| \geq \cdots$. That is, we can assume that the first ruling contains the most number of points, the second ruling contains the same number or less, and so on.

Example 1.1.43. After relabelling the horizontal and vertical rulings according to Conventions 1.1.41 and 1.1.42, the set of points X of Example 1.1.40 can be drawn as in Figure 1.2.

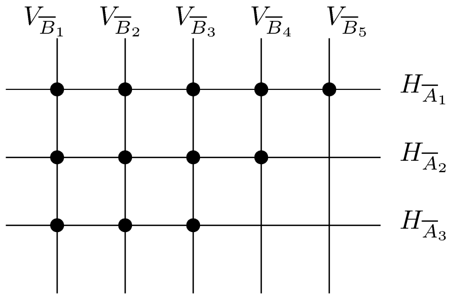


Figure 1.2: The set of points X after relabelling the rulings.

Using the previous notation and conventions, we recall the following definitions [59, Definitions 3.11, 3.12 and 3.13].

Definition 1.1.44. Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of distinct points and suppose that $\pi_1(X) = \{A_1, \dots, A_h\}$ and $\pi_2(X) = \{B_1, \dots, B_\nu\}$.
For $i = 1, \dots, h$, set $\alpha_i := |\pi_1^{-1}(A_i) \cap X|$, and let $\alpha_X := (\alpha_1, \dots, \alpha_h)$.
For $j = 1, \dots, \nu$, set $\beta_j := |\pi_2^{-1}(B_j) \cap X|$, and let $\beta_X := (\beta_1, \dots, \beta_\nu)$.

The number α_i counts the number of points in X whose first coordinate is A_i, that is the number of points of X that lie on the horizontal ruling H_i. Analogously, the number β_j counts the number of points in X whose second coordinate is B_j, that is the number of points of X that lie on the vertical ruling V_j. By Convention 1.1.42 , we have $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_h$, and similarly, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_\nu$.

Every set of points X can now associated with two tuples α_X and β_X, both of which are partitions of $|X|$.

Definition 1.1.45. A tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers is a *partition* of an integer s if $\sum_{i=1}^{r} \lambda_i = s$ and $\lambda_i \geq \lambda_{i+1}$ for every i. We write $\lambda = (\lambda_1, \dots, \lambda_r) \vdash s$.

The *conjugate* of λ is the tuple $\lambda^\star = (\lambda_1^\star, \dots, \lambda_{\lambda_1}^\star)$ where $\lambda_i^\star = \#\{\lambda_j \in \lambda \mid \lambda_j \geq i\}$. Furthermore, $\lambda^\star \vdash s$.

Definition 1.1.46. To any partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash s$ we can associate the following diagram: on an $r \times \lambda_1$ grid, place λ_1 points on the first horizontal line, λ_2 points on the second, and so on, where the points are left justified. The resulting diagram is called the *Ferrers diagram of* λ.

Definition 1.1.47. Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of distinct points. We say that X *resembles a Ferrers diagram* if after we apply Convention 1.1.42, the set of points looks like a Ferrers diagram.

Applying Lemma 3.17, Theorem 3.21 and Theorem 4.11 in [59], we have

Lemma 1.1.48. *Let Y be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Y is ACM if and only if Y resembles a Ferrers diagram.*

The next definition introduces an important property that allows us to characterize ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We will generalize this property for varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Definition 2.2.9).

Definition 1.1.49. A set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ satisfies *property ($\star$)* if whenever $A \times B$ and $A' \times B'$ are in X with $A \neq A'$ and $B \neq B'$, then either $A \times B'$ or $A' \times B$ (or both) is also in X.

We now recall two different characterizations of arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ [59, Theorem 4.11].

Theorem 1.1.50. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of distinct points. Then the following are equivalent:*

(i) *X is an ACM set of points.*

(ii) $\alpha^{\star}_X = \beta_X$.

(iii) *X satisfies property ($\star$).*

Note that, by the previous theorem, we can determine if a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$ is ACM directly from a combinatorial description of the points.

Furthermore, when X is ACM, the combinatorial description of the points allows us to determine the bigraded minimal free resolution of $I(X)$ [59, Theorem 5.3].

Theorem 1.1.51. *Suppose that $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is an ACM set of points with $\alpha_X = (\alpha_1, \ldots, \alpha_h)$. Then the bigraded minimal free resolution of $I(X)$ has the form*

$$0 \longrightarrow \bigoplus_{(v_1,v_2)\in V_X} R(-v_1,-v_2) \longrightarrow \bigoplus_{(c_1,c_2)\in C_X} R(-c_1,-c_2) \longrightarrow I(X) \longrightarrow 0,$$

where

$$C_X := \{(h,0),(0,\alpha_1)\} \cup \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\},$$

and

$$V_X := \{(h,\alpha_h)\} \cup \{(i-1,\alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$

1.2 Regularity of edge ideals

In this section we recall many recent results on the regularity of ordinary powers of squarefree monomial ideals and we survey how to find bounds and compute the exact value for the regularity in terms of combinatorial data from associated simplicial complexes and/or hypergraphs. We refer to [19] for all definitions, results and techniques recalled in this section.

1.2.1 Combinatorial framework

We now illustrate the general framework and recall basic notation and terminology from commutative algebra and combinatorics (for more details, see [13], [19], [70], [83], [93]).

Let $R = \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial ring over $\mathbf{k}$, and let $\mathbf{m}$ be the maximal homogeneous ideal in R. By abusing of notation, we identify a subset V of the vertices $X = \{x_1, \ldots, x_n\}$ with the squarefree monomial $x^V = \prod_{x \in V} x$ in the polynomial ring R.

Definition 1.2.1. A *simplicial complex* Δ over the vertex set $X = \{x_1, \ldots, x_n\}$ is a collection of subsets of X such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. Elements of Δ are called *faces*. Maximal faces (with respect to the inclusion) are called *facets*.

For $F \in \Delta$, the *dimension of* F is defined to be $\dim F = |F| - 1$. The *dimension of* Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$.

The complex is called *pure* if all of its facets are of the same dimension.

Definition 1.2.2. Let Δ be a simplicial complex and let $Y \subseteq X$ be a subset of its vertices. The *induced subcomplex of* Δ *on* Y, denoted by $\Delta[Y]$, is the simplicial complex with vertex set Y and faces $\{F \in \Delta \mid F \subseteq Y\}$.

Example 1.2.3. The simplicial complex Δ in Figure 1.3 is of dimension 3. The facet $\{a, b, c, d\}$ is of dimension 3, the facet $\{e, f, g\}$ is of dimension 2 and the facet $\{d, e\}$ is of dimension 1.

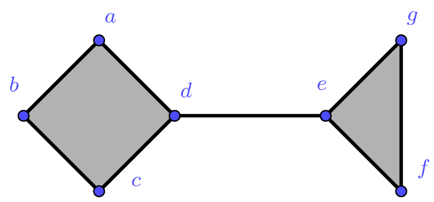


Figure 1.3: A nonpure simplicial complex.

Definition 1.2.4. A *hypergraph* $H = (X, \mathcal{E})$ over the vertex set $X = \{x_1, \ldots, x_n\}$ consists of X and a collection $\mathcal{E}$ of nonempty subsets of X which are called *edges* of H.

An *isolated vertex* is a vertex that does not belong to any edge. An *isolated loop* is an edge consisting of a single vertex.

Definition 1.2.5. A hypergraph H is *simple* if there is no nontrivial containment between any pair of its edges.

We shall assume that hypergraphs under consideration are simple and have no isolated vertices.

Definition 1.2.6. Let $H = (X, \mathcal{E})$ be a simple hypergraph. An edge $E \in \mathcal{E}$ is *incident* to a vertex $x \in X$ if $x \in E$. The *degree* of the vertex x, denoted by $\deg(x)$, is the number of edges incident to x.

Definition 1.2.7. Let $Y \subseteq X$ be a subset of the vertices in H. The *induced subhypergraph of H on Y*, denoted by $H[Y]$, is the hypergraph with vertex set Y and edge set $\{E \in \mathcal{E} \mid E \subseteq Y\}$.

Definition 1.2.8. Let H be a simple hypergraph. A collection C of edges of H is called a *matching* if the edges in C are pairwise disjoint. The maximum size of a matching in H is called its *matching number*.

Definition 1.2.9. Let H be a simple hypergraph. A collection C of edges of H is called an *induced matching* if C is a matching, and C consists of all edges of the induced subhypergraph $H[\cup_{E \in C} E]$ of H. The maximum size of an induced matching in H is called its *induced matching number* and it is denoted by $\nu(H)$.

Example 1.2.10. Let $H = (X, \mathcal{E})$, with vertex set $X = \{a, b, c, d, e, f\}$ and edge set $\mathcal{E} = \{\{a, b, c, d\}, \{d, e\}, \{f\}\}$. The collection $\{\{a, b, c, d\}, \{f\}\}$ forms an induced matching in H.

Definition 1.2.11. Let $H = (X, \mathcal{E})$ be a simple hypergraph and let x be a vertex in X. We call set of *neighbors* of x the set

$$N(x) = \{\ V \subseteq X \quad s.t. \quad V \cup \{x\} \in \mathcal{E}\}$$

and we denote $N[x] = N(x) \cup \{x\}$.
Let E be an edge in H. We call set of *neighbors* of E the set

$$N(E) = \{x \in X \mid \exists F \subseteq E \ s.t. \ F \cup \{x\} \in \mathcal{E}\}$$

and we denote $N[E] = N(E) \cup E$.

Definition 1.2.12. Let $H = (X, \mathcal{E})$ be a simple hypergraph and let E be an edge in H. We call *deletion of E from H*, and we denote it by $H \setminus E$, the hypergraph obtained by deleting E from the edge set of H (but the vertices are remained).

For a subset $Y \subseteq X$ of vertices in H, we define $H \setminus Y$ to be the subhypergraph of H obtained by deleting the vertices of Y and their incident edges.

We also define H_E to be the induced subhypergraph of H over the vertex set $X \setminus N[E]$.

Definition 1.2.13. Let $H = (X, \mathcal{E})$ be a simple hypergraph. A collection of vertices V in H is called an *independent set* if there is no edge $E \in \mathcal{E}$ such that $E \subseteq V$.

Definition 1.2.14. Let $H = (X, \mathcal{E})$ be a simple hypergraph. The *independence complex of* H, denoted by $\Delta(H)$, is the simplicial complex whose faces are independent sets in H.

Definition 1.2.15. A *graph* is a hypergraph in which all edges are of cardinality 2.

The *complement* of a graph G, denoted by G^c, is the graph with the same vertex set and an edge E is in G^c if and only if E is not in G.

Definition 1.2.16. Let $G = (V, E)$ be a graph.

1. G is called a path with l vertices, denoted by P_l, if $V = \{v_1, \ldots, v_l\}$ and $\{v_i, v_{i+1}\} \in E$ for all $1 \leq i \leq l - 1$.

2. G is called a cycle with n vertices, denoted by C_n, if $V = \{v_1, \ldots, v_n\}$ and $\{v_i, v_{i+1}\} \in E$ for all $1 \leq i \leq n - 1$ and $\{v_n, v_1\} \in E$.

3. G is called a dumbbell graph if G contains two cycles C_n and C_m joined by a path P_l of l vertices. We denote it by $C_n \cdot P_l \cdot C_m$.

Definition 1.2.17. A *chord* is an edge joining two not adjacent vertices in a cycle. A *minimal cycle* is a cycle without chords.

Definition 1.2.18. A graph G is called *chordal* when all its minimal cycles have length three, or equivalently, if it has no induced cycles of length ≥ 4.

Remark 1.2.19. [14, Remark 2.12] Let P_l be a path of l vertices, then we have

$$\nu(P_l) = \lfloor \frac{l+1}{3} \rfloor.$$

Remark 1.2.20. [14, Remark 2.13] Let C_n be a cycle of n vertices, then we have

$$\nu(C_n) = \lfloor \frac{n}{3} \rfloor.$$

A maximal induced matching of C_n is completely determined by just choosing a first edge, and then we go (for instance) in clockwise direction by taking the third consecutive edge after the last one chosen. Thus, we shall use $r = n \bmod 3$ to give a specific characterization of the structure of the maximal induced matching. Depending on r we can assume the following:

1. when $r = 0$, the edges x_1x_2 and x_1x_n do not belong to a maximal induced matching of C_n;

2. when $r = 1$, the edges x_1x_2, x_1x_n and $x_{n-1}x_n$ do not belong to a maximal induced matching of C_n;

3. when $r = 2$, the edges x_1x_2, x_2x_3, x_1x_n and $x_{n-1}x_n$ do not belong to a maximal induced matching of C_n.

1.2.2 Stanley-Reisner ideals, edge ideals, cover ideals and Alexander duality

The correspondences between squarefree monomial ideals and simplicial complexes and/or simple hypergraphs arise via the Stanley-Reisner ideal and edge ideal constructions and allow us to pass back and forth from commutative algebra to combinatorics.

Definition 1.2.21. Let Δ be a simplicial complex on X. The *Stanley-Reisner ideal* of Δ is defined to be

$$I_\Delta = (x^F \mid F \subseteq X \text{ is not a face of } \Delta).$$

The edge ideal construction was introduced in [100] for graphs and later generalized to hypergraph in [66].

Definition 1.2.22. Let H be a simple hypergraph on X. The *edge ideal of* H is defined to be

$$I(H) = (x^E \mid E \subseteq X \text{ is an edge in } H).$$

Definition 1.2.23. Let H be a simple hypergraph on X. The *cover ideal of* H is defined to be

$$J(H) = \bigcap_{\{x_i,x_j\}\in E} (x_i, x_j).$$

Remark 1.2.24. Recall that the Stanley-Reisner ideal of the independence complex Δ of H is the edge ideal of H, that is $I_\Delta = I(H)$.

Another significant tool in the study of squarefree monomial ideals is the Alexander duality theory for simplicial complexes.

Definition 1.2.25. Let Δ be a simplicial complex over the vertex set X. The *Alexander dual of* Δ, denoted by $\Delta^\vee$, is the simplicial complex over X with faces

$$\{X \setminus F \mid F \notin \Delta\}.$$

Notice that $\Delta^{\vee\vee} = \Delta$.

If $I = I_\Delta$, then we shall denote by $I^\vee$ the Stanley-Reisner ideal of the Alexander dual $\Delta^\vee$. If $I = I(H)$, then we shall denote by $H^\vee$ the simple hypergraph corresponding to $I^\vee$.

If G is a graph, it is a well known fact that each $I(G)$ and $J(G)$ is the Alexander dual of the other, that is $I(G)^\vee = J(G)$.

It is celebrated result of Terai (see [94]) that the regularity of a squarefree monomial ideal can be related to the projective dimension of its Alexander dual.

Theorem 1.2.26. *Let $I \subseteq R$ be a squarefree monomial ideal. Then*

$$\operatorname{reg}(I) = pd\,(R/I^\vee).$$

1.2.3 Regularity of powers of squarefree monomial ideals

The notion of the Castelnuovo-Mumford regularity can be defined in various way. First, we give the general definition using local cohomology modules and then we give a definition via the minimal free resolution in the specialized context of polynomial rings. We refer to [19] for the following definitions and results.

Definition 1.2.27. Ler R be a standard graded algebra over a Noetherian commutative ring with identity and let $\mathbf{m}$ be its maximal homogeneous ideal. Let M be a finitely generated graded R-module. For $i \geq 0$, let

$$a^i(M) = \begin{cases} \max\{l \in \mathbb{Z} \mid [H^i_{\mathbf{m}}(M)]_l \neq 0\} & \text{if } \; H^i_{\mathbf{m}}(M) \neq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The *regularity* of M is defined to be

$$\operatorname{reg}(M) = \max_{i \geq o}\{a^i(M)\}.$$

Note that the regularity of M is well-defined since $a^i(M) = 0$ for $i > \dim M$.

When R is a standard graded polynomial ring, the regularity of an R-module can also be computated via the minimal free resolution (see [25], [36]).

Definition 1.2.28. Let R be a standard graded polynomial ring over a field and let $\mathbf{m}$ be its maximal homogeneous ideal. Let M be a finitely generated graded R-module and let

$$0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{pj}(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0j}(M)} \to M \to 0$$

be its minimal free resolution. Then the regularity of M is given by

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{ij}(M) \neq 0\}.$$

Remark 1.2.29. By looking at the minimal free resolution, it is easy to see that $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$, so we can work with $\operatorname{reg}(I)$ and $\operatorname{reg}(R/I)$ interchangeably.

Remark 1.2.30. It is clear from the previous definition that the regularity of M gives an upper bound for generating degrees of M.

Now we recall one of the most powerful results about regularity of powers of homogeneous ideals, that was independently proved by Cutkosky, Trung and Herzog (see [32]) and Kodiyalam (see [79]).

Theorem 1.2.31. *Let R be a standard graded algebra over a Noetherian commutative ring with identity. Let $I \subseteq R$ be a homogeneous ideal. Then there exist constants a and b such that*

$$\operatorname{reg} I^q = aq + b \quad \textit{for all} \quad q \gg 0.$$

Moreover,

$$a = \min\{d(J) \mid J \textit{ is a minimal reduction of } I\},$$

where $J \subseteq I$ is a reduction of I if $I^{s+1} = JI^s$ for some (and all) $s \geq 0$, and $d(J)$ denotes the maximal generating degree of J.

Determine the constants b and $q_0 = \min\{t \in \mathbb{Z} \mid \operatorname{reg} I^q = aq + b \; \forall \; q \geq t\}$ remains a wide open problem. When we restrict this problem to the case where $I = I(G)$ is the edge ideal of a simple graph G, it is known that for $q \gg 0$,

$$regI^q = 2q + b.$$

When I is the edge ideal of a particular graph G, like a forest, a cycle or a unicyclic graph, the problem of determining the constants b and $q_0 = \min\{t \in \mathbb{Z} \mid regI^q = 2q + b \; \forall q \geq t \}$ has been solved. The regularity of the edge ideal of a forest was first computed by Zheng (see [104]).

Theorem 1.2.32. *[104, Theorem 2.18] Let G be a forest, then*

$$\operatorname{reg} I(G) = \nu(G) + 1.$$

The previous result was extended to chordal graphs in [66].

Theorem 1.2.33. *[66, Theorem 6.8] Let G be a chordal graph, then*

$$\operatorname{reg} I(G) = \nu(G) + 1.$$

In [78] Katzman first noticed that the previous equality is a lower bound for general graphs.

Theorem 1.2.34. *[78, Corollary 1.2] Let G be a graph, then*

$$\operatorname{reg} I(G) \geq \nu(G) + 1.$$

This result is extended for all simple hypergraphs in [85].

The decycling number of a graph is an important combinatorial invariant which can be used to obtain an upper bound for the regularity of the edge ideal of a graph.

Definition 1.2.35. For a graph G and $D \subset V(G)$, if $G \setminus D$ is acyclic, i.e. contains no induced cycle, then D is said to be a *decycling set* of G. The size of a smallest decycling set of G is called the *decycling number* of G and denoted by $\nabla(G)$.

Theorem 1.2.36. *[16, Theorem 4.11] Let G be a graph, then*

$$\operatorname{reg} I(G) \leq \nu(G) + \nabla(G) + 1.$$

Theorem 1.2.37. *[85, Corollary 3.9] Let H be a simple hypergraph. Suppose that $\{E_1, \dots, E_s\}$ forms an induced matching in H. Then*

$$\operatorname{reg}(H) \geq \sum_{i=1}^{s}(|E_i| - 1) + 1.$$

Remark 1.2.38. Note that if H consists of disjoint edges then the bound in Theorem 1.2.37 becomes an equality.

We have also obtained the explicit computation of the regularity for special classes of squarefree monomial ideals that have combinatorial structures that force them to have small regularity. Note that a squarefree monomial ideal has regularity 1 if and only if it is generated by a collection of variables. Thus, it's natural starting to consider ideals with regularity at least 2. The following result was originally stated and proved by Wegner (see [101]) using topological language, and re-stated in terms of monomial ideals by Fröberg (see [37] , [48] and [64]).

Theorem 1.2.39. *Let G be a simple graph. Then* $\operatorname{reg} I(G) = 2$ *if and only if G^C is a chordal graph.*

It is still an open problem to give a combinatorial characterization for squarefree monomial ideals I such that $\operatorname{reg}(I) = 3$ or to classify simple graphs G such that $\operatorname{reg} I(G) = 3$, and only few partial results have been obtained to today.

In [14] Beyarslan, Hà and Trung provided a formula for the regularity of powers of the edge ideal of a forest or a cycle in terms of its induced matching number.

Theorem 1.2.40. *[14, Theorem 4.5] Let G be a graph with edge ideal $I(G)$ and let $\nu(G)$ denote its induced matching number. Then, for all $q \geq 1$, we have*

$$\operatorname{reg} I(G)^q \geq 2q + \nu(G) - 1.$$

Theorem 1.2.41. *[14, Theorem 4.7] Let G be a forest and let $I = I(G)$ be its edge ideal. Then for all $q \geq 1$, we have*

$$regI^q = 2q + \nu(G) - 1.$$

Theorem 1.2.42. *[14, Theorem 5.2] Let C_n denote the n-cycle and let $I = I(C_n)$ be its edge ideal. Let $\nu = \lfloor \frac{n}{3} \rfloor$ be the induced matching number of C_n. Then*

$$\operatorname{reg} I = \begin{cases} \nu + 1 & \text{if } n \equiv 0, 1 \quad (mod\, 3), \\ \nu + 2 & \text{if } n \equiv 2 \quad (mod\, 3). \end{cases}$$

Moreover, for all $q \geq 2$

$$\operatorname{reg} I(C_n)^q = 2q + \nu(C_n) - 1.$$

In addition they prove an upper bound for graphs which contain Hamiltonian path. We recall that a *Hamiltonian path* of a graph G is a path that goes through each vertex of G exactly once.

Theorem 1.2.43. *[14, Theorem 3.1] Let G be a graph on n vertices. If G contains a Hamiltonian path, then*

$$\operatorname{reg} I(G) \leq \lfloor \frac{n+1}{3} \rfloor + 1.$$

Theorem 1.2.44. *[3, Theorem 1.2] Let G be a unicyclic graph (i.e., a graph having exactly one cycle) and let $I = I(G)$ be its edge ideal. Then for all $q \gg 1$, we have*

$$\operatorname{reg} I^q = 2q + \operatorname{reg} I - 2.$$

The simplest situation for an edge ideal is when its powers have linear resolution.

Definition 1.2.45. Let I be an ideal of R. We say that I has a *d-linear resolution* if I is generated by homogeneous elements of degree d and $reg(I) = d$. That is, the graded minimal free resolution of I is of the form:

$$0 \to R(-d-s)^{\beta_s} \to \cdots \to R(-d-1)^{\beta_1} \to R(-d)^{\beta_0} \to I \to 0.$$

Among all the interesting problems in Catslnuovo-Mumford regularity, classification of ideals with linear resolution is of great importance. Proving that a class of ideals has a d-linear resolution is difficult in general. However, some classes of ideals with linear resolution may be found in [6], [27], [39], [71], [104]. It is a nice result (see [48], [101]) that the edge ideal of a graph G has a linear resolution if and only if G^c is chordal.

Theorem 1.2.46. *[48, Theorem 1] Let G be a graph. Then $I(G)$ has a linear resolution if and only if G^c is a chordal graph.*

Theorem 1.2.47. *[35, Theorem 3] Let G be a graph. Then $I(G)$ has a linear resolution if and only if $J(G)$ is CM.*

It also follows from [71] that if $I(G)$ has a linear resolution then so does $I(G)^q$ for all $q \gg 1$. It is, thus, of interest to characterize graphs whose (sufficiently large) powers have linear resolutions. It is known (see [88]) that if a power of $I(G)$ has a linear resolution then G^c has no induced 4-cycles. It is often difficult to get the exact value for the asymptotic linear function $\operatorname{reg} I^q$. Linear bounds are also of interest. In [14, Theorem 4.5] it was given a general lower bound for $\operatorname{reg} I^q$ that was inspired by a result of Katzman (see [78]), who proved the bound when $q = 1$, i.e. for the edge ideal itself. Unfortunately, there has not been any satisfactory general upper bound. For the special class of bipartite graphs, it was obtained the following upper bound for $\operatorname{reg} I^q$.

Theorem 1.2.48. *[75, Theorem 1.1] Let G be a bipartite graph and let $I = I(G)$ its edge ideal. Then for all $q \gg 1$, we have*

$$\operatorname{reg} I^q \leq 2q + \textit{co-chord}(G) - 1.$$

We recall that the *co-chordal number of G*, denoted by co-chord(G), is the least number of co-chordal subgraphs of G (subgraphs whose complements are chordal) whose union is G.

1.2.4 Inductive results on regularity of powers of edge ideals

In this section, we recall some of the most important inductive results that relate the regularity of (powers of) a squarefree monomial ideal corresponding to a hypergraph to that one of smaller ideals corresponding to subhypergraphs (for more details, see [19]).

Theorem 1.2.49. *Let G be a graph and let H be an induced subgraph of G. Then for any $s \geq 1$ and any $i, j \geq 0$, we have*

$$\beta_{i,j}(I(H)^q) \leq \beta_{i,j}(I(G)^q).$$

Corollary 1.2.50. *Let G be a graph and let H be an induced subgraph of G. Then, for all $q \geq 1$,*

$$\operatorname{reg} I(H)^q \leq \operatorname{reg} I(G)^q.$$

For a subset V of the vertices in a hypergraph H, let $H\colon V$ and $H+V$ denote the hypergraphs corresponding to the squarefree monomial ideals $I(H)\colon x^V$ and $I(H) + x^V$, respectively. We have the following inductive bounds (see [19]).

Theorem 1.2.51. *Let H be a simple hypergraph and let V be a collection of d vertices in H. Then*

$$\operatorname{reg}(H) \leq \max\{\operatorname{reg}(H\colon V) + d, \operatorname{reg}(H+V)\}. \tag{1.1}$$

Theorem 1.2.52. *Let H be a simple hypergraph and let E be an edge of cardinality d in H. Then*

$$\operatorname{reg}(H) \leq \max\{\ d,\ \operatorname{reg}(H \setminus E),\ \operatorname{reg}(H_E) + d - 1\ \}. \tag{1.2}$$

Remark 1.2.53. [62, Lemma 3.1, Theorems 3.4 and 3.5] In particular, if $G = (V, E)$ is a graph, for all $x \in V$ and $e \in E$ we have:

(i) $\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G \setminus x), \operatorname{reg} I(G \setminus N[x]) + 1\}$;

(ii) $\operatorname{reg} I(G) \leq \max\{2, \operatorname{reg} I(G \setminus e), \operatorname{reg} I(G_e) + 1\}$.

Kalai and Meshulam (see [77]) proved the following powerful result that allows us to use induction even when we do not necessarily split the edges of H into disjoint subsets. This result was later extended to arbitrary (not necessary squarefree) monomial ideals by Herzog (see [69]).

Theorem 1.2.54. *Let $I_1, \ldots, I_s$ be monomial ideals in R. Then*

$$\operatorname{reg}\left(R/\sum_{i=1}^{s} I_i\right) \leq \sum_{i=1}^{s} \operatorname{reg}(R/I_i).$$

In particular, for edge ideals of a hypergraph and subhypergraphs, we have the following inductive bound.

Corollary 1.2.55. *Let H and $H_1, \ldots, H_s$ be simple hypergraphs over the same vertex set X such that $\mathcal{E}(H) = \bigcup_{i=1}^{s} \mathcal{E}(H_i)$. Then*

$$\operatorname{reg}(R/I(H)) \leq \sum_{i=1}^{s} \operatorname{reg}(R/I(H_i)).$$

In one of his recent works (see [10]), Banerjee provides us an inductive method to work with regularity of powers of edge ideals. We recall the following definitions and theorems (see [10]) that will be crucial to our treatment in the last chapter of this thesis.

Theorem 1.2.56. *[10, Theorem 5.2] Let G be a graph and let s be a positive integer. Denote the set of minimal monomial generators of $I(G)^s$ by $\{m_1, \ldots, m_k\}$. Then*

$$\operatorname{reg} I(G)^{s+1} \leq \max\{\operatorname{reg} I(G)^s, \operatorname{reg}(I(G)^{s+1} : m_l) + 2s, 1 \leq l \leq k\}.$$

Definition 1.2.57. Let $G = (V, E)$ be a graph with edge ideal $I = I(G)$. Two vertices u and v in G are said to be *even-connected* with respect to an s-fold product $M = x^{e_1} \ldots x^{e_s}$, where $e_1, \ldots, e_s$ are edges in G, if there is a path $p_0, \ldots, p_{2l+1}$, for some $l \geq 1$, in G such that the following conditions hold:

1) $p_0 \equiv u$ and $p_{2l+1} \equiv v$;

2) for all $0 \leq j \leq l-1$, $\{p_{2j+1}, p_{2j+2}\} = e_i$ for some i;

3) for all i, $|\ \{j \mid \{p_{2j+1}, p_{2j+2}\} = e_i\}\ | \leq |\ \{t \mid e_t = e_i\}\ |$.

Definition 1.2.58. The edges $e_1 = v_{1,1}v_{1,2}, \ldots, e_q = v_{q,1}v_{q,2}$ are in an *even-connected position*, if for all $1 \leq i \leq q-1$, the vertex $x_{i,2}$ is connected to the vertex $x_{i+1,1}$ and there exist $u \in N(e_1)$ and $v \in N(e_q)$ such that u and v are even-connected with respect to $x_{e_1} \cdots x_{e_q}$.

Theorem 1.2.59. *[10, Theorems 6.1 and 6.7] Let $G = (V, E)$ be a graph with edge ideal $I = I(G)$, and let $s \geq 1$ be an integer. Let $M = x^{e_1} \dots x^{e_s}$ be a minimal generator of I^s. Then $(I^{s+1} \colon M)$ is minimally generated by monomials of degree 2, and uv (u and v may be the same) is a minimal generator of $(I^{s+1} \colon M)$ if and only if either $\{u, v\} \in E$ or u and v are even-connected with respect to M.*

Remark 1.2.60. [10, Lemma 6.11] Let $(I^{s+1} \colon M)^{\text{pol}}$ be the polarization of the ideal $(I^{s+1} \colon M)$ (see e.g. [70, Section 1.6]). From the previous theorem we can construct a graph G' whose edge ideal is given by $(I^{s+1} \colon M)^{\text{pol}}$. The new graph G' is given by:

(i) All the vertices and edges of G.

(ii) Any two vertices u, v, $u \neq v$ that are even-connected with respect to M are connected by an edge in G'.

(iii) For every vertex u which is even-connected to itself with respect to M, there is a new vertex u' which is connected to u by an edge and not connected to any other vertex (so uu' is a whisker).

Chapter 2

Special arrangements of lines: codimension two ACM varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

This chapter is based upon the joint project with G. Favacchio and E. Guardo.

In this chapter we investigate a special arrangements of lines in multiprojective spaces, i.e., we study codimension two arithmetically Cohen-Macaulay (ACM) varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, called *varieties of lines* in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We also show a connection between their ACM property and a combinatorial commutative algebra result.

2.1 Varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let $R := \mathbf{k}[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}]$ be the $\mathbb{N}^3-$graded ring with the trigrading induced on it by setting $\deg x_{i,j} = e_i$ for $i = 1, 2, 3$ (see Definition 1.1.1).

Using the notation fixed in Chapter 1, let $P := ((a_0, a_1), (b_0, b_1), (c_0, c_1))$ be a point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $I_P := (a_1x_{1,0} - a_0x_{1,1}, b_1x_{2,0} - b_0x_{2,1}, c_1x_{3,0} - c_0x_{3,1})$ be the defining ideal of P. Note that I_P is a height three prime ideal generated by homogeneous linear forms of different multidegrees.

Throughout this chapter, linear forms are denoted by capital letters. In particular, we use A_i to denote a linear form of degree $(1, 0, 0)$, B_j a linear form of degree $(0, 1, 0)$, and C_k a linear form of degree $(0, 0, 1)$. We denote by $\mathcal{L}(A_i)$, $\mathcal{L}(B_j)$ and $\mathcal{L}(C_k)$ the respective hyperplanes of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and we say that a hyperplane in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is of type e_i if it is defined by a form of degree e_i.

We recall the following definition [60, Definition 2.2].

Definition 2.1.1. Let $F, G \in R$ be two homogeneous linear forms of different degree. In $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the variety $\mathcal{L}$ defined by the ideal $(F, G) \subseteq R$ is called a *line* of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and we denote it by $\mathcal{L}(F, G)$.

We say that a line $\mathcal{L}(F,G)$ is of type e_i+e_j, with $i \neq j$, if $\{\deg F, \deg G\} = \{e_i, e_j\}$.

In particular, if $A \in R_{1,0,0}$, $B \in R_{0,1,0}$ and $C \in R_{0,0,1}$, then we denote by $\mathcal{L}(A,B)$ the variety in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the ideal $(A,B) \subseteq R$, and we call it a *line of type* $(1,1,0)$. Analogously, we call the variety $\mathcal{L}(A,C)$ a *line of type* $(1,0,1)$ and the variety $\mathcal{L}(B,C)$ a *line of type* $(0,1,1)$. We also refer to lines of type e_1+e_2, e_1+e_3 and e_2+e_3 by writing *lines having direction* e_3, e_2 and e_1, respectively.

Definition 2.1.2. We say that $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a *variety of lines* if it is given by a finite union of distinct lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.1.3. Given $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ a variety of lines, we denote by $\mathcal{H}_1(X) := \{\mathcal{L}(A_1), \ldots, \mathcal{L}(A_{d_1})\}$, $\mathcal{H}_2(X) := \{\mathcal{L}(B_1), \ldots, \mathcal{L}(B_{d_2})\}$ and $\mathcal{H}_3(X) := \{\mathcal{L}(C_1), \ldots, \mathcal{L}(C_{d_3})\}$ the hyperplanes of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ containing some lines of X. In particular:

$$X := \bigcup_{(i,j)\in U_3(X)} \mathcal{L}(A_i,B_j) \cup \bigcup_{(i,k)\in U_2(X)} \mathcal{L}(A_i,C_k) \cup \bigcup_{(j,k)\in U_1(X)} \mathcal{L}(B_j,C_k)$$

where $U_3(X) \subseteq [d_1] \times [d_2], U_2(X) \subseteq [d_1] \times [d_3]$ and $U_1(X) \subseteq [d_2] \times [d_3]$ are sets of ordered pairs of integers, with $[n] := \{1,2,\ldots,n\} \subset \mathbb{N}$.

For $i = 1,2,3$, we denote by X_i the set of lines of X having direction e_i and we call $U_i(X)$ the *index set of* X_i.

Thus, the ideal defining X is

$$I_X = \bigcap_{(i,j)\in U_3(X)} (A_i,B_j) \cap \bigcap_{(i,k)\in U_2(X)} (A_i,C_k) \cap \bigcap_{(j,k)\in U_1(X)} (B_j,C_k).$$

Example 2.1.4. Let consider the following variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} X =& \mathcal{L}(A_1,B_2) \cup \mathcal{L}(A_1,B_3) \cup \mathcal{L}(A_1,B_5) \cup \mathcal{L}(A_2,B_1) \cup \mathcal{L}(A_2,B_2) \cup \\ & \cup \mathcal{L}(A_1,C_1) \cup \mathcal{L}(A_1,C_4) \cup \mathcal{L}(A_2,C_3) \cup \mathcal{L}(B_1,C_2) \cup \mathcal{L}(B_2,C_3). \end{aligned}$$

The variety X consists of 10 lines: 5 lines of type $(1,1,0)$, 3 lines of type $(1,0,1)$ and 2 lines of type $(0,1,1)$. In particular, the set of lines of X having *direction* e_3 is

$$X_3 = \{\mathcal{L}(A_1,B_2), \mathcal{L}(A_1,B_3), \mathcal{L}(A_1,B_5), \mathcal{L}(A_2,B_1), \mathcal{L}(A_2,B_2)\}$$

and the *index set* of X_3 is

$$U_3 = \{(1,2),(1,3),(1,5),(2,1),(2,2)\} \subseteq [2] \times [5].$$

Analogously, $X_2 = \{\mathcal{L}(A_1,C_1), \mathcal{L}(A_1,C_4), \mathcal{L}(A_2,C_3)\}$ is the set of lines of X having *direction* e_2, with *index set* $U_2 = \{(1,1),(1,4),(2,3)\} \subseteq [2] \times [4]$ and $X_1 = \{\mathcal{L}(B_1,C_2), \mathcal{L}(B_2,C_3)\}$ is the set of lines of X having *direction* e_1, with *index set* $U_1 = \{(1,2),(2,3)\} \subseteq [5] \times [4]$.

In this thesis we are interested in a combinatorial characterization of ACM varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and their Hilbert function. In $\mathbb{P}^1 \times \mathbb{P}^1$, the key concept to describe combinatorially ACM sets of points was the notion of a *Ferrers diagram* (see Definitions 1.1.45, 1.1.46, 1.1.47 and see for instance [59] for more details).

We adapt Definition 1.1.46 to our context.

Construction 2.1.5. *Let X be a variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and consider the set X_3 of lines of X of type $(1,1,0)$ indexed by $U_3(X) \subseteq [d_1] \times [d_2]$. We represent X_3 as a $d_1 \times d_2$ grid, where the horizontal lines are labeled by the $\mathcal{L}(A_i)$'s for $i = 1, \ldots, d_1$ and the vertical lines by the $\mathcal{L}(B_j)$'s for $j = 1, \ldots, d_2$. By abuse of notation, we denote the horizontal lines by $\mathcal{L}(A_i)$ and the vertical lines by $\mathcal{L}(B_j)$. Then, a line $\mathcal{L}(A_i, B_j) \in X_3$ is drawn as the intersection point of $\mathcal{L}(A_i)$ and $\mathcal{L}(B_j)$ in the grid. Similarly, we can construct a $d_1 \times d_3$ grid representing X_2 and a $d_2 \times d_3$ grid representing X_1.*

Definition 2.1.6. Let X be a variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $h \in \{1, 2, 3\}$. We say that X *resembles a Ferrers diagram with respect to the direction e_h* if the grid representing the lines of X_h, constructed as above, resembles a Ferrers diagram.

Definition 2.1.7. A finite subset $U = \{(u_i, u_j)\} \subseteq \mathbb{N}^2$ *resembles a Ferrers diagram* if it satisfies the following property:

$$(u_i, u_j) \in U \Rightarrow (u_h, u_k) \in U \quad \forall\, 1 \leq h \leq i \;,\; 1 \leq k \leq j.$$

Remark 2.1.8. Note that Definition 2.1.6 is equivalent to saying that the index set $U_h(X) \subset \mathbb{N}^2$ resembles a Ferrers diagram as Definition 2.1.7.

Remark 2.1.9. Construction 2.1.5 makes clear the connection between X_h ($h \in \{1, 2, 3\}$) and a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. X_h is a cone of a set of distinct points on a hyperplane of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. So, we can look at it as a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with associated grid as described in the construction.

Example 2.1.10. Let X be the following variety of 15 lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} X =\{&\mathcal{L}(A_1, B_2), \mathcal{L}(A_1, B_4), \mathcal{L}(A_1, B_5), \mathcal{L}(A_2, B_2), \mathcal{L}(A_2, B_3), \\ &\mathcal{L}(A_2, B_4), \mathcal{L}(A_2, B_5), \mathcal{L}(A_3, B_1), \mathcal{L}(A_3, B_2), \mathcal{L}(A_3, B_3), \\ &\mathcal{L}(A_3, B_4), \mathcal{L}(A_3, B_5), \mathcal{L}(A_4, B_4), \mathcal{L}(B_1, C_1), \mathcal{L}(B_2, C_2)\}. \end{aligned}$$

Then,

$$\begin{aligned} X_3 =\{&\mathcal{L}(A_1, B_2), \mathcal{L}(A_1, B_4), \mathcal{L}(A_1, B_5), \mathcal{L}(A_2, B_2), \mathcal{L}(A_2, B_3), \\ &\mathcal{L}(A_2, B_4), \mathcal{L}(A_2, B_5), \mathcal{L}(A_3, B_1), \mathcal{L}(A_3, B_2), \mathcal{L}(A_3, B_3), \\ &\mathcal{L}(A_3, B_4), \mathcal{L}(A_3, B_5), \mathcal{L}(A_4, B_4)\}. \end{aligned}$$

Using Construction 2.1.5, we can represent the set X_3 of lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the intersection points in a 4×5 grid as in Figure 2.1.

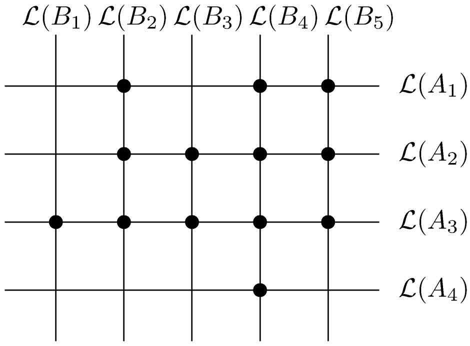


Figure 2.1: The set of lines X_3.

After relabelling, we see that X_3 resembles a Ferrers diagram of type $(5, 4, 3, 1)$. Then, using Lemma 1.1.48, X_3 is ACM (see Figure 2.2).

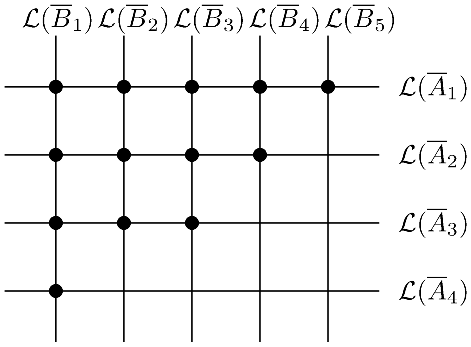


Figure 2.2: The set of lines X_3 resembling a Ferrers diagram.

Example 2.1.11. Let X be the variety of lines as in Example 2.1.10. We have $X_1 = \{\mathcal{L}(B_1, C_1), \mathcal{L}(B_2, C_2)\}$ and the 2×2 grid representing X_1 does not resemble any Ferrers diagram (see Figure 2.3). Thus X does not resemble a Ferrers diagram with respect to the direction e_1. Hence, from Lemma 1.1.48, X_1 is not ACM.

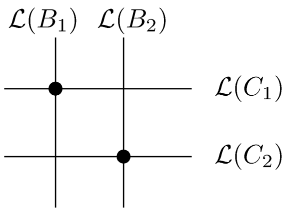


Figure 2.3: The set of lines X_1.

Since Ferrers diagrams play a crucial role in the characterization of the ACM property for a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (see for instance [60]), it is natural for us to investigate the same property for a variety of lines $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (since X has also codimension 2). In the next section, we will show that the ACM property of X depends on the X_i's (see Corollary 2.2.6), but the ACM-ness of the X_i's is not sufficient to ensure that X is also ACM (see Remark 2.2.7).

2.2 A combinatorial characterization of ACM varieties of lines

In this section, we study the ACM property for varieties of lines from a combinatorial point of view. We refer to [70] for all the introductory material on monomial ideals.

The next lemma can be recovered from [98, Proposition 3.2].

Lemma 2.2.1. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be an ACM variety of lines. Then, there exist three forms A, B and C of degree $(1,0,0), (0,1,0)$ and $(0,0,1)$, respectively, such that $(\bar{A}, \bar{B}, \bar{C})$ is a regular sequence in R/I_X.*

Proof. Let $A \in R_{1,0,0}$ be such that $\mathcal{L}(A) \notin \mathcal{H}_1(X)$. We claim that $\bar{A}$ is a nonzero divisor of R/I_X. Indeed, take $F \in R$ a homogeneous form such that $AF \in I_X$. Then $AF \in I_{\mathcal{L}}$, for any line $\mathcal{L} \in X$. Since $I_{\mathcal{L}}$ is a prime ideal and $A \notin I_{\mathcal{L}}$, then we get $F \in I_{\mathcal{L}}$, for any $\mathcal{L} \in X$.

Now we prove the existence of the linear form B. Since X is ACM, then $J := I_X + (A)$ is CM. Moreover, J is homogeneous and its height is 3. Take the primary decomposition of J, say $J = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, and let $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ for $i = 1, \ldots, t$. The set of the nonzero divisors of R/J is then $\bigcup_i \bar{\mathfrak{p}}_i$. In order to prove that there exists an element $B \in R_{0,1,0}$ nonzero divisor of R/J, it is enough to show that $(\bigcup_i \mathfrak{p}_i)_{0,1,0} \subsetneq R_{0,1,0}$. Since $R_{0,1,0}$ is a K-vector space over an infinite field, it is not a union of a finite number of its proper subspaces, and so it is enough to show that $(\mathfrak{p}_i)_{0,1,0} \subsetneq R_{0,1,0}$ for each $i = 1, \ldots, t$.

Let $i \in \{1, \ldots, t\}$, then we have $I_X \subseteq J \subseteq \mathfrak{p}_i$. Therefore, there exists $\mathcal{L} \in X$ such that $I_X \subseteq I_{\mathcal{L}} \subseteq \mathfrak{p}_i$. This implies $\mathfrak{p}_i = I_{\mathcal{L}} + (A)$. Since $I_{\mathcal{L}} \neq R_{0,1,0}$ we are done.

Analogously we prove the existence of a form $C \in R_{0,0,1}$. □

We set the notation for this section. Let X be a variety of lines and I_X its defining ideal

$$I_X = \bigcap_{(i,j)\in U_3(X)} (A_i, B_j) \cap \bigcap_{(i,k)\in U_2(X)} (A_i, C_k) \cap \bigcap_{(j,k)\in U_1(X)} (B_j, C_k) \subseteq R.$$

We construct a new polynomial ring in $d_1 + d_2 + d_3$ variables where each variable corresponds to a hyperplane containing some lines of X. We denote by $S := \mathbf{k}[a_1, \dots, a_{d_1}, b_1, \dots, b_{d_2}, c_1, \dots, c_{d_3}]$ the polynomial ring in $d_1+d_2+d_3$ variables and $\deg a_i = (1,0,0)$, $\deg b_j = (0,1,0)$, $\deg c_k = (0,0,1)$. We set

$$J_X = \bigcap_{(i,j)\in U_3(X)} (a_i, b_j) \cap \bigcap_{(i,k)\in U_2(X)} (a_i, c_k) \cap \bigcap_{(j,k)\in U_1(X)} (b_j, c_k) \subseteq S.$$

Then J_X is a height 2 monomial ideal of S, and its associated primes correspond to the components of X.

The next lemma is crucial since, as its consequence, we can connect homological invariants between ACM varieties of lines and some height 2 monomial ideals. Similar arguments were also used in [45, Theorem 3.2].

Lemma 2.2.2. *Let X be a variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X is ACM if and only if $J_X \subseteq S$ is CM.*

Proof. Set $T := S[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}]$. Consider J_X as an ideal, say $\overline{J}_X$, in the ring T. Since J_X is a height 2 monomial ideal in S, then $\overline{J}_X$, being a cone, continues to be a height 2 monomial ideal. Moreover, $\overline{J}_X$ has the same primary decomposition as J_X. Consider the linear forms $a_i - A_i$, $b_j - B_j$, $c_k - C_k$ and let L be the ideal generated by all these linear forms.

Assume J_X is CM. Thus, in the quotient $T/(\overline{J}_X, L)$ we can view the addition of each linear form in L as a proper hyperplane section. We have that R/I_X and $T/(\overline{J}_X, L)$ both have height 2 and $R/I_X \cong T/(\overline{J}_X, L)$. Then, since J_X is CM, we get that X is ACM.

On the other hand, if X is ACM, then, applying Lemma 2.2.1, there exists a sequence of linear forms $(A, B, C) \subseteq R$ that is regular in the quotient R/I_X. Let $\mathfrak{q} := (A, B, C) \subseteq R$ be the ideal generated by these three linear forms. Consider the ideal $(I_X + \mathfrak{q})/\mathfrak{q} \subseteq R/\mathfrak{q}$, that can be viewed as a codimension 2 monomial ideal in a polynomial ring in three variables. Since a Hilbert-Burch matrix of I_X has the same "structure"as the Hilbert-Burch matrix of a monomial ideal, i.e., it is a matrix with only two non zero entries in each column (see for instance [47, Lemma 3.21] or [87, Theorem 1.5]), then I_X is generated by some products among the linear forms defining the lines of X. Since the addition of each linear form in L can be seen as a proper hyperplane section, we also have $R/I_X \cong T/(\overline{J}_X, L)$. Then J_X is CM.

□

Corollary 2.2.3. *Let X be an ACM variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then I_X is generated by products of linear forms.*

As a consequence of Lemma 2.2.2, it is interesting to further investigate the structure of the monomial ideal J_X associated to X. Now we refer to [70, 99] and to Chapter 1 of this thesis for all preliminaries and for further results on graphs.

Remark 2.2.4. Let X be a variety of lines. Let $G_X = (V_X, E_X)$ be the graph with vertex set

$$V_X := \{a_1, \dots, a_{d_1}, b_1, \dots, b_{d_2}, c_1, \dots, c_{d_3}\}$$

and edge set

$$
\begin{aligned}
E_X :=& \Big\{\{a_i, b_j\} \subseteq V_X \mid \mathcal{L}(A_i, B_j) \in X_3\Big\} \cup \\
& \Big\{\{a_i, c_k\} \subseteq V_X \mid \mathcal{L}(A_i, C_k) \in X_2\Big\} \cup \\
& \Big\{\{b_j, c_k\} \subseteq V_X \mid \mathcal{L}(B_j, C_k) \in X_1\Big\}.
\end{aligned}
$$

Then, we note that the monomial ideal J_X is the cover ideal of the graph G_X:

$$J_X = J(G_X) \subseteq S,$$

that is, the Stanley-Reisner ideal of the simplicial complex (see [70, Lemma 1.5.4])

$$\Delta_X := \langle V_X \setminus e \mid e \in E_X \rangle.$$

A useful application of Remark 2.2.4 is the following lemma.

Lemma 2.2.5. *Let X be an ACM variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathcal{H} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a hyperplane containing some lines of X. Then the variety of lines $Y = \{\mathcal{L} \in X \mid \mathcal{L} \not\subset \mathcal{H}\}$ is ACM.*

Proof. Let H be the linear form defining $\mathcal{H}$. Denoted by z the variable of S corresponding to H (the linear form H is one of the forms A_i, B_j, C_k and z is the corresponding variable among a_i, b_j, c_k). We have

i) $J_X : z = \bigcap\limits_{\substack{p \,\in\, ass(J_X) \\ z \,\notin\, p}} \mathfrak{p}$. Both are monomial ideals, so the equality easily follows by checking the inclusions for monomials.

ii) $J_X : z$ is the Stanley-Reisner ideal of the simplicial complex $link_{\Delta_X}(z)$ (see [70, Sections 1.5.2 and 8.1.1]). Indeed, the Stanley-Reisner ideal of the *link* of z in Δ_X is generated by monomials corresponding to the elements $F \subseteq V_X$ such that $\{z\} \cup F \notin \Delta_X$. All these monomials are in $J_X : z = I_{\Delta_X} : z$ and vice versa.

Then, in order to prove the statement, it is enough to show that $J_X : z$ is CM. From Lemma 2.2.2, we have that J_X is CM, so the statement follows by [70, Corollary 8.1.8]. $\square$

Corollary 2.2.6. *If X is an ACM variety of lines, then X resembles a Ferrers diagram with respect to the direction e_h, for each $h = 1, 2, 3$.*

Proof. We show that $U_1(X)$ resembles a Ferrers diagram. Analogously, one can show the same for $U_2(X)$ and $U_3(X)$. Let us consider the variety of lines X_1 consisting of the lines of X of type $(0,1,1)$. Since $I_{X_1} = I_{X \setminus \{\mathcal{L}(A_1), \ldots, \mathcal{L}(A_{d_1})\}}$, X_1 preserves the ACM property by Lemma 2.2.5. Moreover, $X_1 = \bigcup\limits_{(j,k) \in U_1(X)} \mathcal{L}(B_j, C_k)$, i.e., it is a cone of an ACM set of distinct points on a hyperplane of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, see Remark 2.1.9. A well known characterization, see for instance [59, Theorem 4.11], shows that this set of points resembles a Ferrers diagram. Using Remark 2.1.8, $U_1(X)$ resembles a Ferrers diagram. Then, the statement follows from Lemma 1.1.48.

□

Remark 2.2.7. From previous corollary, if there exists $i \in \{1,2,3\}$ such that X_i is not ACM, then X is not ACM. The following example shows that even if all X_i's are ACM, X may not be ACM.

Example 2.2.8. Let us consider the following variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 2.4):

$$X = \{\mathcal{L}(A_1, B_1), \mathcal{L}(A_2, C_1), \mathcal{L}(B_3, C_3)\}.$$

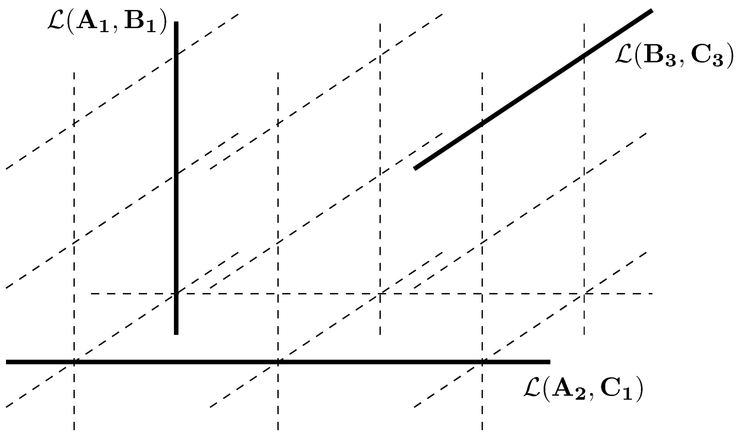


Figure 2.4: The variety of lines X (in bold).

It is clear that the sets $X_1 = \{\mathcal{L}(B_3, C_3)\}$, $X_2 = \{\mathcal{L}(A_2, C_1)\}$ and $X_3 = \{\mathcal{L}(A_1, B_1)\}$ resemble a Ferrers diagram, so each of them is ACM. But, in this case, X is not ACM. This follows for instance from Lemma 2.2.2 and from [70, Lemma 9.1.12].

The next definition introduces a property for varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in analogy to the known $(\star)$-property defined for sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (see Definition 1.1.49).

Definition 2.2.9. Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a variety of lines. We say that X has the *($\star$)-property* (or explicitly, *star property*) if given any two lines L_1, $L_2 \in X$, there exists $L_3 \in X$ such that L_1, L_3 and L_2, L_3 are coplanar.

We slight generalize this property for varieties of lines.

Definition 2.2.10. Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a variety of lines. Let $n \geq 4$, $n \in \mathbb{N}$, we say that X has the *n-hyperplanes ($\star$) property* (for short, $Hyp_n(\star)$-property) if given n hyperplanes $H_1, H_2, \ldots, H_n$ such that $\mathcal{L}(H_i, H_j) \in X$ for any $j \neq i-1, i, i+1$ then $\mathcal{L}(H_u, H_{u+1}) \in X$ for some $u \in \{1, 2, \ldots, n\}$, where $H_0 = H_n$ and $H_{n+1} = H_1$.

Remark 2.2.11. Note that if $n > 6$, then X has the $Hyp_n(\star)$-property. Indeed, among $n > 6$ hyperplanes there are at least three of the same type and so the condition $\mathcal{L}(H_i, H_j) \in X$ for any $j \neq i-1, i, i+1$ (where $H_0 = H_n$ and $H_{n+1} = H_1$) fails to be true.

Remark 2.2.12. Note that the $Hyp_4(\star)$-property is equivalent to ($\star$)-property as Definition 2.2.9.

Example 2.2.13. Let us consider the following variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 2.5):

$$\begin{aligned} X = &\mathcal{L}(A_1, B_1) \cup \mathcal{L}(A_1, B_2) \cup \mathcal{L}(A_1, B_3) \cup \mathcal{L}(A_2, B_1) \cup \\ &\mathcal{L}(A_2, B_2) \cup \mathcal{L}(A_1, C_1) \cup \mathcal{L}(A_1, C_2) \cup \mathcal{L}(A_2, C_1) \cup \\ &\mathcal{L}(B_1, C_1) \cup \mathcal{L}(B_1, C_2) \cup \mathcal{L}(B_2, C_1) \cup \mathcal{L}(B_3, C_1). \end{aligned}$$

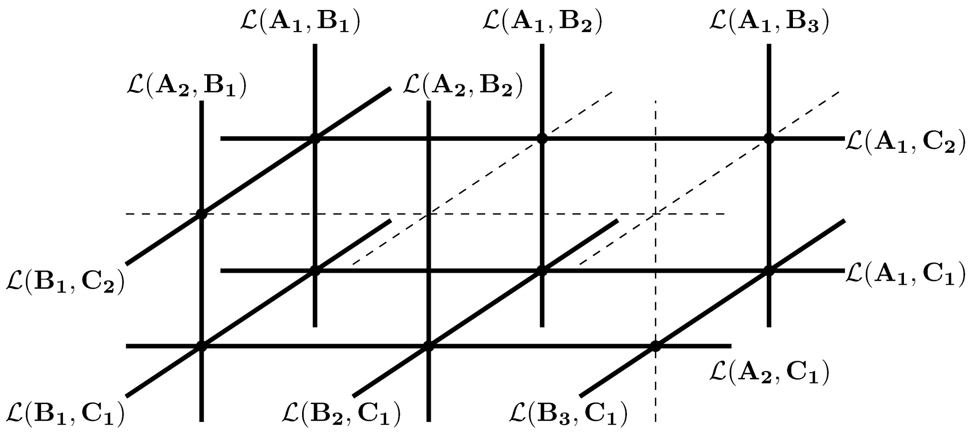


Figure 2.5: The variety of lines X (in bold).

Then X has the $Hyp_4(\star)$-property. Indeed, if we take the 4 hyperplanes $\mathcal{L}(A_1)$, $\mathcal{L}(A_2)$, $\mathcal{L}(B_1), \mathcal{L}(B_2)$, we have that $\mathcal{L}(A_1, B_1), \mathcal{L}(A_2, B_2) \in X$ and also $\mathcal{L}(A_1, B_2) \in X$; if we take the 4 hyperplanes $\mathcal{L}(A_1), \mathcal{L}(A_2)$, $\mathcal{L}(B_1), \mathcal{L}(C_1)$, we have that $\mathcal{L}(A_1, B_1)$, $\mathcal{L}(A_2, C_1) \in X$ and also $\mathcal{L}(B_1, C_1) \in X$; and so on, if we take any two lines in X, there exists a third line in X that is coplanar with the other two.

Example 2.2.14. Let us consider the following variety of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 2.6):

$$\begin{aligned} X = &\mathcal{L}(A_1, B_1) \cup \mathcal{L}(A_1, B_2) \cup \mathcal{L}(A_1, B_3) \cup \mathcal{L}(A_2, B_2) \cup \mathcal{L}(A_1, C_1) \cup \\ &\mathcal{L}(A_1, C_2) \cup \mathcal{L}(A_2, C_1) \cup \mathcal{L}(B_1, C_1) \cup \mathcal{L}(B_3, C_1). \end{aligned}$$

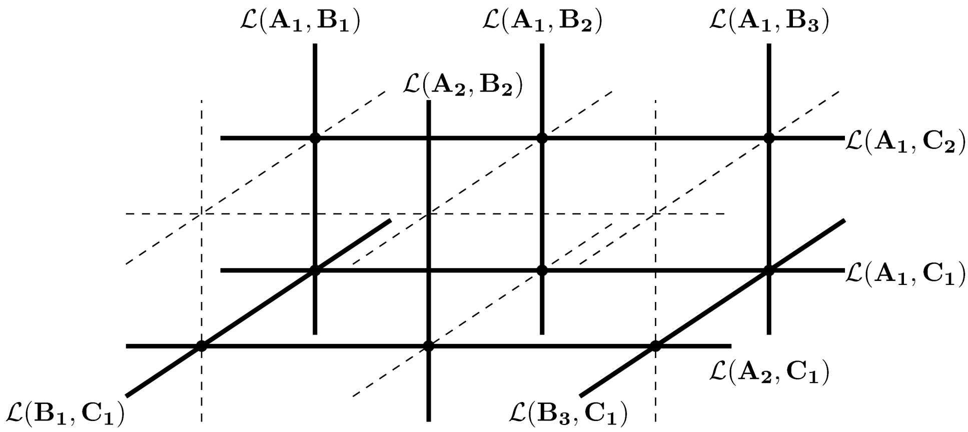


Figure 2.6: The variety of lines X (in bold).

Then X has the $Hyp_5(\star)$-property. Indeed, if we take the 5 hyperplanes $\mathcal{L}(A_1)$, $\mathcal{L}(A_2)$, $\mathcal{L}(B_1), \mathcal{L}(B_2)$, $\mathcal{L}(C_1)$, we have that the 5 lines $\mathcal{L}(A_1, B_1)$, $\mathcal{L}(A_1, B_2)$, $\mathcal{L}(A_2, B_2)$, $\mathcal{L}(A_2, C_1)$, $\mathcal{L}(B_1, C_1) \in X$ and also $\mathcal{L}(A_1, C_1) \in X$; if we take the 5 hyperplanes $\mathcal{L}(A_1), \mathcal{L}(A_2)$, $\mathcal{L}(B_3), \mathcal{L}(B_2), \mathcal{L}(C_1)$, we have that $\mathcal{L}(A_1, B_3)$, $\mathcal{L}(A_1, B_2)$, $\mathcal{L}(A_2, B_2)$, $\mathcal{L}(A_2, C_1)$, $\mathcal{L}(B_3, C_1) \in X$ and also $\mathcal{L}(A_1, C_1) \in X$; and so on, if we take any 5 hyperplanes $H_1, \ldots, H_5$ among $\mathcal{L}(A_1), \mathcal{L}(A_2), \mathcal{L}(B_1), \mathcal{L}(B_2)$, $\mathcal{L}(B_3), \mathcal{L}(C_1), \mathcal{L}(C_2)$ such that $\mathcal{L}(H_i, H_j) \in X$ for any $j \neq i-1, i, i+1$, then there exists $u \in \{1, \ldots, 5\}$ such that $\mathcal{L}(H_u, H_{u+1}) \in X$, where $H_0 = H_5$ and $H_6 = H_1$.

Note that if we take $\mathcal{L}(B_1), \mathcal{L}(B_2), \mathcal{L}(B_3)$ among the 5 hyperplanes we choose, the condition $\mathcal{L}(H_i, H_j) \in X$ for any $j \neq i-1, i, i+1$ fails to be true and then there is nothing to verify.

The following theorem is the main result of this section.

Theorem 2.2.15. *Let X be a variety of lines. Then X is ACM if and only if X has the $Hyp_n(\star)$-property for $n = 4, 5, 6$.*

Proof. Let I_X be the ideal defining the variety of lines $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. From Lemma 2.2.2, X is ACM if and only if $J_X \subseteq S$ is CM. From Remark 2.2.4, the ideal J_X is the cover ideal of the graph G_X, i.e., $J_X = J(G_X)$. From Theorem 1.2.47, the face ideal $I(G_X)$ has a linear resolution and then, using Theorem 1.2.46, G_X^c is a chordal graph, that is, X has the $Hyp_n(\star)$-property for any n. Remark 2.2.11 completes the proof. $\square$

2.3 A numerical characterization of the ACM property

Since we are interested in the study of the ACM property for varieties of lines X, from now on we assume that $U_h(X)$ resembles a Ferrers diagram for each $h = 1, 2, 3$. In order to give an alternative characterization of the ACM property we introduce the following notation.

Definition 2.3.1. Let $P = P_{ijk} = \mathcal{L}(A_i) \cap \mathcal{L}(B_j) \cap \mathcal{L}(C_k)$ be a point of a variety of lines X. We call the *multiplicity of* P the number of lines of X passing through the point P and we denote it by μ_{ijk}.

Remark 2.3.2. Since at most three lines of X (one of each type) pass through the point P, $\mu_{ijk} \leq 3$.

Definition 2.3.3. Given a variety of lines X, we define a 3-dimensional matrix $M_X := (\mu_{ijk}) \in \mathbb{N}^{d_1 \times d_2 \times d_3}$ whose (i, j, k)-entry is the multiplicity of $P_{i,j,k}$. We call it the *matrix of the multiplicities* of X.

We also define

Definition 2.3.4. $M_X^{(3)} := (\mu_{ij0}) \in \mathbb{N}^{d_1 \times d_2}$, where

$$\mu_{ij0} := \begin{cases} 1 & \text{if } (i,j) \in U_3(X), \text{ i.e., } \mathcal{L}(A_i, B_j) \in X \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, $M_X^{(2)} := (\mu_{i0k}) \in \mathbb{N}^{d_1 \times d_3}$, where $\mu_{i0k} := \begin{cases} 1 & \text{if } \mathcal{L}(A_i, C_k) \in X \\ 0 & \text{otherwise} \end{cases}$

and $M_X^{(1)} := (\mu_{0jk}) \in \mathbb{N}^{d_2 \times d_3}$, where $\mu_{0jk} := \begin{cases} 1 & \text{if } \mathcal{L}(B_j, C_k) \in X \\ 0 & \text{otherwise} \end{cases}$.

Example 2.3.5. Let us consider $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as Figure 2.7:

$$\begin{aligned} X =& \mathcal{L}(A_1, B_1) \cup \mathcal{L}(A_1, B_2) \cup \mathcal{L}(A_2, B_2) \cup \mathcal{L}(A_1, C_1) \cup \mathcal{L}(A_2, C_1) \\ & \mathcal{L}(A_2, C_2) \cup \mathcal{L}(B_1, C_1) \cup \mathcal{L}(B_1, C_2) \cup \mathcal{L}(B_2, C_2). \end{aligned}$$

We have

$$\mu_{111} = \mu_{222} = 3 \quad , \quad \mu_{121} = \mu_{221} = \mu_{211} = \mu_{112} = \mu_{122} = \mu_{212} = 2$$

and

$$M_X^{(3)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_X^{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_X^{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

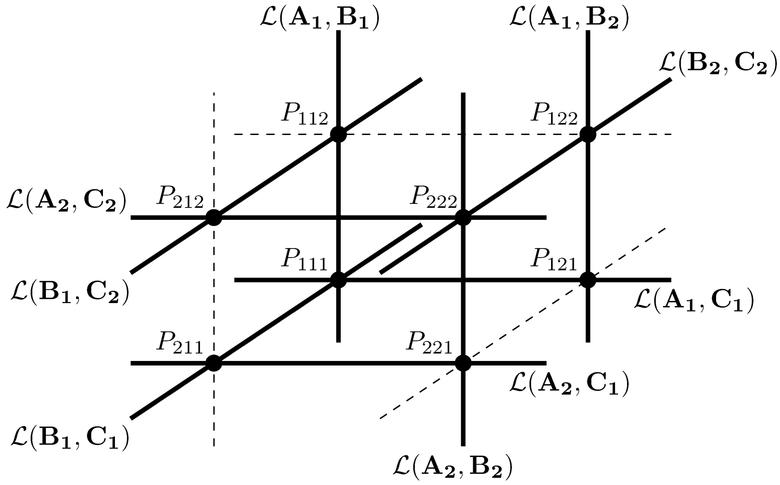


Figure 2.7: The variety of lines X (in bold).

Now we provide a criterion to establish if X is ACM or not just looking at the matrices of the multiplicities M_X, $M_X^{(1)}$, $M_X^{(2)}$ and $M_X^{(3)}$.

Proposition 2.3.6. *Let X be a variety of lines. Then X has the $Hyp_6(\star)$-property if and only if for all $a_1, a_2 \in [d_1], b_1, b_2 \in [d_2], c_1, c_2 \in [d_3]$*

$$\textit{either } \begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_2c_1} \\ \mu_{a_2b_1c_1} & \mu_{a_2b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \textit{ or } \begin{pmatrix} \mu_{a_1b_1c_2} & \mu_{a_1b_2c_2} \\ \mu_{a_2b_1c_2} & \mu_{a_2b_2c_2} \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}.$$

Proof. If X does not have the $Hyp_6(\star)$-property, then there exist six planes, say $\mathcal{L}(A_1), \mathcal{L}(A_2), \mathcal{L}(B_1), \mathcal{L}(B_2), \mathcal{L}(C_1), \mathcal{L}(C_2)$, such that the lines $\mathcal{L}(A_1, B_1)$, $\mathcal{L}(A_1, B_2)$, $\mathcal{L}(A_1, C_1)$, $\mathcal{L}(A_2, B_2)$, $\mathcal{L}(A_2, C_1)$, $\mathcal{L}(A_2, C_2)$, $\mathcal{L}(B_1, C_1)$, $\mathcal{L}(B_1, C_2)$, $\mathcal{L}(B_2, C_2)$ belong to X and $\mathcal{L}(A_2, B_1)$, $\mathcal{L}(B_2, C_1)$, $\mathcal{L}(A_1, C_2) \notin X$. Then we have that

$$\begin{pmatrix} \mu_{111} & \mu_{121} \\ \mu_{211} & \mu_{221} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_{112} & \mu_{122} \\ \mu_{212} & \mu_{222} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}.$$

On the other hand if

$$\begin{pmatrix} \mu_{111} & \mu_{121} \\ \mu_{211} & \mu_{221} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_{112} & \mu_{122} \\ \mu_{212} & \mu_{222} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix},$$

then it is easy to check that X does not have the $Hyp_6(\star)$-property since $\mathcal{L}(A_1, B_1), \mathcal{L}(A_1, B_2), \mathcal{L}(A_1, C_1), \mathcal{L}(A_2, B_2), \mathcal{L}(A_2, C_1), \mathcal{L}(A_2, C_2), \mathcal{L}(B_1, C_1)$, $\mathcal{L}(B_1, C_2)$, $\mathcal{L}(B_2, C_2) \in X$ and $\mathcal{L}(A_2, B_1)$, $\mathcal{L}(B_2, C_1)$, $\mathcal{L}(A_1, C_2) \notin X$. $\square$

Proposition 2.3.7. *Let X be a variety of lines. Then X has the $Hyp_5(\star)$-property if and only if for all $a_1, a_2 \in [d_1], b_1, b_2 \in [d_2], c_1, c_2 \in [d_3]$ the following three conditions hold:*

1) *either* $\begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_2c_1} \\ \mu_{a_2b_1c_1} & \mu_{a_2b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ *or* $\begin{pmatrix} \mu_{a_1b_10} & \mu_{a_1b_20} \\ \mu_{a_2b_10} & \mu_{a_2b_20} \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

2) *either* $\begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_1c_2} \\ \mu_{a_2b_1c_1} & \mu_{a_2b_1c_2} \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ *or* $\begin{pmatrix} \mu_{a_10c_1} & \mu_{a_10c_2} \\ \mu_{a_20c_1} & \mu_{a_20c_2} \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

3) *either* $\begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_1c_2} \\ \mu_{a_1b_2c_1} & \mu_{a_1b_2c_2} \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ *or* $\begin{pmatrix} \mu_{0b_1c_1} & \mu_{0b_1c_2} \\ \mu_{0b_2c_1} & \mu_{0b_2c_2} \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof. If X does not have the $Hyp_5(\star)$-property, we say, without loss of generality, that exist five planes $\mathcal{L}(A_1), \mathcal{L}(A_2), \mathcal{L}(B_1), \mathcal{L}(B_2), \mathcal{L}(C_1)$ such that, among all, only the lines $\mathcal{L}(A_2, B_1)$, $\mathcal{L}(A_1, C_1)$, $\mathcal{L}(B_2, C_1) \notin X$. Then we have

$$\begin{pmatrix} \mu_{111} & \mu_{121} \\ \mu_{211} & \mu_{221} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_{110} & \mu_{120} \\ \mu_{210} & \mu_{220} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, assume, for instance, we have the following equalities

$$\begin{pmatrix} \mu_{111} & \mu_{121} \\ \mu_{211} & \mu_{221} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_{110} & \mu_{120} \\ \mu_{210} & \mu_{220} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From the previous equalities, we get $\mathcal{L}(A_1, C_1) \notin X$ thus, since $\mu_{111} = 2$, we have $\mathcal{L}(A_1, B_1), \mathcal{L}(B_1, C_1) \in X$. Analogously $\mathcal{L}(A_1, B_2) \in X$ and so, since $\mu_{121} = 1$, we have $\mathcal{L}(B_2, C_1) \notin X$. Moreover $\mathcal{L}(A_2, B_2) \in X$ and so, since $\mu_{221} = 2$, we have $\mathcal{L}(A_2, C_1) \in X$. Finally $\mathcal{L}(A_2, B_1) \notin X$ since $\mu_{210} = 0$. So X does not have the $Hyp_5(\star)$-property. □

Proposition 2.3.8. *Let X be a variety of lines. Then X has the $Hyp_4(\star)$-property if and only if for all $a_1, a_2 \in [d_1], b_1, b_2 \in [d_2], c_1, c_2 \in [d_3]$ the following three conditions hold:*

1) *either* $\begin{pmatrix} \mu_{a_1b_1c_1} \\ \mu_{a_2b_1c_1} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} \mu_{a_1b_10} \\ \mu_{a_2b_10} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2) *either* $\begin{pmatrix} \mu_{a_1b_1c_1} \\ \mu_{a_1b_1c_2} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} \mu_{a_10c_1} \\ \mu_{a_10c_2} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3) *either* $\begin{pmatrix} \mu_{a_1b_1c_1} \\ \mu_{a_1b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} \mu_{0b_1c_1} \\ \mu_{0b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Proof. Suppose that X does not have the $Hyp_4(\star)$-property. Since we are assuming there do not exist four planes $\mathcal{L}(A_1), \mathcal{L}(A_2), \mathcal{L}(B_1), \mathcal{L}(B_2)$ such that $\mathcal{L}(A_1, B_1)$, $\mathcal{L}(A_2, B_2) \in X$ and $\mathcal{L}(A_1, B_2)$ or $\mathcal{L}(A_2, B_1) \notin X$, then, X fails the $Hyp_4(\star)$-property if, without loss of generality, there exist four planes $\mathcal{L}(A_1), \mathcal{L}(A_2), \mathcal{L}(B_1)$, $\mathcal{L}(C_1)$ such that, among all, only the lines $\mathcal{L}(A_2, B_1)$, $\mathcal{L}(B_1, C_1), \mathcal{L}(A_1, C_1) \notin X$. Then we have

$$\begin{pmatrix} \mu_{111} \\ \mu_{211} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_{110} \\ \mu_{210} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the other hand, assume, for instance, we have the following equalities

$$\begin{pmatrix} \mu_{111} \\ \mu_{211} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_{110} \\ \mu_{210} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From the previous equalities, we get $\mathcal{L}(A_1, B_1) \in X$ thus, since $\mu_{111} = 1$, we have $\mathcal{L}(A_1, C_1), \mathcal{L}(B_1, C_1) \notin X$. Analogously $\mathcal{L}(A_2, B_1) \notin X$ and so, since $\mu_{211} = 1$, we have $\mathcal{L}(A_2, C_1) \in X$. So X does not have the $Hyp_4(\star)$-property. □

Example 2.3.9. Let X be as in Example 2.3.5 (see Figure 2.7). We observe that

$$\begin{pmatrix} \mu_{111} & \mu_{121} \\ \mu_{211} & \mu_{221} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \text{and} \begin{pmatrix} \mu_{112} & \mu_{122} \\ \mu_{212} & \mu_{222} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

and then, by Proposition 2.3.6, we have that X does not have the $Hyp_6(\star)$-property and so, by Theorem 2.2.15, X is not ACM.

Example 2.3.10. Let us consider the variety $W = X \cup \mathcal{L}(A_2, B_1)$, where X is as Example 2.3.5.

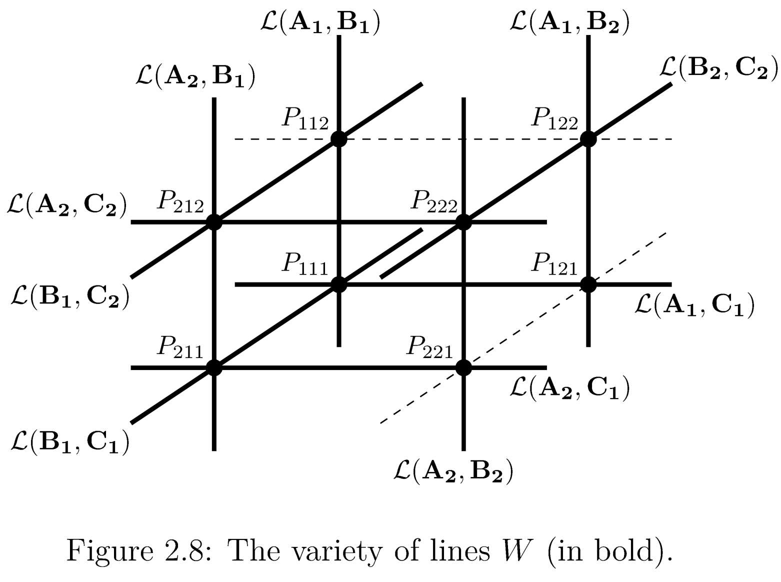


Figure 2.8: The variety of lines W (in bold).

We have $\mu_{111} = \mu_{222} = \mu_{211} = \mu_{212} = 3,\ \mu_{121} = \mu_{221} = \mu_{112} = \mu_{122} = 2.$

And for all $a_1, a_2 \in [2],\ b_1, b_2 \in [2],\ c_1, c_2 \in [2]$, we have:

$$\begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_2c_1} \\ \mu_{a_2b_1c_1} & \mu_{a_2b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_2c_1} \\ \mu_{a_2b_1c_1} & \mu_{a_2b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_1c_2} \\ \mu_{a_2b_1c_1} & \mu_{a_2b_1c_2} \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} \mu_{a_1b_1c_1} & \mu_{a_1b_1c_2} \\ \mu_{a_1b_2c_1} & \mu_{a_1b_2c_2} \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} \mu_{a_1b_1c_1} \\ \mu_{a_2b_1c_1} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \mu_{a_1b_1c_1} \\ \mu_{a_1b_1c_2} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \mu_{a_1b_1c_1} \\ \mu_{a_1b_2c_1} \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So, by Propositions 2.3.6, 2.3.7 and 2.3.8, the variety of lines W has the $Hyp_n(\star)$-property for $n = 4, 5, 6$ and then, by Theorem 2.2.15, W is ACM.

2.4 The Hilbert function of ACM codimension two varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

In this section we study the Hilbert function of varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We start from the following specific case.

Definition 2.4.1. If X is a variety of lines such that the index sets $U_1(X)$, $U_2(X)$ and $U_3(X)$ are Ferrers diagram, then we call X a *Ferrers variety of lines.* That is, after renaming, we assume that if $\mathcal{L}(A_i, B_j) \in U_h(X)$, then $\mathcal{L}(A_{i'}, B_{j'}) \in U_h(X)$ for every $1 \leq i' \leq i,\ 1 \leq j' \leq j$ and for each direction $h = 1, 2, 3$.

Lemma 2.4.2. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a Ferrers variety of lines. Then X is ACM.*

Proof. The variety of lines X has the $Hyp_n(\star)$-property for $n = 4, 5, 6$ and then, by Theorem 2.2.15, is ACM. ☐

Now, let X be a Ferrers variety of lines and let $X_3 = \bigcup_{(r,s)\in U_3(X)} \mathcal{L}(A_r, B_s)$ be the variety of lines consisting of the lines of X of type $(1, 1, 0)$. Since $U_3(X)$ is a Ferrers diagram, the variety X_3 is ACM (in $\mathbb{P}^1 \times \mathbb{P}^1$) and we can explicitly write out a set of minimal generators of I_{X_3}, see Remark 2.1.9 and [59]. We denote by $D_3(X) := \{(a_{3,1}, b_{3,1}, 0), \ldots, (a_{3,t_1}, b_{3,t_1}, 0)\}$ the set of the degrees of the minimal generators of I_{X_3}.

Analogously, if we consider the varieties of lines X_1 and X_2 consisting of the lines of X of types $(0, 1, 1)$ and $(1, 0, 1)$, respectively, we obtain the sets of degrees $D_1(X) = \{(0, b_{1,1}, c_{1,1}), \dots, (0, b_{1,t_2}, c_{1,t_2})\}$ and $D_2(X) = \{(a_{2,1}, 0, c_{2,1}), \dots, (a_{2,t_3}, 0, c_{2,t_3})\}$. Then we denote by

$$\begin{aligned} D(X) := \{(&\max\{a_{3,i}, a_{2,k}\}, \max\{b_{3,i}, b_{1,j}\}, \max\{c_{1,j}, c_{2,k}\}) \mid \\ &\forall\ (a_{3,i}, b_{3,i}, 0) \in D_3(X), (a_{2,k}, 0, c_{2,k}) \in D_2(X), \\ &(0, b_{1,j}, c_{1,j}) \in D_1(X)\}. \end{aligned}$$

Finally, we denote by $\hat{D}(X)$ the set of the minimal elements of $D(X)$ with respect to the natural partial order $\preceq$ on the elements of $\mathbb{N}^3$.

Theorem 2.4.3. *Let X be a Ferrers variety of lines,*

$$X = \bigcup_{\substack{i\in[a]\\ j\in[b]}} \mathcal{L}(A_i, B_j) \cup \bigcup_{\substack{i\in[a]\\ k\in[c]}} \mathcal{L}(A_i, C_k) \cup \bigcup_{\substack{j\in[b]\\ k\in[c]}} \mathcal{L}(B_j, C_k).$$

Then I_X is minimally generated by the following set of forms

$$\left\{ \prod_{i\le a} A_i \prod_{j\le b} B_j \prod_{k\le c} C_k \mid \textit{for each } (a,b,c) \in \hat{D}(X) \right\}.$$

Proof. First, we prove that if $(a, b, c) \in \hat{D}(X)$, then $\prod_{i\le a} A_i \prod_{j\le b} B_j \prod_{k\le c} C_k \in I_X$. Indeed $(a, b, c) \in \hat{D}(X)$ implies $\prod_{i\le a} A_i \prod_{j\le b} B_j \in I_{X_3}$, $\prod_{j\le b} B_j \prod_{k\le c} C_k \in I_{X_1}$, $\prod_{i\le a} A_i \prod_{k\le c} C_k \in I_{X_2}$ and they are not necessarily minimal elements of the respective ideal. Thus

$$\prod_{i\le a} A_i \prod_{j\le b} B_j \prod_{k\le c} C_k \in I_{X_1} \cap I_{X_2} \cap I_{X_3} = I_X.$$

Now, we show that if $(a, b, c) \in \hat{D}(X)$ and $a > 0$, then

$$\prod_{i\le a-1} A_i \prod_{j\le b} B_j \prod_{k\le c} C_k \notin I_X.$$

This fact follows by contradiction. Indeed if $\prod_{i\le a-1} A_i \prod_{j\le b} B_j \prod_{k\le c} C_k \in I_X$, then $(a-1, b, 0), (a-1, 0, c), (0, b, c)$ are degrees of some (not necessarily minimal) elements in the ideal and therefore there is an element in $D(X)$ less than or equal to $(a-1, b, c)$, contradicting the minimality of $(a, b, c) \in \hat{D}(X)$. Analogously, it can be easily showed that if $(a, b, c) \in \hat{D}(X)$ and $b > 0$ (or $c > 0$), then $\prod_{i\le a} A_i \prod_{j\le b-1} B_j \prod_{k\le c} C_k \notin I_X$ (or $\prod_{i\le a} A_i \prod_{j\le b} B_j \prod_{k\le c-1} C_k \notin I_X$). Finally, we claim that I_X is minimally generated by the forms $\prod_{i\le a} A_i \prod_{j\le b} B_j \prod_{k\le c} C_k$ with $(a, b, c) \in \hat{D}(X)$. Take a form $F \in I_X$, without loss of generality we can assume that it is product of linear forms

$$F := \prod_{i\in\mathcal{A}} A_i \prod_{j\in\mathcal{B}} B_j \prod_{k\in\mathcal{C}} C_k.$$

By contradiction we assume A_i divides F and A_{i-1} does not divide F. Then $\prod_{i\in\mathcal{A}} A_i \prod_{j\in\mathcal{B}} B_j \in I_{X_3}$. Then $F \in (\prod_{i\le a'} A_i \prod_{j\le b'} B_j)$ for some a', b'. Repeating the same argument with respect to the other two directions we get the proof. The minimality come from the minimality of the degrees in $\hat{D}(X)$. □

The following corollary is an immediate consequence of Theorem 2.4.3 and the ACM property. Set

$$\langle \hat{D}(X) \rangle := \{(i,j,k) \mid (i,j,k) \geq (a,b,c), \text{ for some } (a,b,c) \in \hat{D}(X)\}.$$

Corollary 2.4.4. *Let X be a Ferrers variety of lines. Then*

$$\Delta H_X(i,j,k) = \begin{cases} 0 & \text{if } (i,j,k) \in \langle \hat{D}(X) \rangle \\ 1 & \text{otherwise} \end{cases}.$$

Example 2.4.5. Let us consider the following variety of lines

$$X = \{\mathcal{L}(A_i, B_j) \cup \mathcal{L}(A_i, C_k) \cup \mathcal{L}(B_j, C_k) \mid 1 \leq i \leq 4,\ 1 \leq j \leq 3,\ 1 \leq k \leq 2\}.$$

In this case $D_3(X) = \{(4,0,0), (0,3,0)\}$, $D_2(X) = \{(4,0,0), (0,0,2)\}$ and $D_1(X) = \{(0,3,0), (0,0,2)\}$. So we have $D(X) = \{(4,3,2), (4,3,0), (4,0,2), (0,3,2)\}$ and $\hat{D}(X) = \{(4,3,0), (4,0,2), (0,3,2)\}$. Therefore, from Theorem 2.4.3, a minimal set of generators of I_X is given by:

$$A_1A_2A_3A_4B_1B_2B_3,\ A_1A_2A_3A_4C_1C_2,\ B_1B_2B_3C_1C_2$$

and

$$\Delta H_X(i,j,k) = \begin{cases} 0 & \text{if } (i,j,k) \geq (4,3,0) \text{ or } (4,0,2) \text{ or } (0,3,2) \\ 1 & \text{otherwise} \end{cases}.$$

2.5 Case study: grids of lines

In the previous sections we focused on the study of special arrangements of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ having the ACM property. Recall that for a point $P \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ there are exactly three lines passing through P, one for each direction. We have the following definition.

Definition 2.5.1. Let $\mathcal{Y}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We call *grid of lines* arising from $\mathcal{Y}$, and denote it by $X_{\mathcal{Y}}$, the set containing all the lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ passing through some point of $\mathcal{Y}$.

In other words, if $\mathcal{Y}$ is a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then

$$X_{\mathcal{Y}} := \bigcup_{P_{ijk} \in \mathcal{Y}} \mathcal{L}(A_i, B_j) \cup \mathcal{L}(A_i, C_k) \cup \mathcal{L}(B_j, C_k)$$

where $P_{ijk} := \mathcal{L}(A_i) \cap \mathcal{L}(B_j) \cap \mathcal{L}(C_k)$.

Example 2.5.2. Let $\mathcal{Y} = \{P_{122}, P_{212}, P_{221}\}$ be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let us consider the grid of lines $X_{\mathcal{Y}}$ arising from $\mathcal{Y}$ (see Figure 2.9):

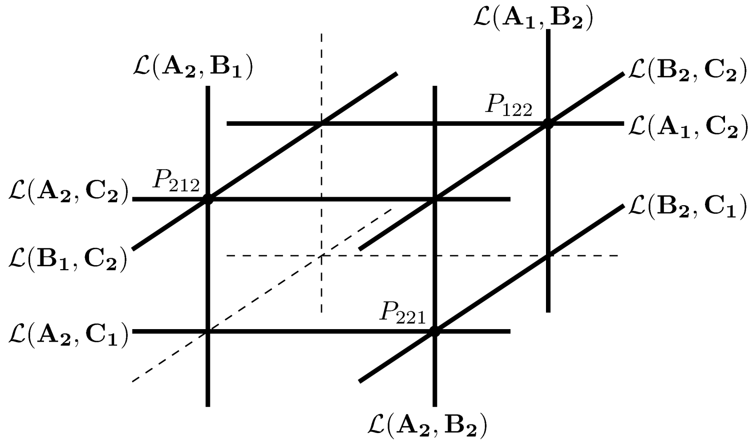


Figure 2.9: The grid of lines $X_{\mathcal{Y}}$ (in bold).

The grid $X_{\mathcal{Y}}$ is formed by 9 lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} X =& \mathcal{L}(A_1, B_2) \cup \mathcal{L}(A_2, B_1) \cup \mathcal{L}(A_2, B_2) \cup \mathcal{L}(A_1, C_2) \cup \mathcal{L}(A_2, C_1) \cup \\ & \mathcal{L}(A_2, C_2) \cup \mathcal{L}(B_1, C_2) \cup \mathcal{L}(B_2, C_1) \cup \mathcal{L}(B_2, C_2). \end{aligned}$$

The next example shows that, even if $\mathcal{Y}$ is an ACM set of points, $X_{\mathcal{Y}}$ could be not ACM.

Example 2.5.3. Suppose $\mathcal{Y} := \{P_{112}, P_{122}, P_{121}, P_{212}\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. According to [45], $\mathcal{Y}$ is an ACM set of points. Let us consider the grid of lines $X_{\mathcal{Y}}$ arising from $\mathcal{Y}$ (see Figure 2.10). We have $\mathcal{L}(A_2, B_1), \mathcal{L}(B_2, C_1) \in X_{\mathcal{Y}}$ and $\mathcal{L}(A_2, B_2), \mathcal{L}(A_2, C_1)$, $\mathcal{L}(B_1, C_1) \notin X_{\mathcal{Y}}$, that is, $X_{\mathcal{Y}}$ does not have the $Hyp_4(\star)$-property. Thus $X_{\mathcal{Y}}$ is not ACM.

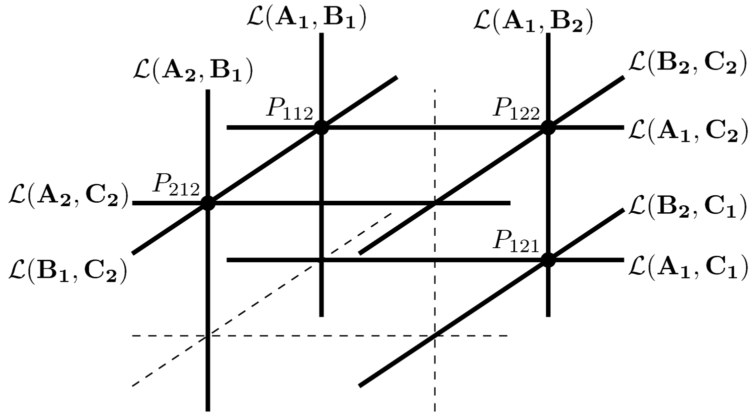


Figure 2.10: The grid of lines $X_{\mathcal{Y}}$ arising from $\mathcal{Y}$ (in bold).

It is interesting to ask which sets of points $\mathcal{Y} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ lead to an ACM grid of lines $X_{\mathcal{Y}}$. A special class of CM rings are complete intersections (see Definition 1.1.34 and Remark 1.1.35).

Recall that in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, a set of points $\mathcal{C}$ is a complete intersection of points of type (a_1, a_2, a_3) if $I_{\mathcal{C}} = (F_1, F_2, F_3)$ is a complete intersection and $\deg F_i = a_i \mathbf{e_i}$ for $i = 1, 2, 3$, where each F_i is product of linear forms.

Example 2.5.4. Let $\mathcal{C} = \{P_{111}, P_{112}, P_{121}, P_{122}, P_{211}, P_{212}, P_{221}, P_{222}\}$ be a complete intersection of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of type $(2, 2, 2)$. The grid of lines arising from $\mathcal{C}$ (see Figure 2.11) is formed by 12 lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$: 4 lines of type $(1, 1, 0)$, 4 lines of type $(0, 1, 1)$ and 4 lines of type $(1, 0, 1)$.

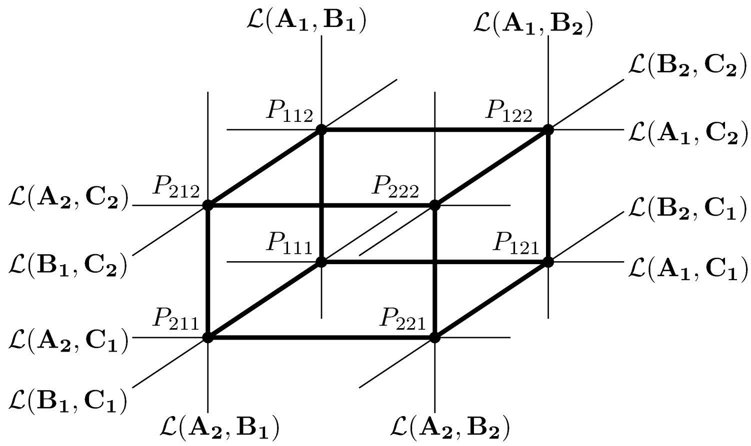


Figure 2.11: The grid $X_{\mathcal{C}}$ arising from a CI of points of type $(2, 2, 2)$.

Lemma 2.5.5. *Let* $\mathcal{C} := \{\mathcal{L}(A_i) \cap \mathcal{L}(B_j) \cap \mathcal{L}(C_k) \mid i \in [a], j \in [b], k \in [c]\}$ *be a complete intersection of points in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *of type* (a, b, c) *and let* $X := X_{\mathcal{C}}$ *be the grid of lines arising from* $\mathcal{C}$*. Then a set of minimal generators of the ideal defining* X *is*

$$I_X = \Big(\prod_{i\in[a]} A_i \cdot \prod_{j\in[b]} B_j \ , \ \prod_{i\in[a]} A_i \cdot \prod_{k\in[c]} C_k \ , \ \prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k \Big).$$

Proof. Of course we have

$$I_X \supseteq \Big(\prod_{i\in[a]} A_i \cdot \prod_{j\in[b]} B_j \ , \ \prod_{i\in[a]} A_i \cdot \prod_{k\in[c]} C_k \ , \ \prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k \Big).$$

On the other hand we have

$$I_X = \left(\prod_{i\in[a]} A_i, \prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k \right) \bigcap \left(\prod_{j\in[b]} B_j, \prod_{k\in[c]} C_k \right).$$

Thus if $F \in I_X$ is a multihomogeneous form

$$F = F_0 \prod_{i\in[a]} A_i + F_1 \prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k.$$

Since $F_1 \prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k \in \left(\prod_{j\in[b]} B_j, \prod_{k\in[c]} C_k \right)$ we get

$$F_0 \prod_{i\in[a]} A_i \in \left(\prod_{j\in[b]} B_j, \prod_{k\in[c]} C_k \right),$$

therefore, by a multigrading argument,

$$F_0 \in \left(\prod_{j\in[b]} B_j, \prod_{k\in[c]} C_k \right)$$

and we are done. □

Remark 2.5.6. The ideal defining a grid of lines arising from a complete intersection of points of type (a,b,c) is generated by 3 polynomials of degree $(a,b,0),(a,0,c)$ and $(0,b,c)$, respectively. This does not characterize the ideals of these particular grids of lines. Indeed, take

$$Y = \mathcal{L}(A_1,B_1) \cup \mathcal{L}(A_1,C_1) \cup \mathcal{L}(B_2,C_1),$$

one can check that I_Y is minimally generated by three forms of degree $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ but Y is not a grid of lines arising from a CI of points.

Remark 2.5.7. Let $\mathcal{H}_1 := \{\mathcal{L}(A_1),\ldots,\mathcal{L}(A_a)\}$, $\mathcal{H}_2 := \{\mathcal{L}(B_1),\ldots,\mathcal{L}(B_b)\}$ and $\mathcal{H}_3 := \{\mathcal{L}(C_1),\ldots,\mathcal{L}(C_c)\}$ be sets of hyperplanes defined by forms of degree $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively. Let J be the ideal generated by $\prod_{i\in[a]} A_i \cdot \prod_{j\in[b]} B_j$, $\prod_{i\in[a]} A_i \cdot \prod_{k\in[c]} C_k$ and $\prod_{j\in[b]} B_j \cdot \prod_{k\in[c]} C_k$. Then J is the ideal of a grid of lines arising from a CI of points of type (a,b,c).

Theorem 2.5.8. *Let $\mathcal{C} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a complete intersection of points of type (a,b,c). Then $X_{\mathcal{C}}$ is ACM and a trigraded minimal free resolution of $I_{X_{\mathcal{C}}}$ is*

$$0 \to R^2(-a,-b,-c) \to R(-a,-b,0) \oplus R(-a,0,-c) \oplus R(0,-b,-c) \to I_{X_{\mathcal{C}}} \to 0.$$

Proof. The grid of lines $X := X_{\mathcal{C}}$ has the $Hyp_n(\star)$-property for $n = 4,5,6$ and then, by Theorem 2.2.15, X is ACM. Moreover, by Lemma 2.5.5, a set of minimal generators of I_X is

$$I_X = \left(\prod_{i\in[a]} A_i \prod_{j\in[b]} B_j \;,\; \prod_{i\in[a]} A_i \prod_{k\in[c]} C_k \;,\; \prod_{j\in[b]} B_j \prod_{k\in[c]} C_k \right),$$

and then a Hilbert-Burch matrix of I_X is

$$\begin{pmatrix} \prod_{i\in[a]} A_i & \prod_{i\in[a]} A_i \\ \prod_{j\in[b]} B_j & 0 \\ 0 & \prod_{k\in[c]} C_k \end{pmatrix}.$$

□

Example 2.5.9. If $\mathcal{C} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a complete intersection of points of type $(2, 3, 2)$, then the grid $X_{\mathcal{C}}$ (see Figure 2.12) is formed by 6 lines of type $(1, 1, 0)$, 4 lines of type $(1, 0, 1)$ and 6 lines of type $(0, 1, 1)$:

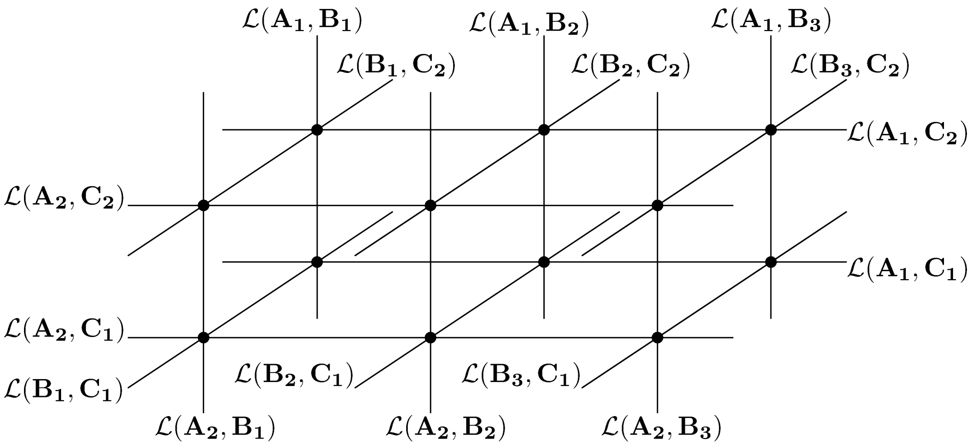


Figure 2.12: The grid of lines $X_{\mathcal{C}}$ arising from a CI of points of type $(2, 3, 2)$.

By Lemma 2.5.5, a set of minimal generators of $I_{X_{\mathcal{C}}}$ is

$$I_{X_{\mathcal{C}}} = (A_1A_2B_1B_2B_3, A_1A_2C_1C_2, B_1B_2B_3C_1C_2).$$

In particular, by Theorem 2.5.8, $I_{X_{\mathcal{C}}}$ has a trigraded minimal free resolution of the following type

$$0 \to R^2(-2, -3, -2) \to R(-2, -3, 0) \oplus R(-2, 0, -2) \oplus R(0, -3, -2) \to I_{X_{\mathcal{C}}} \to 0.$$

Corollary 2.5.10. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a grid of lines arising from a complete intersection of points of type (a, b, c). Then the first difference of Hilbert function of R/I_X is:*

$$\Delta H_X(i, j, k) = \begin{cases} 0 & \text{if } (i, j, k) \geq (a, b, 0) \text{ or } (i, j, k) \geq (a, 0, c) \\ & \text{or } (i, j, k) \geq (0, b, c) \\ 1 & \text{otherwise.} \end{cases}$$

Proof. From the trigaded minimal free resolution of Theorem 2.5.8 we have that the Hilbert function of R/I_X is:

$$\begin{aligned} H_X(i,j,k) = &(i+1)(j+1)(k+1) - (i+1)(j-b+1)_+(k-c+1)_+ + \\ &-(i-a+1)_+(j+1)(k-c+1)_+ + \\ &-(i-a+1)_+(j-b+1)_+(k+1) + \\ &+2(i-a+1)_+(j-b+1)_+(k-c+1)_+ \ , \end{aligned}$$

where $(n)_+ := \max\{n, 0\}$.

Suppose, to fix ideas, that $(i,j,k) \geq (a,b,0)$. There are two cases to consider: $k \geq c$ and $k < c$. Using the previous equality and by the definition of first difference of Hilbert function, it's easy to check that in each case $\Delta H_X(i,j,k) = 0$. Moreover, if $(i,j,k) \ngeq (a,b,0),(a,0,c),(0,b,c)$, we have that $H_X(i,j,k) = (i+1)(j+1)(k+1)$ and then $\Delta H_X(i,j,k) = 1$. □

Corollary 2.5.11. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a grid of lines arising from a complete intersection of points of type (a,b,c). Then we have*

$$\operatorname{reg}(I_X) = a + b + c - 1.$$

Proof. We can consider the ideal I_X as homogeneous in the normal sense, i.e., a homogeneous ideal in a $\mathbb{N}^1$-graded ring $S = \mathbf{k}[x_1, \ldots, x_6]$. By Theorem 2.5.8, a minimal free resolution of $I_X \subseteq S$ is

$$0 \to S^2(-a-b-c) \to S(-a-b) \oplus S(-a-c) \oplus S(-b-c) \to I_X \to 0.$$

By Definition 1.2.27), we have

$$\operatorname{reg}(I_X) = \max\{a+b, a+c, b+c, a+b+c-1\} = a+b+c-1.$$

□

Remark 2.5.12. Note that a grid of lines arising from a complete intersection of points is a Ferrers variety of lines. So the statements of Lemma 2.5.5 and Corollary 2.5.10 are in agreement with the statements of Theorem 2.4.3 and Corollary 2.4.4 for the particular case of grids of lines arising from a CI of points.

Example 2.5.13. Let us consider the following grid of lines arising from a CI of points of type $(4,3,2)$:

$$X = \{\mathcal{L}(A_i,B_j) \cup \mathcal{L}(A_i,C_k) \cup \mathcal{L}(B_j,C_k) \mid i = 1,2,3,4 \ , \ j = 1,2,3 \ , \ k = 1,2\}.$$

It is formed by 26 lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$: 12 lines of type $(1,1,0)$, 6 lines of type $(0,1,1)$ and 8 lines of type $(1,0,1)$.
By Lemma 2.5.5, a set of minimal generators of I_X is given by:

$$A_1A_2A_3A_4B_1B_2B_3, \ A_1A_2A_3A_4C_1C_2, \ B_1B_2B_3C_1C_2,$$

of degree $(4,3,0)$, $(4,0,2)$ and $(0,3,2)$, respectively. By Theorem 2.5.8, I_X is ACM and its generators came from the following Hilbert-Burch matrix:

$$\begin{pmatrix} A_1A_2A_3A_4 & 0 \\ B_1B_2B_3 & B_1B_2B_3 \\ 0 & C_1C_2 \end{pmatrix}.$$

The trigraded minimal free resolution of I_X is:

$$0 \to R^2(-4,-3,-2) \to R(-4,-3,0) \oplus R(-4,0,-2) \oplus R(0,-3,-2) \to I_X \to 0.$$

By Corollary 2.5.11, we have $\operatorname{reg}(I_X) = 8$.

By Corollary 2.5.10, the first difference of Hilbert function of R/I_X is then given by:

$$\forall\, i = 0,1,2,3, \quad \Delta H_X(i,j,k) = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \ldots \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\ 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ 5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ \vdots & & \vdots & & \vdots & & \vdots & \ddots \end{array}$$

$$\forall\, i \geq 4, \quad \Delta H_X(i,j,k) = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \ldots \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\ \vdots & & \vdots & & \vdots & & \vdots & \ddots \end{array}$$

The following example shows that there exists an ACM grid of lines $X_{\mathcal{Y}}$ arising from a non-ACM set of points $\mathcal{Y}$.

Example 2.5.14. The following set of points

$$\mathcal{Y} := \{P_{111}, P_{121}, P_{211}, P_{122}, P_{212}, P_{222}\}$$

is not an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see [45]). However, $X_{\mathcal{Y}} = X_{\mathcal{C}}$ where $\mathcal{C} := \{P_{ijk} \mid 1 \leq i,j,k \leq 2\}$ (see Figure 2.13), and then $X_{\mathcal{Y}}$ is an ACM grid of lines.

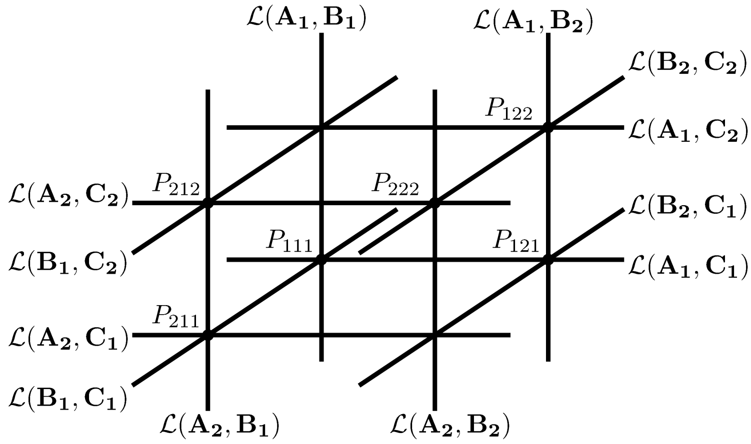


Figure 2.13: The grid $X_{\mathcal{Y}}$ arising from a non-ACM set of points $\mathcal{Y}$.

2.6 Case study: complete intersections of lines

From Theorem 2.5.8, we note that the ideal $I_{X_\mathcal{C}}$ is generated by three forms that do not form a regular sequence. That is, even if $\mathcal{C}$ is a complete intersection of points, then its associated variety of lines $X_\mathcal{C}$ is not a complete intersection of lines. Thus, it is natural to study which varieties of lines are defined by a complete intersection, i.e., their defining ideal has only two generators. Theorem 2.6.2 and Remark 2.6.5 will describe complete intersections of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Remark 2.6.1. If X is an ACM variety of lines, from Corollary 2.2.3, I_X is generated by products of linear forms. Then

$$I_X \supseteq \left(\prod_{i\in[a]} A_i \prod_{j\in[b]} B_j \ , \ \prod_{i\in[a]} A_i \prod_{k\in[c]} C_k \ , \ \prod_{j\in[b]} B_j \prod_{k\in[c]} C_k \right).$$

So any set of minimal generators of I_X contains one element of degree $(a_3, b_3, 0)$, one element of degree $(a_2, 0, c_2)$ and one element of degree $(0, b_1, c_1)$.

Theorem 2.6.2. *Let X be a variety of lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then the ideal I_X is a complete intersection if and only if $I_X = (F_1, F_2)$, with $\deg F_1 = a\boldsymbol{e}_i$ and $\deg F_2 = b\boldsymbol{e}_j + c\boldsymbol{e}_k$ with $j, k \neq i$, for some $a, b, c \in \mathbb{N}$.*

Proof. One implication is trivial. Let I_X be a complete intersection, i.e., I_X is generated by a regular sequence of length 2, then X is ACM. So, from Remark 2.6.1, any set of minimal generators of I_X contains one element G_1 of degree $(0, b_1, c_1)$, one element G_2 of degree $(a_2, 0, c_2)$ and one element G_3 of degree $(a_3, b_3, 0)$ for some integers a_i, b_j, c_k. Since I_X is a complete intersection, one of these three generators say, without loss of generality, the one of degree $(0, b_1, c_1)$, is not minimal, i.e. $G_1 \in (G_2, G_3)$. This easily implies $a_2 a_3 = 0$. $\square$

Corollary 2.6.3. *Let X be a complete intersection of lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. To fix ideas, suppose that $I_X = (F_1, F_2)$, with $\deg F_1 = a\boldsymbol{e}_1$ and $\deg F_2 = b\boldsymbol{e}_2 + c\boldsymbol{e}_3$. Then a trigraded minimal free resolution of I_X is*

$$0 \to R(-a,-b,-c) \to R(-a,0,0) \oplus R(0,-b,-c) \to I_X \to 0.$$

Corollary 2.6.4. *Let X be a complete intersection of lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. To fix ideas, suppose that $I_X = (F_1, F_2)$, with $\deg F_1 = a\boldsymbol{e}_1$ and $\deg F_2 = b\boldsymbol{e}_2 + c\boldsymbol{e}_3$. Then we have*

$$\operatorname{reg}(I_X) = a + b + c - 1.$$

Proof. We can consider the ideal I_X as homogeneous in the normal sense, i.e., a homogeneous ideal in a $\mathbb{N}^1$-graded ring $S = \mathbf{k}[x_1, \ldots, x_6]$. By Corollary 2.6.3, a minimal free resolution of $I_X \subseteq S$ is

$$0 \to S(-a-b-c) \to S(-a) \oplus S(-b-c) \to I_X \to 0.$$

By Definition 1.2.27), we have

$$\operatorname{reg}(I_X) = \max\{a, b+c, a+b+c-1\} = a+b+c-1.$$

□

Remark 2.6.5. From Theorem 2.6.2, a complete intersection of lines X is then obtained from a grid arising from a complete intersection of points by removing either all the lines having direction $\mathbf{e}_i$ for some i, or all the lines having direction $\mathbf{e}_i$ and $\mathbf{e}_j$ with $i \neq j$. For instance, if we remove all the lines having direction e_1, from Remark 2.6.1, we have

$$I_X = \left(\prod_{i\in[a]} A_i, \prod_{j\in[b]} B_j \prod_{k\in[c]} C_k \right) = \bigcap_{\substack{i\in[a]\\ j\in[b]}} (A_i, B_j) \cap \bigcap_{\substack{i\in[a]\\ k\in[c]}} (A_i, C_k).$$

Example 2.6.6. Let X be the set of lines of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ obtained by a grid of lines $X_{\mathcal{C}}$ arising from a complete intersection of points $\mathcal{C}$ of type $(2,3,2)$ removing all the lines having direction e_2 (see Figure 2.14). That is,

$$X = \bigcup_{\substack{i\in[2]\\ j\in[3]}} \mathcal{L}(A_i, B_j) \bigcup_{\substack{j\in[3]\\ k\in[2]}} \mathcal{L}(B_j, C_k).$$

Then the ideal I_X is a complete intersection and it is generated by the regular sequence $F_1 = B_1B_2B_3$ and $F_2 = A_1A_2C_1C_2$ of degree $(0,3,0)$ and $(2,0,2)$, respectively. By Corollary 2.6.4 we also have $\operatorname{reg}(I_X) = 6$.

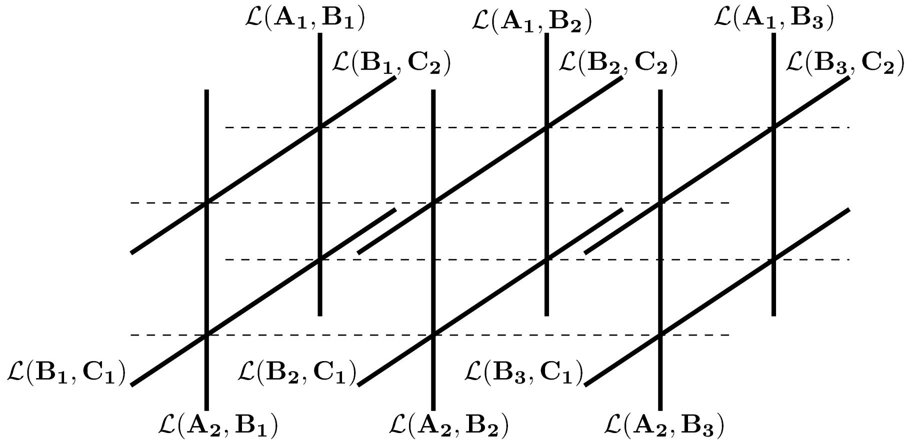


Figure 2.14: A complete intersection of lines (in bold).

We end this chapter with three research topics that are currently investigating.

1. Guida, Orecchia and Ramella, in [61], studied the *complete grids* of lines in $\mathbb{P}^3$, whose defining ideal is the 1-lifting ideal of a specific monomial ideal J in a polynomial ring S in three variables. In particular, from [61, Example 4.9] and Corollary 2.4.4, we noted that the first difference of the Hilbert function of the ideal I_{X_C} of a grid of lines arising from a complete intersection of points of type $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in degree (i, j, k) is equal to 1 if and only if (i, j, k) belongs to the order ideal $N(J) \subseteq \mathbb{N}^3$ of the specific monomial ideal $J = (x_1^2x_2^2, x_1^2x_3^2, x_2^2x_3^2)$ in S. So we make the following question:

   Question 2.6.7. For complete intersection of points of type (a, b, c), is the Hilbert function related to the order ideal of $J = (x_1^a x_2^b, x_1^a x_3^c, x_2^b x_3^c)$?

2. Let us consider the ACM varieties of lines X and the Ferrers variety of lines X' as in Figure 2.15 and Figure 2.16, respectively. We have that, for each $h = 1, 2, 3$, X_h and X'_h have the same Hilbert functions. We also get $H_X = H_{X'}$.

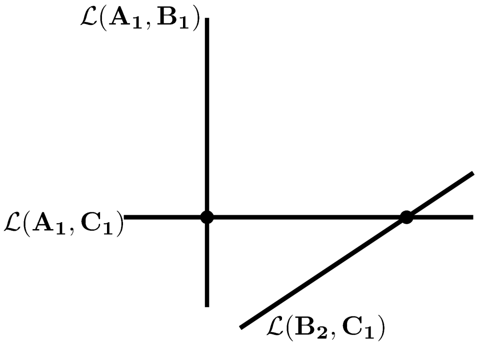


Figure 2.15: The ACM variety X.

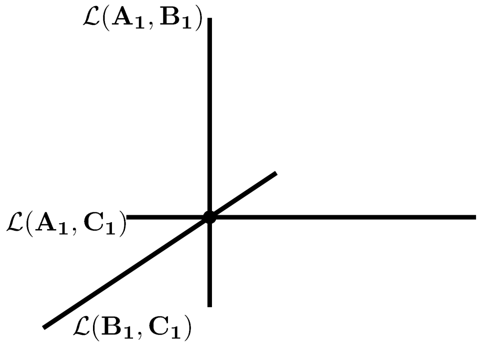


Figure 2.16: The Ferrers variety X'.

According to many experimental computations using CoCoA, [1], we ask the following question:

Question 2.6.8. Let X be an ACM variety of lines and X' be a Ferrers variety of lines such that, for $h = 1, 2, 3$, X_h and X'_h have the same Hilbert functions. Is it true that $H_X = H_{X'}$?

3. In this chapter we have computed the regularity of the defining ideals of grids of lines and complete intersections of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. A future research topic to explore is the computation of the regularity of other more general classes of varieties of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Chapter 3

Regularity of bicyclic graphs and their powers

This chapter is based upon the joint project with Y. Cid-Ruiz, S. Jafari and N. Nemati.

The purpose of this chapter is to extend the results of [3] to the family of bicyclic graphs (i.e. graphs with exactly two cycles). In particular, let $I(G)$ be the edge ideal of a bicyclic graph G. We characterize the Castelnuovo-Mumford regularity of $I(G)$ in terms of the induced matching number of G. The simplest case of the family of bicyclic graphs is that of dumbbell graphs. A dumbbell graph $C_n \cdot P_l \cdot C_m$ is a graph consisting of two cycles C_n and C_m connected with a path P_l, where n, m, and l are the number of vertices (see Example 3.1.1). For dumbbell graphs, we explicitly compute the induced matching number. Moreover, we prove that $\operatorname{reg} I(G)^q = 2q + \operatorname{reg} I(G) - 2$, for all $q \geq 1$, when G is a dumbbell graph with a connecting path having no more than two vertices.

By abuse of notation, we think of the vertices of $G = (V, E)$ as the variables of $R = K[x_1, \ldots, x_r]$. Following this notation, we consider the edges of G as squarefree monomials of degree two. When there is no confusion, we use e to denote an edge and x_e for the monomial correspond to e. If we need to specify the vertices of an edge, we use $e_{i,j} = x_i x_j$.

3.1 Regularity and induced matching number of a dumbbell graph

In this section we compute the induced matching number of a dumbbell graph and the regularity of its edge ideal. Recall that $C_n \cdot P_l \cdot C_m$ denotes the graph constructed by joining two cycles C_n and C_m via a path P_l. In this section, we denote the vertices of C_n, C_m and P_l by $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ and $\{z_1, \ldots, z_l\}$, respectively. We make the identifications $x_1 = z_1$ and $y_1 = z_l$.

Example 3.1.1. Two simple cases when $l = 2$ and $l = 1$ are the following:

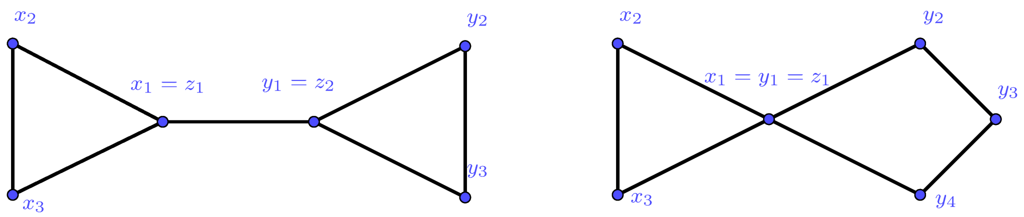
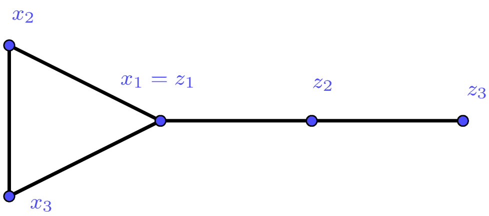


Figure 3.1: The graphs $C_3 \cdot P_2 \cdot C_3$ and $C_3 \cdot P_1 \cdot C_4$.

Notation 3.1.2. Let ξ_3 be the function defined as below

$$\xi_3(n) = \begin{cases} 1 & \text{if} \quad n \equiv 0, 1 \pmod 3, \\ 0 & \text{if} \quad n \equiv 2 \quad\pmod 3. \end{cases}$$

Let $C_n \cdot P_l$ be the graph given by connecting the path P_l to the cycle C_n. For instance, the graph $C_3 \cdot P_3$ can be illustrated as:



Proposition 3.1.3. *Let $n \geq 3$ and $l \geq 1$. Then*

$$\nu(C_n \cdot P_l) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) + 1}{3} \right\rfloor.$$

Proof. <u>Case 1:</u> From Remark 1.2.20, in the case $n \equiv 2 \pmod 3$ we have that in clockwise and anticlockwise directions the two consecutive edges to the vertex x_1 are not chosen in a maximal induced matching of C_n. Then, we can choose the edges in P_l without any constraint coming from the maximal induced matching chosen in C_n, and so we have

$$\nu(C_n \cdot P_l) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{l+1}{3} \rfloor.$$

<u>Case 2:</u> It remain to consider the case $\xi_3(n) = 1$, i.e., $n \equiv 0, 1 \pmod 3$. Let $\mathcal{M}$ be an induced matching of maximal size in G. We analyze separately the two cases of whether $z_1 z_2$ (the edge adjacent to the cycle C_n) is in $\mathcal{M}$ or not.

Suppose that $z_1 z_2$ is not an edge of $\mathcal{M}$. Then $\mathcal{M}$ can be considered as the union of a maximal matching of C_n as introduced in Remark 1.2.20 and a maximal matching of the path $P_l \setminus z_1$. Thus

$$|\mathcal{M}| = \nu(C_n) + \nu(P_{l-1}) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor.$$

If $z_1z_2 \in \mathcal{M}$, then none of the edges incident to the vertices in $N_{C_n}[x_1] = \{x_1, x_2, x_n\}$ are in $\mathcal{M}_{C_n} := \{e \in \mathcal{M} \mid e \in C_n\}$. Hence $|\mathcal{M}_{C_n}| = \nu(P_{n-3})$, and since $n \equiv 0, 1 \pmod 3$, then it follows $|\mathcal{M}_{C_n}| = \lfloor \frac{n-2}{3} \rfloor = \lfloor \frac{n}{3} \rfloor - 1$. From $z_1z_2 \in \mathcal{M}$ we get $|\mathcal{M}_{P_l}| = \nu(P_l) = \lfloor \frac{l+1}{3} \rfloor$. So, by joining both computations we get

$$|\mathcal{M}| = \lfloor \frac{n}{3} \rfloor - 1 + \lfloor \frac{l+1}{3} \rfloor = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{l-2}{3} \rfloor.$$

Therefore, we obtain that

$$\nu(C_n \cdot P_l) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor.$$

□

Theorem 3.1.4. *Let $n, m \geq 3$ and $l \geq 1$. Then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Proof. We use the same argument as in Proposition 3.1.3. By Remark 1.2.20 we have that when either $n \equiv 2 \pmod 3$ or $m \equiv 2 \pmod 3$, then the maximal induced matching in C_n or in C_m does not affect the way we choose edges in the path P_l.

In the case $n \equiv 0, 1 \pmod 3$ we can choose a maximal induced matching that does not use the edge connected to the cycle C_n, which is the same as saying that we are not going to use one extreme vertex of the path P_l. Similarly, when $m \equiv 0, 1 \pmod 3$ we can drop the other extreme vertex. □

The aim of the rest of this section is to explicitly compute the regularity of $I(C_n \cdot P_l \cdot C_m)$ in term of the induced matching number. We divide the section into three subsections depending on the value of l mod 3. The base of our computations is given by the following proposition.

Proposition 3.1.5. *Let $n, m \geq 3$ and $l \geq 1$. Then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) - \nu(C_n \cdot P_l \cdot C_m) = \operatorname{reg} I(C_n \cdot P_{l+3} \cdot C_m) - \nu(C_n \cdot P_{l+3} \cdot C_m).$$

Proof. From the formula obtained in Theorem 3.1.4 or [80, Lemma 1], we have the equality

$$\nu(C_n \cdot P_{l+3} \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1.$$

We can apply the Lozin transformation (e.g., see [16], [80]) to any of the vertices in the bridge P_l. Then from [16, Theorem 1.1] we have

$$\operatorname{reg} I(C_n \cdot P_{l+3} \cdot C_m) = \operatorname{reg} I(C_n \cdot P_l \cdot C_m) + 1.$$

Thus, the statement of the proposition follows by subtracting these equalities. □

From the previous proposition, it follows that we only need to consider the cases $l = 1$, $l = 2$ and $l = 3$. We treat each case in a separate subsection. In the following theorem we compute the regularity of the edge ideal of the dumbbell $C_n \cdot P_l \cdot C_m$.

This theorem is proved in the next three sections.

Theorem 3.1.6. *Let $m, n \geq 3$ and $l \geq 1$.*

(i) If $l \equiv 0, 1$ (mod 3), then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \ (mod\ 3), \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) If $l \equiv 2$ (mod 3), then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \textit{if } n \equiv 0, 1 \ (mod\ 3), \\ & \quad m \equiv 2 \ (mod\ 3); \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Follows from Proposition 3.1.5, and Theorem 3.1.8, Theorem 3.1.14, and Theorem 3.1.16, giving below. □

3.1.1 The case $l = 1$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_1 \cdot C_m$.

Proposition 3.1.7. *Let $n, m \geq 3$. Then*

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \leq \max\left\{ \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2 \right\}.$$

Moreover, $\operatorname{reg} I(C_n \cdot P_1 \cdot C_m)$ *is equal to one of these terms.*

Proof. We use [33, Lemma 3.2], that gives an improved version of the exact sequence coming from deleting the vertex z_1. We have

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \in \Big\{ \operatorname{reg} I\big((C_n \cdot P_1 \cdot C_m) \setminus z_1\big), \\ \operatorname{reg} I\big((C_n \cdot P_1 \cdot C_m) \setminus N[z_1]\big) + 1 \Big\}.$$

Since $(C_n \cdot P_1 \cdot C_m) \setminus z_1 = P_{n-1} \cup P_{m-1}$ and $(C_n \cdot P_1 \cdot C_m) \setminus N[z_1] = P_{n-3} \cup P_{m-3}$, we get the result by applying Theorem 1.2.32. □

Theorem 3.1.8. *Let $n, m \geq 3$. Then*

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_1 \cdot C_m) + 2 & \text{if } n \equiv 2 \ (mod\ 3), \ m \equiv 2 \ (mod\ 3); \\ \nu(C_n \cdot P_1 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$. Since $\lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k}{3} \rfloor$ when $k \equiv 2 \pmod 3$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Thus Proposition 3.1.7 yields

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2. \tag{3.1}$$

Consider the induced subgraph $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\}$ where x_n is in C_n and it is incident to x_1 (e.g., see x_3 in Example 3.1.1). In fact, H is the graph given by joining C_m and a path P_{n-1}, that is, $H = C_m \cdot P_{n-1}$. Now from Proposition 3.1.3, we have that $\nu(H) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. By Corollary 1.2.50, we get $\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \geq \operatorname{reg} I(H)$. From [3, Theorem 1.1], we have $\operatorname{reg} I(H) = \nu(H) + 2$. Therefore, the equality holds in Equation 3.1. The proof of this part is complete since Theorem 3.1.4 yields $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$.

For any case distinct to $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1.$$

Therefore, from Proposition 3.1.7, we have

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1. \tag{3.2}$$

From Theorem 3.1.4, we have $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. Moreover, Theorem 1.2.34 gives $\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \geq \nu(C_n \cdot P_1 \cdot C_m) + 1$. Thus, the equality in Equation 3.2 holds. Therefore the proof is complete. □

3.1.2 The case $l = 2$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_2 \cdot C_m$.

Remark 3.1.9. The regularity of $I(C_n)$ is given in Theorem 1.2.42. For simplicity of notation, we use the equivalent formula $\operatorname{reg} I(C_n) = \lfloor \frac{n-2}{3} \rfloor + 2$.

Proposition 3.1.10. *Let $n, m \geq 3$. Then*

$$\nu(C_n \cdot P_2 \cdot C_m) \leq \operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) \leq \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2. \tag{3.3}$$

Proof. We only need to prove the inequality on the right since the lower bound is given due to Theorem 1.2.34 and $\operatorname{reg}(J) - 1 = \operatorname{reg}(\frac{R}{J})$ for any ideal $J \subset R$. In the original graph $C_n \cdot P_2 \cdot C_m$ we shall remove the edge that connects the two cycles C_n and C_m. The set of vertices of C_n and C_m

are given respectively by $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$, and we assume that the edge $e = x_1y_1$ is the bridge between the two cycles. Also, we denote by $C_n \cup C_m$ the resulting graph given as the disjoint union of the two cycles C_n and C_m. Thus Remark 1.2.53(ii) yields the inequality

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) \leq \max\left\{\operatorname{reg}\left(\frac{R}{I(C_n \cup C_m) : e}\right)+1, \operatorname{reg}\left(\frac{R}{I(C_n \cup C_m)}\right)\right\}.$$

From [74, Lemma 3.2] we have that the regularity of the two disjoint cycles $C_n \cup C_m$ is given by

$$\operatorname{reg}\left(\frac{R}{I(C_n \cup C_m)}\right) = \operatorname{reg}\left(\frac{R}{I(C_n)}\right) + \operatorname{reg}\left(\frac{R}{I(C_m)}\right),$$

and using Remark 3.1.9 we get the equality

$$\operatorname{reg}\left(\frac{R}{I(C_n \cup C_m)}\right) = \left\lfloor\frac{n-2}{3}\right\rfloor + \left\lfloor\frac{m-2}{3}\right\rfloor + 2.$$

Consider the graph $H = \{x_2, x_n\} \cup P_{n-3} \cup \{y_2, y_m\} \cup P_{m-3}$, where $\{x_2, x_n\}$ and $\{y_2, y_m\}$ are incident vertices of graph $C_n \cdot P_2 \cdot C_m$ to x_1 and y_1, respectively (see Example 3.1.1). Moreover, P_{n-3} is the path with vertices $x_3, \ldots, x_{n-1}$ and P_{m-3} is the path with vertices $y_3, \ldots, y_{m-1}$. It is easy to see that $\operatorname{reg} I(H) = \operatorname{reg} I(C_n \cup C_m) : (e)$. Hence from Remark 1.2.19, Theorem 1.2.54 and again [74, Lemma 3.2] we get

$$\operatorname{reg}\left(\frac{R}{I(C_n \cup C_m) : (e)}\right) + 1 = \left\lfloor\frac{n-2}{3}\right\rfloor + \left\lfloor\frac{m-2}{3}\right\rfloor + 1,$$

that proves the proposition. □

As a result of the previous proposition, we can prove the following corollary.

Corollary 3.1.11. *Let $n \equiv 0, 1$ (mod 3) and $m \equiv 0, 1$ (mod 3). Then*

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor.$$

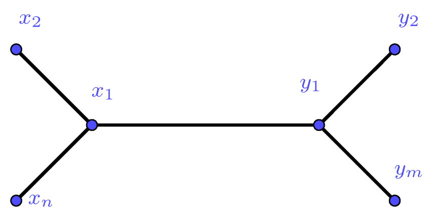
Proof. We note that $\lfloor\frac{k}{3}\rfloor = \lfloor\frac{k-2}{3}\rfloor + 1$ when $k \equiv 0, 1 \pmod 3$. From Theorem 3.1.4, in Equation 3.3 the lower and upper bound coincide for these cases. So the equality is established. □

Now we have only three more cases left to deal with, i.e., the case $n \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$, the case $n \equiv 1 \pmod 3$, $m \equiv 2 \pmod 3$, and the case $n \equiv 2 \pmod 3$, $m \equiv 2 \pmod 3$.

Lemma 3.1.12. *Let* $n \equiv 2$ *(mod* 3*) and* $m \equiv 2$ *(mod* 3*). Then*

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. We shall divide the graph into three subgraphs H_1, H_2 and H_3. We make $H_1 = C_n \setminus \{x_1\}$ and $H_2 = C_m \setminus \{y_1\}$. The subgraph H_3 is defined by taking the bridge $e = x_1 y_1$ and the neighboring vertices $\{x_2, x_n, y_2, y_m\}$, i.e. the graph below.



Using this decomposition and Theorem 1.2.54 we get the inequality

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) \leq \operatorname{reg}(R/I(H_1)) + \operatorname{reg}(R/I(H_2)) + \operatorname{reg}(R/I(H_3)).$$

Then have that H_1 and H_2 are paths of length $n-1$ and $m-1$ respectively, and using Theorem 1.2.32 we get

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) \leq \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Finally, in the case $n \equiv 2$ (mod 3) and $m \equiv 2$ (mod 3), we have the equality $\nu(C_n \cdot P_2 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$, and the proof follows from Theorem 1.2.34. $\square$

Lemma 3.1.13. *Let* $n \equiv 0, 1$ *(mod* 3*) and* $m \equiv 2$ *(mod* 3*). Then*

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) = \nu(C_n \cdot P_2 \cdot C_m) + 1 = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. In this case we will delete the vertex x_1 from the cycle C_n. We have that $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ is an induced subgraph of $C_n \cdot P_2 \cdot C_m$ which is given as the union of a path of length $n-1$ and a cycle m, i.e., $H = P_{n-1} \cup C_m$. From Corollary 1.2.50 we get that

$$\operatorname{reg}(R/I(C_n \cdot P_2 \cdot C_m)) \geq \operatorname{reg}(R/I(H)) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

It follows from Proposition 3.1.10 and the fact that $\lfloor k/3 \rfloor = \lfloor (k-2)/3 \rfloor + 1$ when $k \equiv 0, 1$ (mod 3) that

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

$\square$

Theorem 3.1.14. *Let $n, m \geq 3$. Then*

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_2 \cdot C_m) + 2 & \text{if } n \equiv 0, 1 \ (mod\ 3), \\ & \quad m \equiv 2 \ (mod\ 3); \\ \nu(C_n \cdot P_2 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. This theorem follows by Corollary 3.1.11, Lemma 3.1.12, and Lemma 3.1.13. ☐

3.1.3 The case $l = 3$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_3 \cdot C_m$.

Proposition 3.1.15. *Let $n, m \geq 3$. Then*

(i) $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \leq \nu(C_n \cdot P_3 \cdot C_m) + 2,$ *if $n, m \equiv 2 \ (mod\ 3)$;*

(ii) $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) = \nu(C_n \cdot P_3 \cdot C_m) + 1,$ *otherwise.*

Proof. Let $E(P_3) = \{e, e'\}$ be the set of the edges of P_3, where $e = z_1 z_2$ and $e' = z_2 z_3$ are connected to C_n and C_m, respectively.
Since $\operatorname{reg}(I(C_n \cup (e' \cdot C_m)) : e) = \operatorname{reg}(I(P_{n-3} \cup P_{m-1}))$, then Remark 1.2.53 (2) yields the inequality

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) \leq \max\left\{ \operatorname{reg}\left(\frac{R}{I(P_{n-3} \cup P_{m-1})}\right) + 1, \operatorname{reg}\left(\frac{R}{I(C_n \cup (e' \cdot C_m))}\right)\right\}.$$

From Proposition 3.1.3 and [3, Lemma 3.2], it also follows that

$$\operatorname{reg}(I(e' \cdot C_m)) = \lfloor \frac{m}{3} \rfloor + \lfloor \frac{3 - \xi_3(m)}{3} \rfloor + 1.$$

Thus, using Remark 3.1.9, [74, Lemma 3.2] and Theorem 1.2.32, we get

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) \leq \max\left\{ \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1, \left\lfloor \frac{n-2}{3} \right\rfloor + 1 + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{3 - \xi_3(m)}{3} \right\rfloor \right\}.$$

On the other hand, from Theorem 3.1.4 we have that

$$\nu(C_n \cdot P_3 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{4 - \xi_3(n) - \xi_3(m)}{3} \rfloor.$$

Therefore, we can check that $\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) \leq \nu(C_n \cdot P_3 \cdot C_m) + 1$ when $n, m \equiv 2$ (mod 3), and that $\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) = \nu(C_n \cdot P_3 \cdot C_m)$ in all the remaining cases. ☐

Theorem 3.1.16. *Let $n, m \geq 3$. Then*

$$\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_3 \cdot C_m) + 2 & \text{if } n, m \equiv 2 \ (mod\ 3), \\ \nu(C_n \cdot P_3 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Using Proposition 3.1.15, we only need to prove that $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \geq \nu(C_n \cdot P_3 \cdot C_m) + 2$ in the case $n, m \equiv 2 \pmod 3$. Hence, we assume $n, m \equiv 2 \pmod 3$. Let z_2 be the middle vertex of $C_n \cdot P_3 \cdot C_m$. By deleting z_2 we see that $H = (C_n \cdot P_3 \cdot C_m) \setminus z_2 = C_n \cup C_m$ is an induced subgraph of $C_n \cdot P_3 \cdot C_m$. From Theorem 1.2.42 and [74, Lemma 3.2], we have that

$$\operatorname{reg} I(H) = \operatorname{reg} I(C_n) + \operatorname{reg} I(C_m) - 1 = \nu(C_n) + \nu(C_m) + 3.$$

Since $\nu(C_n \cdot P_3 \cdot C_m) = \nu(C_n) + \nu(C_m) + 1$, then using Corollary 1.2.50 we get

$$\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \geq \operatorname{reg} I(H) = \nu(C_n \cdot P_3 \cdot C_m) + 2.$$

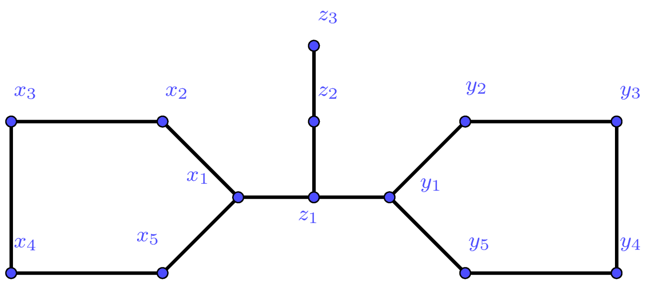
□

3.2 Combinatorial characterization of $\operatorname{reg}(I(G))$ in terms of $\nu(G)$

Let G be a general bicyclic graph. Then its decycling number (see Definition 1.2.35) is smaller or equal than 2, and so from Theorem 1.2.34 and Theorem 1.2.36, we get

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 3.$$

Example 3.2.1. The following graph G



has regularity $\operatorname{reg} I(G) = 6$ and induced matching number $\nu(G) = 3$.

In this section, we give a combinatorial characterization of the bicyclic graphs with regularity $\nu(G)+1$, $\nu(G)+2$ and $\nu(G)+3$. For the rest of this chapter, we shall use the term "the dumbbell" of the bicyclic graph G, and it denotes the unique subgraph of G of the form $C_n \cdot P_l \cdot C_m$. The theorem below contains the characterization that we found.

Theorem 3.2.2. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold:*

(I) If $n, m \equiv 0, 1$ (mod 3), then

$$\operatorname{reg} I(G) = \nu(G) + 1.$$

(II) If $n \equiv 0, 1$ (mod 3) and $m \equiv 2$ (mod 3), then

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2,$$

and $\operatorname{reg} I(G) = \nu(G) + 2$ if and only if $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

(III) If $n, m \equiv 2$ (mod 3) and $l \geq 3$, then

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 3.$$

Moreover:

(i) $\operatorname{reg} I(G) = \nu(G) + 3$ if and only if $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$.

(ii) $\operatorname{reg} I(G) = \nu(G) + 1$ if and only if the following conditions hold:

(a) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;

(b) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;

(c) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

(IV) If $n, m \equiv 2$ (mod 3) and $l \leq 2$, then

$$\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2.$$

Moreover, if x is an edge on P_l and if $\mathcal{L}_x(G)$ is the Lozin transformation of G with respect to x, then $\operatorname{reg} I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:

(a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;

(b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;

(c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. This statement summaries our work below. In particular, statement *(I)* follows from Proposition 3.2.4. In Theorem 3.2.13, *(II)* is proved. By Theorem 3.2.18 and Theorem 3.2.23, we get *(III)*. Finally, from Corollary 3.2.24, we obtain *(IV)*. □

The following simple remark will be crucial in our treatment.

Remark 3.2.3. [3, Observation 2.1] Let G be a graph with a leaf y and its unique neighbor x, say $e = \{x, y\}$. If $\{e_1, \ldots, e_s\}$ is an induced matching in $G \setminus N[x]$, then $\{e_1, \ldots, e_s, e\}$ is an induced matching in G. So we have

$$\nu(G \setminus N[x]) + 1 \leq \nu(G).$$

Proposition 3.2.4. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold:*

(i) When $n, m \equiv 0, 1$ (mod 3), we have $\operatorname{reg} I(G) = \nu(G) + 1$.

(ii) When $n \equiv 0, 1$ (mod 3) and $m \equiv 2$ (mod 3), we have $\operatorname{reg} I(G) \leq \nu(G) + 2$.

(iii) When $l \leq 2$, we have $\operatorname{reg} I(G) \leq \nu(G) + 2$.

Proof. (i) Again, it is enough to prove the upper bound $\operatorname{reg} I(G) \leq \nu(G)+1$. Let E' be the set of edges $E' = E(G) \setminus E(C_n \cdot P_l \cdot C_m)$. We proceed by induction on the cardinality of E'. If $|E'| = 0$ then the statement follows from Theorem 3.1.6, so we assume $|E'| > 0$. There exists a leaf y in G such that $N[y] = \{x\}$. Let $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Then by Remark 1.2.53 we have

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

The graphs G' and G'' can be either bicyclic graphs with the same dumbbell $C_n \cdot P_l \cdot C_m$, or unicyclic graphs with a circle C_r ($r = n$ or $r = m$) of the type $r \equiv 0, 1$ (mod 3), or forests. Using either the induction hypothesis, or [3, Theorem 1.1] and Theorem 1.2.54, then we get $\operatorname{reg} I(G') = \nu(G')+1$ and $\operatorname{reg} I(G'') = \nu(G'') + 1$. Since we have $\nu(G') \leq \nu(G)$ and $\nu(G'') + 1 \leq \nu(G)$ (by Remark 3.2.3), then we obtain the required inequality.

(ii) and (iii) follow by the same inductive argument, only changing the fact that G' and G'' could be unicyclic graphs with cycle C_r of the type $r \equiv 2$ (*mod* 3). □

Remark 3.2.5. The inductive process of the previous proposition cannot conclude $\operatorname{reg} I(G) \leq \nu(G) + 2$ in the case $l \geq 3$. Here we may encounter two disjoint subgraphs G_1 and G_2 with $\operatorname{reg} I(G_i) = \nu(G_i) + 2$, which implies $\operatorname{reg} I(G_1 \cup G_2) = \nu(G_1 \cup G_2) + 3$. This is exactly the case of Example 3.2.1.

An alternative proof of the inequality $\operatorname{reg} I(G) \leq \nu(G) + 3$ can be given by using the same inductive technique of Proposition 3.2.4.

For the rest of this chapter we shall use the following notation.

Notation 3.2.6. Let G be a graph, $H \subset G$ be a subgraph, and v and w be vertices of G. Then, we assume the following:

(i) $d(v, w)$ denotes the length (i.e., the number of edges) of a minimal path between v and w. In particular, $d(v, v) = 0$.

(ii) $d(v, H)$ denotes the minimal distance from the vertex v to the subgraph H, that is
$$d(v, H) = \min\{d(v, w) \mid w \in H\}.$$
In particular, $d(v, H) = 0$ if and only if $v \in H$.

(iii) Let $H^{'} \subset G$ be a subgraph, then the distance between H and $H^{'}$ is given by

$$d(H, H^{'}) = \min\{d(v, H^{'}) \mid v \in H\}.$$

In particular, $d(H, H^{'}) = 0$ if and only if $H \cap H^{'} \neq \emptyset$.

(iv) $\Gamma_G(H)$ denotes the subset of vertices

$$\Gamma_G(H) = \{v \in G \mid d(v, H) = 1\}.$$

(v) In the case $k > 0$, $S_{G,k}(H)$ denotes the induced subgraph given by restricting to the vertex set

$$V(S_{G,k}(H)) = \{v \in G \mid d(v, H) \geq k\}.$$

(vi) $S_{G,0}$ denotes the subgraph given by the vertex set

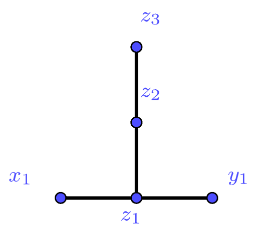
$$V(S_{G,0}(H)) = \{v \in G \mid d(v, H) > 0 \text{ or } \deg(v) \geq 3\}.$$

and the edge set

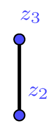
$$E(S_{G,0}(H)) = \{(v, w) \in E(G) \mid v, w \in V(S_{G,0}(H))\} \setminus \{(v, w) \in E(G) \mid v, w \in H\}.$$

We clarify the previous notation in the following example.

Example 3.2.7. (i) Let G be the graph of Example 3.2.1 and $H = C_5 \cup C_5$ be the subgraph given by the two cycles of length 5. Then, we have that $\Gamma_G(H)$ is the set containing the vertex in the middle of the bridge joining the two circles, that $S_{G,0}(H)$ is a graph of the form

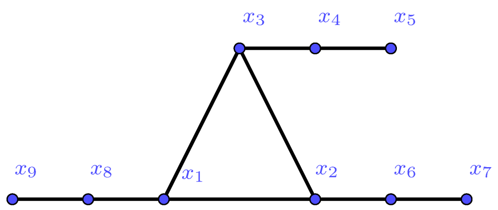


and that the graph

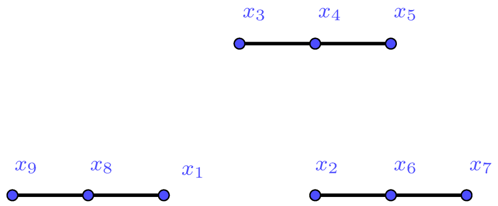


represents $S_{G,2}(H)$.

(ii) Let G be the graph given by



and H be the triangle induced by the vertices $\{x_1, x_2, x_3\}$. Then, we have that $\Gamma_G(H) = \{x_4, x_6, x_8\}$, that $S_{G,0}(H)$ is a graph of the form



and that the graph

represents $S_{G,2}(H)$.

We have already computed $\operatorname{reg} I(G)$ in the case $n, m \equiv 0, 1 \pmod 3$. For the remaining cases we shall divide this section into subsections.

3.2.1 Case I

In this subsection we shall focus on the case $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. This case turns out to be almost identical to a unicyclic graph, and our treatment is influenced by [3, Section 3].

Notation 3.2.8. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. We shall denote by $F_1, \ldots, F_c$ the connected components of $S_{G,0}(C_m)$, and in this case each F_i is either a tree or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod 3$). Then, the graph $S_{G,2}(C_m)$ can be given as the union of the components $H_1, \ldots, H_c$, where each one is defined as

$$H_i = F_i \setminus \{v \in G \mid d(v, C_m) \leq 1\}.$$

We note that each H_i can be a non-connected graph or even the empty graph.

Remark 3.2.9. The following statements hold.

(i) The graph $G \setminus \Gamma_G(C_m)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_m) = C_m \bigcup \left(\bigcup_{i=1}^{c} H_i \right),$$

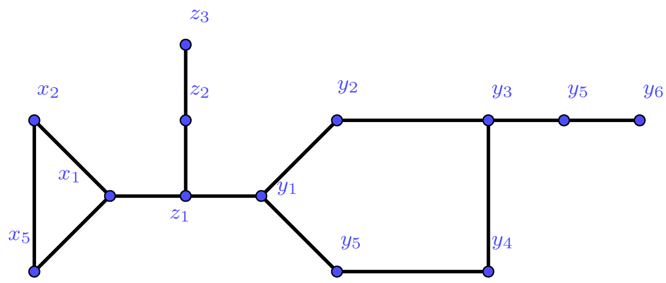
and in particular

$$\nu \left(G \setminus \Gamma_G(C_m) \right) = \nu \left(C_m \right) + \sum_{i=1}^{c} \nu \left(H_i \right)$$

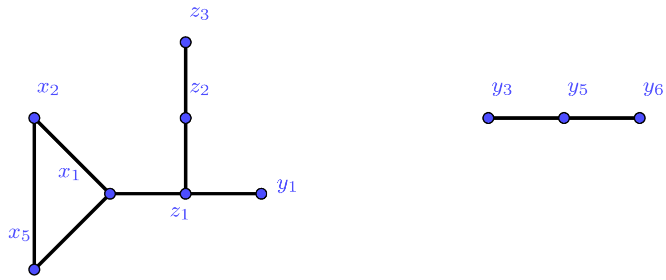
because $d(C_m, H_i) \geq 2$ for all $1 \leq i \leq c$ and $d(H_i, H_j) \geq 2$ for all $1 \leq i < j \leq c$.

(ii) For each $i = 1, \ldots, c$, we have that $|F_i \cap C_m| = 1$.

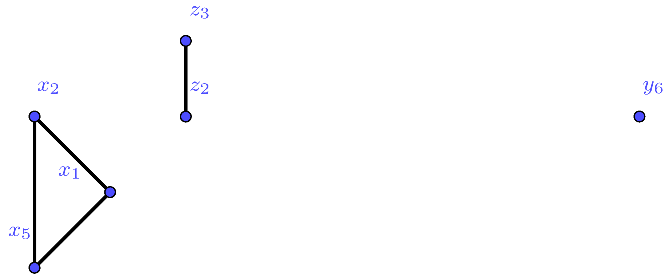
Example 3.2.10. Let G be the graph



and C_5 be the cycle given by $\{y_1, y_2, y_3, y_4, y_5\}$. We have that $\Gamma_G(C_5) = \{z_1, y_5\}$. The graph $S_{G,0}(C_5)$ is given by



with connected components $F_1 = \{y_1, z_1, z_2, z_3, x_1, x_2, x_5\}$ and $F_2 = \{y_3, y_4, y_5\}$. The graph $S_{G,2}(C_5)$ is given by



and following our notations we have $H_1 = \{x_1, x_2, x_5, z_2, z_3\}$ and $H_2 = \{y_6\}$.

Lemma 3.2.11. *Adopt Notation 3.2.8. If* $\nu(H_i) = \nu(F_i)$ *for all* $1 \leq i \leq c$, *then* $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$.

Proof. Follows identically to [3, Lemma 3.5]. □

Proposition 3.2.12. *Adopt Notation 3.2.8. If* $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$, *then* $\operatorname{reg} I(G) = \nu(G) + 1$.

Proof. Once more, we shall only prove that $\operatorname{reg} I(G) \leq \nu(G) + 1$. Assume that $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$. Then the contrapositive of Lemma 3.2.11 implies that there exists some i with $\nu(H_i) < \nu(F_i)$.

Fix i such that $\nu(H_i) < \nu(F_i)$. From Remark 3.2.9 (*ii*), let x be the vertex in $F_i \cap C_m$. Let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Again, we have the inequality

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

Note that both G' and G'' can be either unicyclic graphs with cycle C_n (and $n \equiv 0, 1 \pmod 3$), or forests. Hence, from [3, Theorem 1.1] and Theorem 1.2.32, we get that $\operatorname{reg} I(G') = \nu(G') + 1$ and $\operatorname{reg} I(G'') = \nu(G'') + 1$.

In the case of G', we have that $\operatorname{reg} I(G') = \nu(G') + 1 \leq \nu(G) + 1$. Let H be the induced subgraph of G obtained by deleting the vertices of $F_i \cup N_G[x]$. Then we have $G'' = H \cup H_i$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be maximal induced matchings in H and H_i, respectively, then $\nu(G'') = |\mathcal{M}_1| + |\mathcal{M}_2|$ because $d(H, H_i) \geq 2$. By the condition $\nu(F_i) > \nu(H_i)$, then there exists a maximal induced matching $\mathcal{M}_3$ in F_i, such that $|\mathcal{M}_3| > |\mathcal{M}_2|$. From the fact that $H \cup F_i$ is an induced subgraph in G and $H \cap F_i = \emptyset$, we then get

$$\nu(G) \geq \nu(H \cup F_i) = |\mathcal{M}_1| + |\mathcal{M}_3| > |\mathcal{M}_1| + |\mathcal{M}_2| = \nu(G'').$$

Hence $\operatorname{reg} I(G'') = \nu(G'') + 1 \leq \nu(G)$, and so we get the statement of the proposition. □

Theorem 3.2.13. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$ *such that* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$. *Then the following statements hold.*

(i) $\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2$;

(ii) $\operatorname{reg} I(G) = \nu(G) + 2$ *if and only if* $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

Proof. In Proposition 3.2.4 we proved (*i*). In order to prove (*ii*), we only need to show that $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$ implies $\operatorname{reg} I(G) \geq \nu(G) + 2$, because the reverse inequality follows from Proposition 3.2.12.

From Remark 3.2.9 (i), $G \setminus \Gamma_G(C_m) = C_m \cup (\cup_{i=1}^c H_i)$ where each H_i is either a forest or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod 3$). Then, from [3, Theorem 1.1] and Theorem 1.2.32 we get

$$\begin{aligned}
\operatorname{reg} I(G \setminus \Gamma_G(C_m)) &= \operatorname{reg} I(C_m) + \operatorname{reg} I(\bigcup_{i=1}^{c} H_i) - 1 \\
&= (\nu(C_m) + 2) + (\nu(\bigcup_{i=1}^{c} H_i) + 1) - 1 \\
&= \nu(G \setminus \Gamma_G(C_m)) + 2 \\
&= \nu(G) + 2.
\end{aligned}$$

Finally, since $G \setminus \Gamma_G(C_m)$ is an induced subgraph of G then we have $\operatorname{reg} I(G) \geq \nu(G) + 2$. □

3.2.2 Case II

The object of study of this subsection is the case $n, m \equiv 2 \pmod 3$, $l \geq 3$, and in particular, when $\operatorname{reg} I(G) = \nu(G) + 3$. More specifically, we shall give necessary and sufficient conditions for the equality $\operatorname{reg} I(G) = \nu(G) + 3$.

Notation 3.2.14. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geq 3$. As in Notation 3.2.8, let $F_1, \ldots, F_c$ be the components of the graph $S_{G,0}(C_n)$. We order the F_i's in such a way that F_1 is a unicyclic graph with cycle C_m, and for all $i > 1$ we have that F_i is a tree. The graph $S_{G,2}(C_n)$ can be decomposed into components $H_1, \ldots, H_c$ where

$$H_i = F_i \setminus \{v \in G \mid d(v, C_n) \leq 1\}.$$

Remark 3.2.15. From the previous notation, we get the following simple remarks.

(i) The graph $G \setminus \Gamma_G(C_n)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_n) = C_n \cup \left(\bigcup_{i=1}^{c} H_i \right),$$

and in particular

$$\nu\left(G \setminus \Gamma_G(C_n)\right) = \nu\left(C_n\right) + \sum_{i=1}^{c} \nu\left(H_i\right)$$

because $d(C_n, H_i) \geq 2$ for all $1 \leq i \leq c$ and $d(H_i, H_j) \geq 2$ for all $1 \leq i < j \leq c$.

(ii) The graph $G \setminus \Gamma_G(C_n \cup C_m)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_n \cup C_m) = C_n \cup \left(\bigcup_{i=2}^{c} H_i \right) \cup \left(H_1 \setminus \Gamma_{H_1}(C_m)\right),$$

and in particular

$$\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)).$$

(iii) For each $i = 1, \ldots, c$, we have that $|F_i \cap C_n| = 1$.

(iv) The statement of Lemma 3.2.11 also holds in this case, that is, if $\nu(H_i) = \nu(F_i)$ for all $1 \leq i \leq c$, then $\nu(G \setminus \Gamma_G(C_n)) = \nu(G)$.

(v) Due to the assumption $l \geq 3$, then we have that C_m must be an induced subgraph of H_1. Throughout this subsection and the next one, we shall fundamentally use this fact. It will allow us to inductively "separate" the two cycles C_n and C_m.

Lemma 3.2.16. *Adopt Notation 3.2.14. If $\nu(H_i) = \nu(F_i)$ for all $1 \leq i \leq c$ and $\nu(H_1) = \nu(H_1 \setminus \Gamma_{H_1}(C_m))$, then*

$$\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G).$$

Proof. Since $G \setminus \Gamma_G(C_n \cup C_m)$ is an induced subgraph of G, then we have $\nu(G \setminus \Gamma_G(C_n \cup C_m)) \leq \nu(G)$. From Remark 3.2.15 *(ii)* we get

$$\begin{aligned}\nu(G \setminus \Gamma_G(C_n \cup C_m)) &= \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)) \\ &= \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1) \\ &= \nu(C_n) + \sum_{i=1}^{c} \nu(F_i) \\ &\geq \nu(G),\end{aligned}$$

and so $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$. □

Proposition 3.2.17. *Adopt Notation 3.2.14. If $\nu(G \setminus \Gamma_G(C_n \cup C_m)) < \nu(G)$, then*

$$\operatorname{reg} I(G) \leq \nu(G) + 2.$$

Proof. It follows from the contrapositive of Lemma 3.2.16, that there exists some i with $\nu(H_i) < \nu(F_i)$ or we have $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Then we divide the proof into two cases.

Case 1: In this case, we assume that for some $1 \leq i \leq c$ we have $\nu(H_i) < \nu(F_i)$. This case follows similarly to Proposition 3.2.12. Let x be the vertex in $F_i \cap C_n$. Let us use the notation $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Once more, we have the inequality

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

Note that both G' and G'' are unicyclic graphs, and so we have $\operatorname{reg} I(G') \leq \nu(G') + 2$ and $\operatorname{reg} I(G'') \leq \nu(G'') + 2$ (see Theorem 1.2.36). Since we have $\nu(G') \leq \nu(G)$ and $\nu(G'') + 1 \leq \nu(G)$ (see the proof of Proposition 3.2.12), then the inequality follows in this case.

Case 2: Now we suppose that $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Let x be the vertex in $F_1 \cap C_n$, and set $G' = G \setminus x$ and $G'' = G \setminus N[x]$. We use the inequality

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

The graphs G' and G'' are unicyclic. For the graph G' we have $\operatorname{reg} I(G') \leq \nu(G') + 2 \leq \nu(G) + 2$. The graph G'' can be given as the disjoint union of H_1 and another graph H defined by $H = G \setminus (F_1 \cup N[x])$, that is, $G'' = H \cup H_1$ and $H \cap H_1 = \emptyset$. Since H is a forest, then using [3, Theorem 1.1] we obtain that $\operatorname{reg} I(G'') \leq \nu(G'') + 1$. So we get the inequality $\operatorname{reg} I(G'') + 1 \leq \nu(G'') + 2 \leq \nu(G) + 2$, because G'' is an induced subgraph of G. □

Now we are ready to completely describe the case where $\operatorname{reg} I(G) = \nu(G) + 3$.

Theorem 3.2.18. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. Then $\operatorname{reg} I(G) = \nu(G) + 3$ if and only if the following conditions are satisfied:*

(i) $n \equiv 2 \pmod 3$*;*

(ii) $m \equiv 2 \pmod 3$*;*

(iii) $l \geq 3$*; and*

(iv) $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$*.*

Proof. In Proposition 3.2.4 we proved that the conditions (i), (ii) and (iii) are necessary, and from Proposition 3.2.17 we have that the condition (iv) is also necessary. Hence, we only need to prove that $\operatorname{reg} I(G) = \nu(G) + 3$ under these conditions.

From Remark 3.2.15, and using [3, Theorem 1.1] and Theorem 1.2.32, we can compute

$$\begin{aligned}
\operatorname{reg}\Big(I(G \setminus \Gamma_G(C_n \cup C_m))\Big) &= \operatorname{reg}\Big(I(C_n)\Big) + \operatorname{reg}\Big(I\Big(\bigcup_{i=2}^{c} H_i\Big)\Big) + \\
&\quad + \operatorname{reg}\Big(I\Big(H_1 \setminus \Gamma_{H_1}(C_m)\Big)\Big) - 2 \\
&= (\nu(C_n) + 2) + (\nu(\bigcup_{i=2}^{c} H_i) + 1) + \\
&\quad + (\nu(H_1 \setminus \Gamma_{H_1}(C_m)) + 2) - 2 \\
&= \nu(G \setminus \Gamma_G(C_n \cup C_m)) + 3 \\
&= \nu(G) + 3.
\end{aligned}$$

Since $G \setminus \Gamma_G(C_n \cup C_m)$ is an induced subgraph of G, then we get

$$\operatorname{reg} I(G) \geq \operatorname{reg} I(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G) + 3,$$

and so, from Theorem 1.2.36, the equality it is obtained. □

3.2.3 Case III

In this subsection we assume that G is a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geq 3$. Now that we have characterized when $\operatorname{reg} I(G) = \nu(G) + 3$, we now want to distinguish between $\operatorname{reg} I(G) = \nu(G) + 1$ and $\operatorname{reg} I(G) = \nu(G) + 2$.

Lemma 3.2.19. *Adopt Notation 3.2.14. If $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) = 1$, then*

$$\operatorname{reg} I(G) = \nu(G) + 2.$$

Proof. From Theorem 3.2.18 we have that $\operatorname{reg}(I(G)) \leq \nu(G) + 2$. Using the same method as in Theorem 3.2.18, we can obtain a lower bound

$$\operatorname{reg} I(G) \geq \operatorname{reg} I(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G \setminus \Gamma_G(C_n \cup C_m)) + 3 = \nu(G) + 2,$$

and so the equality follows. □

Lemma 3.2.20. *Adopt Notation 3.2.14. If $\nu(G) = \nu(G \setminus \Gamma_G(C_n))$, then*

$$\operatorname{reg} I(G) \geq \nu(G) + 2.$$

Symmetrically, the same argument holds for C_m.

Proof. The proof follows similarly to Theorem 3.2.13. From Remark 3.2.15 (i), [3, Theorem 1.1] and Theorem 1.2.32 we get

$$\begin{aligned}
\operatorname{reg} I(G \setminus \Gamma_G(C_n)) &= \operatorname{reg} I(C_n) + \operatorname{reg} I\Big(\bigcup_{i=1}^{c} H_i\Big) - 1 \\
&= (\nu(C_n) + 2) + \Big(\nu\Big(\bigcup_{i=1}^{c} H_i\Big) + 1\Big) - 1 \\
&= \nu(G \setminus \Gamma_G(C_n)) + 2 \\
&= \nu(G) + 2.
\end{aligned}$$

So the inequality follows from the fact that $G \setminus \Gamma_G(C_n)$ is an induced subgraph of G. □

The following very simple logical argument will be used several times in the next theorem.

Observation 3.2.21. *Let P_1, P_2, P_3 be boolean values, (i.e., true or false). Assume that P_1 is true if and only if P_2 and P_3 are true, that is*

$$P_1 \iff (P_2 \quad \wedge \quad P_3)\,.$$

Suppose that if P_2 is true, then P_3 is false, that is

$$P_2 \Longrightarrow \neg P_3.$$

Then, P_1 is false.

Notation 3.2.22. Let X be a mathematical expression. Then, $P[X]$ represents a boolean value, which is true if X is satisfied and false otherwise.

Taking into account the induced matching numbers $\nu(G)$, $\nu(G \setminus \Gamma_G(C_n \cup C_m))$, $\nu(G \setminus \Gamma_G(C_n))$ and $\nu(G \setminus \Gamma_G(C_m))$, we can give necessary and sufficient conditions for the equality $\operatorname{reg} I(G) = \nu(G) + 1$.

Theorem 3.2.23. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geq 3$. Then $\operatorname{reg} I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:*

(i) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;

(ii) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;

(iii) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

Proof. From Lemma 3.2.19 and Lemma 3.2.20, we have that the conditions (i), (ii) and (iii) are necessary. Hence, it is enough to prove $\operatorname{reg} I(G) \leq \nu(G) + 1$ under these conditions.

Again, for any $x \in G$ we let $G' = G \setminus x$ and $G'' = G \setminus N[x]$, and we have the upper bound

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G'), \operatorname{reg} I(G'') + 1\}.$$

We shall prove that under the conditions (i), (ii) and (iii) there exists a vertex $x \in C_n$ such that $\operatorname{reg} I(G') \leq \nu(G) + 1$ and $\operatorname{reg} I(G'') + 1 \leq \nu(G) + 1$. We divide the proof into three steps.

Step 1. In this step we prove that for any $x \in C_n$ we have $\operatorname{reg} I(G') \leq \nu(G) + 1$. First we note the following two statements:

- From Theorem 1.2.36 we have that $\operatorname{reg} I(G') \leq \nu(G') + 2$. Hence, $\nu(G') < \nu(G)$ implies that $\operatorname{reg} I(G') \leq \nu(G') + 2 \leq \nu(G) + 1$.
- From [3, Theorem 1.1] we obtain that $\operatorname{reg} I(G') = \nu(G') + 2$ if and only if $\nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))$.

Thus, it follows that

$$\operatorname{reg} I(G') = \nu(G) + 2 \iff \Big(\nu(G) = \nu(G') \text{ and } \nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))\Big).$$

In Observation 3.2.21, let $P_1 = P\Big[\operatorname{reg} I(G') = \nu(G) + 2\Big]$,

$$P_2 = P\Big[\nu(G) = \nu(G')\Big] \text{ and } P_3 = \Big[\nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))\Big].$$

From the logical argument of Observation 3.2.21, if we prove that $\nu(G') = \nu(G)$ implies $\nu(G') > \nu(G' \setminus \Gamma_{G'}(C_m))$, then we will get the required inequality $\operatorname{reg} I(G') \leq \nu(G) + 1$. Assume that $\nu(G) = \nu(G')$. From the hypothesis $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$ and the fact that $G' \setminus \Gamma_{G'}(C_m)$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$, then we get

$$\nu(G') = \nu(G) > \nu(G \setminus \Gamma_G(C_m)) \geq \nu(G' \setminus \Gamma_{G'}(C_m)).$$

Therefore, we have $\operatorname{reg} I(G') \leq \nu(G) + 1$.

Step 2. Since $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$, it follows from Remark 3.2.15 (*iv*) that there exists some $1 \leq i \leq c$ such that $\nu(F_i) > \nu(H_i)$. Following Notation 3.2.14, we have that F_1 is a unicyclic graph containing the cycle C_m and that F_i is a tree for all $i > 1$. In this step, assume $i > 1$ where F_i is a tree and $\nu(F_i) > \nu(H_i)$.

Let x be the vertex in $F_i \cap C_n$ and H be the induced subgraph $H = G \setminus (F_i \cup N_G[x])$. Note that $G'' = H \cup H_i$, $d(H, H_i) \geq 2$ and $d(H, F_i) \geq 2$. Then

$$\nu(G'') = \nu(H) + \nu(H_i) < \nu(H) + \nu(F_i) \leq \nu(G)$$

follows from the condition $\nu(H_i) < \nu(F_i)$. So we have that $\nu(G'') < \nu(G)$.

Let K be the induced subgraph defined by $K = (G \setminus \Gamma_G(C_m)) \setminus (F_i \cup N[x])$. Since $i > 1$, then $F_i \cap F_1 = \emptyset$, and so we get the following statements:

- $G'' \setminus \Gamma_{G''}(C_m) = K \cup H_i$.
- $K \cup F_i$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$.
- We have the following inequalities

$$\nu(G'' \setminus \Gamma_{G''}(C_m)) = \nu(K) + \nu(H_i) < \nu(K) + \nu(F_i) \leq \nu(G \setminus \Gamma_G(C_m)).$$

Again, as in Step 1, [3, Theorem 1.1] and Theorem 1.2.36 yield the following equivalence

$$\operatorname{reg} I(G'') + 1 = \nu(G) + 2 \iff \Big(\nu(G) = \nu(G'') + 1 \text{ and } \nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))\Big).$$

In Observation 3.2.21, let $P_1 = P\Big[\operatorname{reg} I(G'') + 1 = \nu(G) + 2\Big]$, $P_2 = P\Big[\nu(G) = \nu(G'' + 1)\Big]$ and $P_3 = P\Big[\nu(G'') = \nu(G'' \setminus \Gamma_{G'}(C_m))\Big]$. So it is enough to prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$. Assuming $\nu(G) = \nu(G'') + 1$, then we can get

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_m)) - 1 \geq \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Therefore, in this case we have $\operatorname{reg} I(G'') + 1 \le \nu(G) + 1$.

Step 3. In this last step we assume that $\nu(F_1) > \nu(H_1)$ and that $\nu(F_i) = \nu(H_i)$ for all $i > 1$. Let x be the vertex in $F_1 \cap C_n$. Then as in Step 2 we have the statements:

- $\nu(G'') < \nu(G)$.
- $\operatorname{reg} I(G'') + 1 = \nu(G) + 2 \iff \Big(\nu(G) = \nu(G'') + 1$ and $\nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))\Big)$.

Once more, if we prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$, then we obtain that $\operatorname{reg} I(G'') + 1 \le \nu(G) + 1$.

We denote by L the induced subgraph of $G'' \setminus \Gamma_{G''}(C_m)$ given by disconnecting all the trees F_i with $i > 1$, that is

$$L = (G'' \setminus \Gamma_{G''}(C_m)) \setminus \Gamma_G(C_n).$$

From the conditions $\nu(F_i) = \nu(H_i)$ for all $i > 1$, then we get $\nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m))$ (see the proofs of Lemma 3.2.11 or Lemma 3.2.16). We also have that L is an induced subgraph of $G \setminus \Gamma_G(C_n \cup C_m)$ because we have the equality

$$L = (G \setminus \Gamma_G(C_n \cup C_m)) \setminus N[x].$$

Finally, from the hypothesis $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$ we can obtain

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_n \cup C_m)) \ge \nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Therefore, in this case we also have $\operatorname{reg} I(G'') + 1 \le \nu(G) + 1$.

□

3.2.4 Case IV

In this short subsection we deal with the remaining case. We assume G is a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \le 2$.

When $l \le 2$, the two circles are too close to each other, and it is difficult to make a direct analysis (with our methods). Fortunately, with the complete characterization of the case $l \ge 3$, then the problem can be solved with the Lozin transformation. Suppose that x is one vertex on the bridge P_l (at most two), then we can apply the Lozin transformation of G with respect to x, and this can give a bicyclic graph $\mathcal{L}_x(G)$ with dumbbell of the type $C_n \cdot P_k \cdot C_m$ where $k \ge 4$. From [80, Lemma 1] and [16, Theorem 1.1] we get the equality

$$\operatorname{reg}\left(I(\mathcal{L}_x(G))\right) - \nu\left(\mathcal{L}_x(G)\right) = \operatorname{reg}\left(I(G)\right) - \nu\left(G\right), \tag{3.4}$$

therefore we get a characterization in the following corollary.

Corollary 3.2.24. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \leq 2$. Let x be a point on the bridge P_l and let $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to x. Then we have that $\nu(G) + 1 \leq \operatorname{reg} I(G) \leq \nu(G) + 2$, and that $\operatorname{reg} I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:*

(i) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;

(ii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;

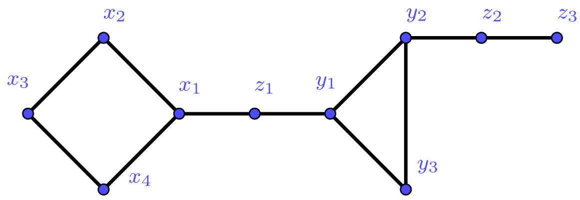
(iii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. It follows from Proposition 3.2.4, Equation 3.4, and Theorem 3.2.23. □

3.2.5 Examples

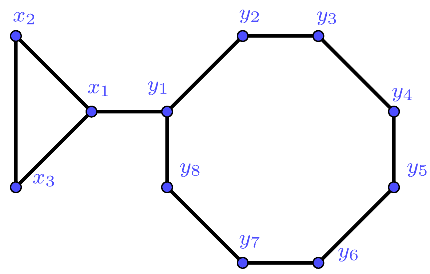
In this last subsection we shall give examples for each one of the statements in the characterization of Theorem 3.2.2.

Example 3.2.25. Statement (I) of Theorem 3.2.2. Let G be the graph below.



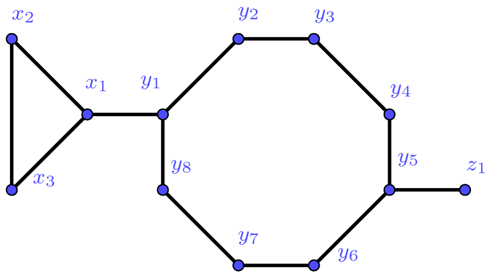
Then we have $\operatorname{reg} I(G) = 4$ and $\nu(G) = 3$.

Example 3.2.26. Statement (II) of Theorem 3.2.2. Let G be the graph below.



Then we have $\operatorname{reg} I(G) = 5$ and $\nu(G) = 3$.

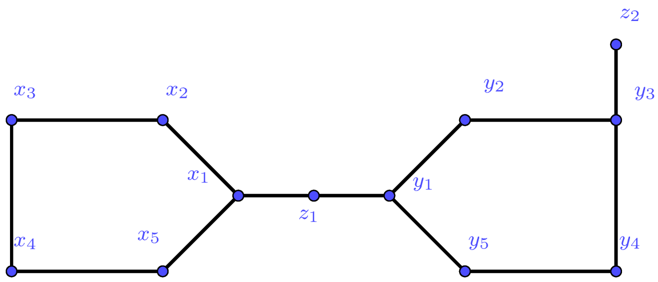
On the other hand, let G be the graph below.



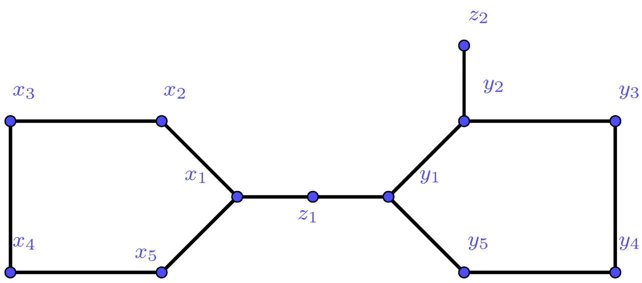
Then we have $\operatorname{reg} I(G) = 5$ and $\nu(G) = 4$.

Example 3.2.27. Statement (III) of Theorem 3.2.2. In Example 3.2.1 we saw a graph G where $\operatorname{reg} I(G) = 6$ and $\nu(G) = 3$.

Let G be the graph below.

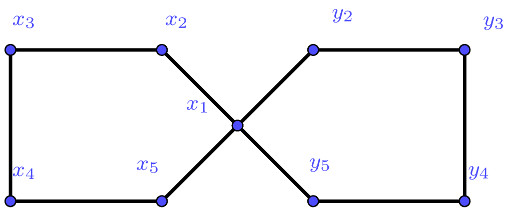


Then we have $\operatorname{reg} I(G) = 5$ and $\nu(G) = 3$. If we move the outer edge to the left, then we get a different result. Let G be the graph below.

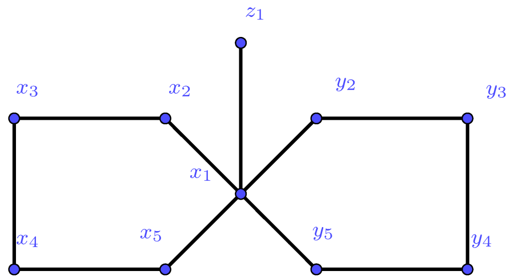


Then we have $\operatorname{reg} I(G) = 5$ and $\nu(G) = 4$.

Example 3.2.28. Statement (IV) of Theorem 3.2.2. Let G be the graph below.



Then we have $\operatorname{reg} I(G) = 4$ and $\nu(G) = 2$. By adding an edge, let G be the graph below.



Then we have $\operatorname{reg} I(G) = 4$ and $\nu(G) = 3$.

3.3 Castelnuovo-Mumford regularity of powers

In this section, we study the regularity of powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leq 2$. Our strategy is to obtain a lower bound and an upper bound for $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$, such that both coincide and are equal to $2q+ \operatorname{reg} I\ (C_n \cdot P_l \cdot C_m)$. To obtain the upper bound, we follow the argument of Banerjee from [10, Theorem 5.2]. To calculate the lower bound, we proceed by looking at "nice" induced subgraphs of $C_n \cdot P_l \cdot C_m$.

As a side result, we answer an interesting question on the behavior of the constant term of the asymptotycally linear regularity function. Let I be an arbitrary ideal generated in degree d and let $b_q := \operatorname{reg}(I^q) - dq$ for $q \geq 1$. An interesting question is the study the sequence $\{b_i\}_{i\geq 1}$. In [38] Eisenbud and Harris proved that if $\dim(R/I) = 0$, then $\{b_i\}_{i\geq 1}$ is a weakly decreasing sequence of non-negative integers. In [11] Banerjee, Beyarslan and Hà conjectured that for any edge ideal, $\{b_i\}_{i\geq 1}$ is a weakly decreasing sequence (see [11, Conjecture 7.11]). For the edge ideal of any dumbbell graph with $l \leq 2$, we prove $b_i = b_1$ for all $i \geq 1$. However, we expect $b_i \leq b_1$ for all $i \geq 1$ for any graph.

Remark 3.3.1. From Theorem 3.1.4 and Theorem 3.1.6, for any $l \leq 2$ we have that

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) \geq \lfloor \frac{n+m+l+1}{3} \rfloor.$$

The previous inequality is not satisfied when $l \geq 3$, because $\operatorname{reg} I(C_4 \cdot P_3 \cdot C_4)$ $= 3$ and $\lfloor \frac{4+4+3+1}{3} \rfloor = 4$.

We will use the notation of even-connection from Banerjee (see Theorem 1.2.56 and Definition 1.2.57). The following lemma is crucial in our treatment of the even-connected vertices, and its proof is similar to [10, Lemma 6.13].

Lemma 3.3.2. *Let G be a graph. As in Remark 1.2.60, let G' be the graph associated to $(I(G)^{q+1}\colon e_1\cdots e_q)^{pol}$. Suppose $u = p_0, p_1, \ldots, p_{2s+1} = v$ is a path that even-connects u and v with respect to the q-fold $e_1\cdots e_q$. Then we have*

$$\bigcup_{i=0}^{2s+1} N_{G'}[p_i] \subset N_{G'}[u] \cup N_{G'}[v].$$

Proof. Let U be the set of vertices $U = \{p_0, p_1, \ldots, p_{2s+1}\}$. For each $1 \leq k \leq s$ we have that $p_{2k-1}p_{2k} = e_{j_k}$ for some $1 \leq j_k \leq q$, i.e., u and v are even connected with respect to the s-fold $e_{j_1}e_{j_2}\cdots e_{j_s}$.

Let w be a vertex even-connected to some vertex $z \in U$ with respect to the q-fold $e_1\cdots e_q$. Then, there exists a path $z = r_0, r_1, \ldots, r_{2t+1} = w$ that even-connects z and w with respect to the q-fold $e_1\cdots e_q$. Let i be the largest integer such that $r_i \in U$. From the fact that $r_0 = z \in U$, we have that the integer i is well defined and $i \geq 0$. Let k be an integer such that $p_k = r_i$.

The proof is now divided into four different cases depending on $i \pmod 2$ and $k \pmod 2$. When i and k are both odd integers, we have that $r_i r_{i+1}$ is equal to some edge of $\{e_1, e_2, \ldots, e_q\}$ and that $p_{k-1}p_k$ is not equal to any edge of $\{e_{j_1}, e_{j_2}, \ldots e_{j_s}\}$. By the definition of i we have

$$\{r_{i+1}, r_{i+2}, \ldots, r_{2t+1}\} \cap U = \emptyset.$$

So, in this case, it follows that

$$u = p_0, \ldots, p_{k-1}, p_k = r_i, r_{i+1}, \ldots, r_{2t+1} = w$$

is a path that even-connects u and w with respect to the q-fold $e_1\cdots e_q$.

The other three cases follow in a similar way. □

Remark 3.3.3. Let $G = C_n \cdot P_l \cdot C_m$. If $(I(G)^{q+1}\colon e_1\cdots e_q)$ is not a square-free monomial ideal and G' is the associated graph, then there exist a vertex x_i which is even-connected to itself. Therefore G' has a leaf. By Lemma 3.3.2 one can see $N_{G'}[x_i]$ contains one of the two cycles. In particular, if we denote the leaf by e, then G'_e is an induced subgraph of a unicyclic graph.

Theorem 3.3.4. *Let $G = C_n \cdot P_l \cdot C_m$ with $l \leq 2$, and $I = I(G)$ be its edge ideal, then*

$$\operatorname{reg}(I^{q+1}\colon e_1\cdots e_q) \leq \operatorname{reg} I$$

for any $1 \leq q$ and any edges $e_1, \ldots, e_q \in E(G)$.

Proof. We split the proof into two cases.

Case 1. First, suppose $(I^{q+1}\colon e_1\cdots e_q)$ is a square-free monomial ideal. In this case $(I^{q+1}\colon e_1\cdots e_q) = I(G')$ where G' is a graph with $V(G) = V(G')$ and $E(G) \subseteq E(G')$. Let $E(G') = E(G) \cup \{a_1, \ldots, a_r\}$. By Remark 1.2.53, we have

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G'_{a_1}) + 1\}.$$

From Lemma 3.3.2, G'_{a_1} is obtained from G' by removing one of the cycles or deleting at least 6 vertices.

Suppose G'_{a_1} is obtained by removing one of the cycles. Without loss of generality assume that C_n is deleted. Then there exists a Hamiltonian path of length $\leq m$ when $l = 2$ and of length $\leq m - 1$ when $l = 1$. From Theorem 1.2.43 and Remark 3.3.1, if C_n has $n \geq 4$ vertices, then we have $\operatorname{reg} I(G'_{a_1}) \leq \operatorname{reg} I(G) - 1$. In the case $n = 3$, there is a Hamiltonian path of length $\leq m - 3$, and so Theorem 1.2.43 and Remark 3.3.1 again imply $\operatorname{reg} I(G'_{a_1}) \leq \operatorname{reg} I(G) - 1$.

Suppose G'_{a_1} is obtained by removing at least 6 vertices. Let H' be the graph given by deleting $N_G[a_1]$. From the assumption of deleting at least 6 vertices we have that $|H'| \leq |G| - 6 \leq n + m + l - 8$. We note that we can add two vertices to H' and connect them in such a way that we obtain a Hamiltonian path. Let H be a graph obtained by adding two vertices and certain edges connecting these two new vertices, such that H has a Hamiltonian path. Note that G'_{a_1} is an induced subgraph of H. Since $|H| \leq n + m + l - 6$, Theorem 1.2.43 yields

$$\operatorname{reg} I(H) \leq \lfloor \frac{n + m + l - 5}{3} \rfloor + 1 = \lfloor \frac{n + m + l + 1}{3} \rfloor - 1.$$

Applying Remark 3.3.1, we get

$$\operatorname{reg} I(G'_{a_1}) \leq \operatorname{reg} I(H) \leq \operatorname{reg} I(G) - 1.$$

Therefore

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G)\}.$$

In the same way, for any subgraph $H = G' \setminus \{a_1, \dots, a_i\}$, we have that

$$\operatorname{reg}(I(H_{a_{i+1}})) \leq \operatorname{reg}(I(G)) - 1.$$

So, we also obtain

$$\operatorname{reg} I(G' \setminus a_1) \leq \max\{\operatorname{reg} I(G' \setminus \{a_1, a_2\}), \operatorname{reg} I(G)\}.$$

By continuing this process, we get $\operatorname{reg} I(G') \leq \operatorname{reg} I(G)$.

Case 2. Suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is not square-free and G' is the graph associated to $(I^{q+1} \colon e_1 \cdots e_q)^{\text{pol}}$. Let $\{b_1, b_2, \dots, b_s\}$ be the subset of edges of $E(G') \setminus E(G)$ that are generated by square monomials , i.e., each b_i is a whisker.

From Remark 1.2.53 we have the inequality

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus b_1), 1 + \operatorname{reg} I(G'_{b_1})\}.$$

Remark 3.3.3 implies that one of the cycles is deleted from G'_{b_1}. Then there exists an edge $e \in G$ such that $d(e, G'_{b_1}) \geq 2$. So, for such an edge e we get that the disjoint union $G'_{b_1} \cup e$ is an induced subgraph of $G' \setminus b_1$. Thus, Remark 1.2.53 and [74, Lemma 3.2] yield that

$$\operatorname{reg}(I(G'_{b_1})) + 1 = \operatorname{reg}(I(G'_{b_1} \cup e)) \leq \operatorname{reg}(I(G')).$$

Therefore, we obtain that $\operatorname{reg} I(G') \leq \operatorname{reg} I(G' \setminus b_1)$.

By applying the same argument, it follows that

$$\begin{aligned}\operatorname{reg} I(G') \leq & \operatorname{reg} I(G' \setminus b_1) \leq \operatorname{reg} I(G' \setminus \{b_1, b_2\}) \leq \cdots \leq \\ \leq & \operatorname{reg} I(G' \setminus \{b_1, \ldots, b_s\}).\end{aligned}$$

Since the graph $G' \setminus \{b_1, \ldots, b_s\}$ has no whiskers, then Step 1 implies that

$$\operatorname{reg} I(G') \leq \operatorname{reg} I(G' \setminus \{b_1, \ldots, b_s\}) \leq \operatorname{reg} I(G).$$

Therefore, the proof is completed. □

Remark 3.3.5. The previous theorem is a generalization of a work done by Yan Gu in [52] for the case $l = 1$.

Theorem 3.3.6. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q \geq 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2,$$

for any $q \geq 1$.

Proof. Using the inequality $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geq 2q + \nu(C_n \cdot P_2 \cdot C_m) - 1$ of [14, Theorem 4.5], for the cases where $\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1$, we get the expected inequality. We now divide the proof into the cases $l = 1$ and $l = 2$.

Case 1. Let $l = 1$. We only need to focus on the case where $n, m \equiv 2 \pmod 3$. Let H be the induced subgraph of $C_n \cdot P_1 \cdot C_m$ mentioned in the proof of Theorem 3.1.8, i.e., $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\} = P_{n-1} \cdot C_m$. Using Theorem 3.1.4, Proposition 3.1.3, and the modularity $n, m \equiv 2 \pmod 3$, we can check that

$$\nu(H) = \nu(C_n \cdot P_1 \cdot C_m)$$

and that

$$\nu(H) = \nu(H \setminus \Gamma_H(C_m)).$$

From Theorem 3.1.8 and [3, Theorem 1.1] we get

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) = \nu(C_n \cdot P_1 \cdot C_m) + 2 = \nu(H) + 2 = \operatorname{reg} I(H).$$

Since H is an induced subgraph of $C_n \cdot P_1 \cdot C_m$, then from [3, Theorem 1.2] and [14, Corollay 4.3] we get the inequality

$$\begin{aligned}\operatorname{reg} I(C_n \cdot P_1 \cdot C_m)^q \geq & \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = \\ = & 2q + \operatorname{reg} I(C_n \cdot P_1 \cdot C_m) - 2.\end{aligned}$$

Case 2. Let $l = 2$. We only need to focus on the cases where $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. We take the same induced subgraph H as in Lemma 3.1.13. The induced subgraph $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ of $C_n \cdot P_2 \cdot C_m$ is given as the union of a path of length $n-1$ and the cycle C_m, i.e., $H = P_{n-1} \cup C_m$.

By Theorem 3.1.14, for the cases $n \equiv 0, 1 \ (\bmod 3)$ and $m \equiv 2 \ (\bmod 3)$, we have

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \nu(C_n \cdot P_2 \cdot C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2,$$

and from [3, Theorem 1.1] we have

$$\operatorname{reg} I(H) = \nu(H) + 2 = \nu(P_{n-1}) + \nu(C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Hence, we get $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \operatorname{reg} I(H)$. Finally, using [3, Theorem 1.2] and [14, Corollary 4.3], we get the inequality

$$\begin{aligned} \operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geq & \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = \\ = & 2q + \operatorname{reg} I(C_n \cdot P_2 \cdot C_m) - 2. \end{aligned}$$

Therefore, the proof is completed. □

Theorem 3.3.7. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geq 1$.

Proof. It follows by Theorem 3.3.4, Theorem 1.2.56 and Theorem 3.3.6. □

Remark 3.3.8. One may ask whether

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

always holds for given n, m, l and q. Unfortunately, this is not the case. In fact, it can be checked that

$$6 = \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5)^2 < 4 + \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5) - 2 = 7.$$

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