Received: 3 September 2021 | Revised: 28 January 2022 | Accepted: 17 February 2022

DOI: 10.1002/jcd.21837

RESEARCH ARTICLE

WILEY

Edge balanced star-hypergraph designs and vertex colorings of path designs

Paola Bonacini | **Lucia Marino**

Mathematics and Computer Science Department, University of Catania, Catania, Italy

Correspondence
Paola Bonacini, Dipartimento di Matematica e Informatica, Università di Catania, Viale A. Doria 6, Catania 95125, Italy.
Email: bonacini@dmi.unict.it

Funding information
Università degli Studi di Catania

Abstract

Let $K_v^{(3)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank 3, defined on a vertex set $X = \{x_1, ..., x_v\}$, so that $\mathcal{E}$ is the set of all triples of X. Let $H^{(3)} = (V, \mathcal{D})$ be a subhypergraph of $K_v^{(3)}$, which means that $V \subseteq X$ and $\mathcal{D} \subseteq \mathcal{E}$. We call *3-edges* the triples of V contained in the family $\mathcal{D}$ and *edges* the pairs of V contained in the 3-edges of $\mathcal{D}$, that we denote by $[x, y]$. A $H^{(3)}$-design Σ is called *edge balanced* if for any $x, y \in X$, $x \neq y$, the number of blocks of Σ containing the edge $[x, y]$ is constant. In this paper, we consider the star hypergraph $S^{(3)}(2, m+2)$, which is a hypergraph with m 3-edges such that all of them have an edge in common. We completely determine the spectrum of edge balanced $S^{(3)}(2, m+2)$-designs for any $m \geq 2$, that is, the set of the orders v for which such a design exists. Then we consider the case $m = 2$ and we denote the hypergraph $S^{(3)}(2, 4)$ by $P^{(3)}(2, 4)$. Starting from any edge-balanced $S^{(3)}\left(2, \frac{v+4}{3}\right)$, with $v \equiv 2 \bmod 3$ sufficiently big, for any $p \in \mathbb{N}$, $\left\lceil \frac{v}{2} \right\rceil \leq p \leq v$, we construct a $P^{(3)}(2, 4)$-design of order $2v$ with feasible set $\{2, 3\} \cup [p, v]$, in the context of proper vertex colorings such that no block is either monochromatic or polychromatic.

KEYWORDS

design, edge balanced, hypergraph, vertex coloring

1 | INTRODUCTION

Let $\lambda K_v^{(r)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank r, defined on a vertex set $X = \{x_1, ..., x_v\}$, so that $\mathcal{E}$ is the set of all subsets of r elements of X and all these sets have multiplicity λ. In this paper, we consider the case $r = 3$. We say that $H^{(3)} = (V, \mathcal{D})$ is a *subhypergraph* of $\lambda K_v^{(3)}$ if $V \subseteq X$ and $\mathcal{D} \subseteq \mathcal{E}$. This means that $H^{(3)}$ is a uniform hypergraph of rank 3. We call *3-edges* the triples of V contained in the family $\mathcal{D}$ and *edges* the pairs of V contained in the 3-edges of $\mathcal{D}$. Such pairs will be denoted by $[x, y]$.

An $H^{(3)}$*-decomposition* of $\lambda K_v^{(3)}$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a collection of hypergraphs all isomorphic to $H^{(3)}$ that partition the edge set of $\lambda K_v^{(3)}$. An $H^{(3)}$-decomposition is also called a $H^{(3)}$*-design of order* v *and index* λ and the elements of $\mathcal{B}$ are called *blocks*.

If $\Sigma = (X, \mathcal{B})$ is a $H^{(3)}$-design, for any $x \in X$ we call *degree of the vertex* x the number $d(x)$ of blocks of $\mathcal{B}$ containing x; for any $x, y \in X$, $x \neq y$, we call *degree of the edge* $[x, y]$ the number $d(x, y)$ of blocks of $\mathcal{B}$ containing the edge $[x, y]$.

Following the classical definition of balanced designs, it is possible to define *balanced* $H^{(3)}$*-designs.*

Definition 1.1. A $H^{(3)}$-design Σ is said to be *balanced* if the degree $d(x)$ of each vertex $x \in X$ is a constant.

In [18], generalizing this idea, the concept of *edge balanced* designs has been introduced.

Definition 1.2. A $H^{(3)}$-design is called *edge balanced* if for any $x, y \in X$, $x \neq y$, the degree $d(x, y)$ is constant.

We will call a balanced hypergraph design *vertex balanced*, to make a distinction with edge balanced hypergraph designs. The concept of balanced G-design, in the case that G is a graph, was introduced by Hell and Rosa in [20]. Later, a lot of work has been done in this field (see e.g., [2,4,5,6,7,10,11,12,21]) both for graph designs and hypergraph designs.

In this paper, we consider star-hypergraphs:

Definition 1.3. A hypergraph $(X, \mathcal{D})$ uniform of rank r is called a *star-hypergraph* if there exists $Y \subset X$ such that $E' \cap E'' = Y$ for any $E', E'' \in \mathcal{D}$. If $|Y| = c$ and $|E| = m$ for all $E \in \mathcal{D}$, we denote such a hypergraph by $S^{(k)}(c, (k - c)m + c)$ and Y is called *center* of the star-hypergraph.

Clearly any $S^{(3)}(1, 2m + 1)$-design is edge balanced of constant degree 1. In this paper, we consider $S^{(3)}(2, m + 2)$-designs and from now on we will take the index $\lambda = 1$. Answering also to a problem given in [18], in the first part of the paper we determine the spectrum of edge balanced $S^{(3)}(2, m + 2)$-designs for any $m \geq 2$, by showing the existence of a cyclic $S^{(3)}(2, m + 2)$-design for any admissible order v. This easily implies that, for any $m \geq 2$, every edge balanced $S^{(3)}(2, m + 2)$-design is also vertex balanced.

In the second part of the paper we consider the case $m = 2$. In this case, coherently with the notation used previously in other papers (see, e.g., [9,8,18]), the hypergraph $S^{(3)}(2, 4)$ will be denoted by $P^{(3)}(2, 4)$. Indeed, continuing the work done in [9], we will consider *Voloshin colorings* of $P^{(3)}(2, 4)$-designs. In general, given a $H^{(3)}$-design $\Sigma = (X, \mathcal{B})$, for some hypergraph $H^{(3)}$, a k-coloring of Σ is a map $\varphi : X \to C$, where C is a set of k colors. A k-coloring is *strict* if exactly k colors are used. From now on, we assume that all our colorings are strict. Motivated by Voloshin's works, it is possible consider these type of colorings:

- colorings such that any block of $\mathcal{B}$ contains at least two vertices of a common color; if Σ is colored in this way, we call it a $CH^{(3)}$-design;
- colorings such that any block of $\mathcal{B}$ contains at least two vertices of different colors; if Σ is colored in this way, we call it a $DH^{(3)}$-design;
- colorings for which Σ is, at the same time, a $CH^{(3)}$ and a $DH^{(3)}$-design; if Σ is colored in this way, we call it a $BH^{(3)}$-design.

In a $CH^{(3)}$-design a block is called *monochromatic* if all its vertices have the same color; in a $DH^{(3)}$-design a block is called *polychromatic* if any two of its vertices have different colors.

Given an $H^{(3)}$-design $\Sigma = (X, \mathcal{B})$, the *feasible set* of Σ is:

$$\Omega(\Sigma) = \{k \mid \exists \ \text{ a } k\text{-coloring of } \Sigma\}.$$

The system Σ is *uncolorable* if $\Omega(\Sigma) = \emptyset$. If Σ is colorable, the minimum and the maximum of $\Omega(\Sigma)$ are the *lower* and *upper chromatic number of* Σ and we denote them by, respectively, $\chi(\Sigma)$ and $\overline{\chi}(\Sigma)$. The feasible set is said to be *broken* if there exists an integer k such that $k \notin \Omega(\Sigma)$ and $i < k < j$ for some $i, j \in \Omega(\Sigma)$ and such an integer k is called a *gap*. In this paper, we will extend such concepts and notations to decompositions of subhypergraphs of the complete hypergraph $K_v^{(3)}$ in hypergraphs all isomorphic to some $H^{(3)}$.

The concept of gaps in feasible sets was introduced by L. Gionfriddo in [15,16,17] in the context of P_3-designs. In [1], gaps in the feasible set for P_4-designs are explored in the context of regular equicolourings. Colorings of Steiner systems, mainly STS, SQS, and $S(2, 4, v)$, have been considered in many papers (see, e.g., [13,14,19,22,23,24]), but the problem in such cases is open.

In [9], feasible sets of $BP^{(3)}(2, 4)$-designs have been studied, determining bounds for lower and upper chromatic numbers and proving the existence of $BP^{(3)}(2, 4)$-designs with the largest possible feasible set. Moreover, in [9] it is proved the existence of uncolorable $BP^{(3)}(2, 4)$-designs for any order $v \geq 28$.

In the second part of this paper, having as a starting point any edge-balanced $S^{(3)}\left(2, \frac{v+4}{3}\right)$-design of sufficiently high order v, with $v \equiv 2 \bmod 3$, we construct in Theorem 5.1 a $BP^{(3)}(2, 4)$-decomposition of the complete multipartite hypergraph $K_{v\times 2}^{(3)}$ (with v partite sets of cardinality 2) with broken feasible set and color classes having a precise correspondence with the partite sets. This general construction easily leads in Theorem 6.1 to $BP^{(3)}(2, 4)$-designs of order $2v$ and broken feasible set. Such a feasible set is of type $\{2, 3\} \cup [p, v]$ for any $p \in \mathbb{N}$, $\left\lceil \frac{v}{2} \right\rceil \leq p \leq v$, with v sufficiently high, where for any $a, b \in \mathbb{N}$, $a \leq b$, we set $[a, b] = \{i \in \mathbb{N} | a \leq i \leq b\}$.

At last let us fix some notation. If $\{x_1, \ldots, x_{m+2}\}$ is the set of vertices and the 3-edge set is

$$\{\{x_i, x_{m+1}, x_{m+2}\} \mid i = 1, \ldots, m\},$$

we denote the hypergraph $S^{(3)}(2, m+2)$ also by $[(x_{m+1}, x_{m+2}), x_1, \ldots, x_m]$.

2 | EDGE BALANCED $S^{(3)}(2, m+2)$ -DESIGNS

If $[(x_{m+1}, x_{m+2}), x_1, ..., x_m]$ is a $S^{(3)}(2, m+2)$, then we say that the edge $[x_{m+1}, x_{m+2}]$ occupies the *central position* and the other edges occupy *lateral positions*. Let $\Sigma = (X, \mathcal{B})$ be a $S^{(3)}(2, m+2)$-design and let $x, y \in X$, $x \neq y$. The *central degree* $C(x, y)$ of $[x, y]$ is the number of blocks of Σ containing the edge $[x, y]$ in the central position. The *lateral degree* $L(x, y)$ of $[x, y]$ is the number of blocks of Σ containing the edge $[x, y]$ in a lateral position. Then we prove that:

Theorem 2.1. *If* $\Sigma = (X, \mathcal{B})$ *is an edge balanced* $S^{(3)}(2, m+2)$*-design of order* v *and index* 1, *then for any* $x, y \in X$, $x \neq y$, *we have:*

- $d(x, y) = \frac{(2m+1)(v-2)}{3m}$,
- $C(x, y) = \frac{v-2}{3m}$,
- $L(x, y) = \frac{2(v-2)}{3}$.

Proof. We know that $|B| = \frac{v(v-1)(v-2)}{6m}$ and that there exists $d \in \mathbb{N}$ such that $d(x, y) = d$ for any $x, y \in X$, $x \neq y$. So we have:

$$d \cdot \binom{v}{2} = (2m+1)|B| \Rightarrow d = \frac{(2m+1)(v-2)}{3m}.$$

Moreover, for any $x, y \in X$, $x \neq y$, we have:

$$\begin{cases} C(x, y) + L(x, y) = d \\ mC(x, y) + L(x, y) = v - 2 \end{cases} \Rightarrow \begin{cases} C(x, y) = \frac{v-2}{3m} \\ L(x, y) = \frac{2(v-2)}{3}. \end{cases}$$

This proves the statement. □

So clearly we also have:

Corollary 2.2. *If* $\Sigma = (X, \mathcal{B})$ *is an edge balanced* $S^{(3)}(2, m+2)$*-design of order* v, *then* $v \equiv 2 \bmod 3m$, $v \geq 3m + 2$.

Moreover, in [18] it is proved the base case of the spectrum of edge balanced $P^{(3)}(2, 4)$-designs:

Theorem 2.3 (Gionfriddo [18, theorem 4.4]). *There exists an edge balanced* $P^{(3)}(2, 4)$*-design of order* 8.

Remark 2.4. Note that if $\Sigma = (X, \mathcal{B})$ is an $S^{(3)}(2, m+2)$-design of order v such that for some $c \in \mathbb{N} C(x, y) = c$ for any $x, y \in X$, $x \neq y$, then Σ is edge balanced.

3 | CYCLIC EDGE-BALANCED $S^{(3)}(2, m+2)$-DESIGNS

Let us consider the complete graph $K_v = (X, E)$ of order v and let $X = \{0, \ldots, v-1\}$. Then it is well known that any edge in E is of the type $\{i, i+r\}$, for some $i \in \{0, \ldots, v-1\}$ and $r \in \left\{1, \ldots, \left\lfloor \frac{v}{2} \right\rfloor\right\}$. In this case, we say that the edge $\{i, i+r\}$ has *difference* r and that it is a *translated form* of the edge $\{0, r\}$.

The natural action of $\mathbb{Z}_v$ on the vertices $X = \{0, \ldots, v-1\}$, defined by $i \to i + j$ for any $j \in \mathbb{Z}_v$ and $i \in \{0, \ldots, v-1\}$, induces an action on the edges. So the edge $\{i, i+r\}$ in the complete graph K_v corresponds to the edge $\{0, r\}$ under this action. Similarly, if $\Sigma = (X, \mathcal{B})$ is a $H^{(3)}$-design, $B, B' \in \mathcal{B}$ and B' corresponds to B under the action of K_v on X, then we say that B' is a translated form of B.

Now we are going to prove the following:

Theorem 3.1. *For any $v \in \mathbb{N}, v = 3m + 2, m \geq 2$ there exists a cyclic edge balanced $S^{(3)}(2, m+2)$-design of order v.*

Proof. Let $v = 3m + 2$, for some $m \geq 2$. By [3, theorem 3.3] we see that base triples in $K_v^{(3)}$ are:

$$\{0, a, a+b\}, \text{ with } a \in \{1, \ldots, m\}, b \in \{a, \ldots, 3m+1-2a\},$$

so that the difference triples in these triples are $\{a, b, a+b\}$. To get a cyclic edge balanced $S^{(3)}(2, m+2)$-design of order v we just need to choose one of the differences in each base triple in the following way for any $a \in \{1, \ldots, m\}$:

$$\begin{cases} \text{for } b \equiv a \bmod 3 \text{ we take the difference } a \\ \text{for } b \equiv a+1 \bmod 3 \text{ we take the difference } b \\ \text{for } b \equiv a+2 \bmod 3 \text{ we take the difference } a+b. \end{cases} \tag{1}$$

If m is odd, we just need to show that any $i \in \left\{1, \ldots, \frac{v-1}{2}\right\}$ is repeated exactly m times in (1) (here we clearly identify $i \in \left\{1, \ldots, \frac{v-1}{2}\right\}$ with $v - i$). In this way, for any $i \in \left\{1, \ldots, \frac{v-1}{2}\right\}$ the m base triples corresponding to i determine a base block (where we do not need to check that the vertices are all different because two distinct base triples determine different triples) and we get a cyclic edge balanced $S^{(3)}(2, m+2)$-design of order v.

If m is even, we need to show that any $i \in \left\{1, \ldots, \frac{v-2}{2}\right\}$ is repeated exactly m times in (1) and that $\frac{v}{2}$ is repeated exactly $\frac{m}{2}$ times. As in the case that m is odd, for any $i \in \left\{1, \ldots, \frac{v-2}{2}\right\}$ the m base triples corresponding to i determine a base block. For each of the $\frac{m}{2}$ base triples corresponding to $\frac{v}{2}$ we take the two translated triples containing the edge $\left\{0, \frac{v}{2}\right\}$ and in this way we get another base block. All these blocks determine a cyclic edge balanced $S^{(3)}(2, m+2)$-design of order v.

To prove this it is sufficient to show that in (1):

- $i \in \{1, \ldots, m+1\}$ is repeated $m - \left\lfloor \frac{i-1}{2} \right\rfloor$ times
- $i \in \{m+2, \ldots, 2m+1\}$ even is repeated $\left\lceil \frac{m}{2} \right\rceil$ times
- $i \in \{m+2, \ldots, 2m+1\}$ odd is repeated $\left\lfloor \frac{m}{2} \right\rfloor$ times
- $i \in \{2m+2, \ldots, 3m-1\}$ is repeated $\left\lfloor \frac{3m+1-i}{2} \right\rfloor$ times (for $m \geq 3$).

It is easy to prove this by induction. Indeed, considering that the base cases $m = 2$ and $m = 3$ are immediate and supposing that the statement holds for $m - 1$, we see that from the $m - 1$ case we have:

- $i \in \{1, \ldots, m\}$ is repeated $m - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor$ times
- $i \in \{m+1, \ldots, 2m-1\}$ even is repeated $\left\lceil \frac{m-1}{2} \right\rceil$ times
- $i \in \{m+1, \ldots, 2m-1\}$ odd is repeated $\left\lfloor \frac{m-1}{2} \right\rfloor$ times
- $i \in \{2m, \ldots, 3m-4\}$ is repeated $\left\lfloor \frac{3m-2-i}{2} \right\rfloor$ times.

When we consider the m case we are adding the following differences:

- $a, 3m - a - 1, 3m - 2a + 1$ for $a \in \{1, \ldots, m-1\}$
- m and $m + 1$ for $a = m$

and it is not difficult to see that the above conditions are satisfied. □

4 | EDGE BALANCED AND VERTEX BALANCED DESIGNS

In this section, we study the possible link between edge balanced and vertex balanced hypergraph designs. Precisely, we want to show the following:

Theorem 4.1. *Let $\Sigma = (X, \mathcal{B})$ be an edge balanced $S^{(3)}(2, m+2)$ design of order v. Then Σ is vertex balanced.*

Proof. For any $x \in X$ we denote with $d(x)$ the number of blocks containing x, with $c(x)$ the number of blocks containing x as an element of degree m (number of triples containing x) and with $l(x)$ the number of blocks containing x as an element of degree 1. Then, recalling the notation given in the beginning of the paper, we have:

$$\sum_{\substack{y \in X \\ y \neq x}} C(x, y) = c(x).$$

Since Σ is edge balanced $C(x, y) = \frac{v-2}{3m}$ and so $c(x) = \frac{(v-1)(v-2)}{3m}$ for any $x \in X$. Moreover, for any $x \in X$:

$$mc(x) + l(x) = \binom{v-1}{2} \Rightarrow l(x) = \frac{(v-1)(v-2)}{6}.$$

So, for any $x \in X$ we have:

$$d(x) = c(x) + l(x) = \frac{(v-1)(v-2)(m+2)}{6m}.$$

This means that Σ is vertex balanced. □

So by Theorem 4.1 we have:

Theorem 4.2. *There exists a vertex balanced $S^{(3)}(2, m+2)$-design of order v for any $m \geq 2$ and any $v \equiv 2 \bmod 3m$, $v \geq 3m+2$.*

At last we show that a vertex balanced hypergraph design is not necessarily edge balanced.

Example 4.3. Let us consider on $X = \{0, 1, ..., 7\}$ the $P^{(3)}(2, 4)$-design having as blocks:

- $[(0, 1), 2, 3]$, $[(0, 1), 4, 5]$, $[(0, 6), 1, 3]$ and their translated forms;
- $[(0, 4), 2, 6]$, $[(2, 6), 0, 4]$, $[(1, 3), 5, 7]$ and $[(5, 7), 1, 3]$.

Let $\mathcal{B}$ be the set of all these blocks. Then, by [3, theorem 3.3] we immediately see that $\Sigma = (X, \mathcal{B})$ is an $P^{(3)}(2, 4)$-design, that is also vertex balanced because $d(x) = 14$ for any $x \in X$. However, Σ is not edge balanced, as, for example, $C(0, 3) = 0$.

5 | DECOMPOSITIONS OF r-PARTITE HYPERGRAPHS WITH BROKEN FEASIBLE SET

Now we are going to consider colorings of $P^{(3)}(2, 4)$-designs. To do this, in this section we consider the following hypergraph. The complete v-partite 3-uniform hypergraph $K^{(3)}_{v \times n}$ is the 3-uniform hypergraph having vertex set $V = X_1 \cup \cdots \cup X_v$, where any $X_i = \{x_{i,1}, ..., x_{i,n}\}$ has cardinality n, and edge set:

$$E = \{\{x_{i,r}, x_{j,s}, x_{k,p}\} | i \neq j, i \neq k, j \neq k, r, s, p \in \{1, ..., n\}\}.$$

Now, let n be even and $v \equiv 2 \bmod 3$, $v \geq 8$. We construct a $P^{(3)}(2, 4)$-decomposition of $K^{(3)}_{v \times n}$ starting from an edge balanced $S^{(3)}\left(2, \frac{v-2}{3} + 2\right)$-design of order v. On $X = \{x_1, ..., x_v\}$ consider an edge balanced $S^{(3)}\left(2, \frac{v-2}{3} + 2\right)$-design $\Sigma = (X, \mathcal{B})$ of order v. Since Σ is edge balanced, for any $i, j \in \{1, ..., v\}$, $i \neq j$, $\{x_i, x_j\}$ occupies a central position in exactly one block of $\mathcal{B}$ by

Theorem 2.1. Moreover for any triple $\{x_i, x_j, x_k\} \in E(K_v^{(3)})$ just one of the couples $\{x_i, x_j\}$, $\{x_i, x_k\}$, $\{x_j, x_k\}$ occupies the central position in a block of $\mathcal{B}$. If $\{x_i, x_j\}$ is such a couple for the triple $\{x_i, x_j, x_k\}$, then we consider the blocks:

$$[(x_{i,r}, x_{j,s}), x_{k,2h+1}, x_{k,2h+2}],\ r, s = 1, \ldots, n \text{ and } h = 0, \ldots, \frac{n}{2} - 1.$$

The set $\mathcal{B}'$ of all these blocks obviously provides a $P^{(3)}(2, 4)$-decomposition of $K_{v\times n}^{(3)}$. Let $\Sigma' = (V, \mathcal{B}')$ such a system of blocks.

Theorem 5.1. *Let $v \in \mathbb{N}$, $v \equiv 2 \bmod 3$, and $v \geq 44$. Then Σ' is a $BP^{(3)}(2, 4)$-decomposition of $K_{v\times 2}^{(3)}$ with feasible set $\{2, 3\} \cup \left[\left\lceil \frac{v}{2} \right\rceil, v\right]$ and color classes that verify the following conditions:*

- *in a 2 and 3-coloring the color classes contain at most two partite sets;*
- *in a k-coloring, with $k \in \left[\left\lceil \frac{v}{2} \right\rceil, v\right]$, any color class is equal either to a partite set or to the union of two partite sets.*

Conversely, any partition of V in k subsets that verifies the above conditions is a k-coloring of Σ'.

Proof. Let $v = 2 + 3m$, for some $m \in \mathbb{N}$, $m \geq 14$. The vertex set is $V = X_1 \cup \cdots \cup X_v$, where the partite sets $X_1, \ldots, X_v$ have two elements each.

Obviously a coloring satisfying one of the conditions of the statement provides a k-coloring of Σ'. We need to prove that there are no other k-colorings for $k \in \{2, 3\} \cup \left[\left\lceil \frac{3m+2}{2} \right\rceil, 3m + 2\right]$ and there are no k-colorings for $k \notin \{2, 3\} \cup \left[\left\lceil \frac{3m+2}{2} \right\rceil, 3m + 2\right]$ (note that for $k = 2, 3$ this is obvious).

Given a k-coloring of Σ' we denote by $A_1, \ldots, A_k$ the color classes. Since the partite sets have just two elements each, we can say that for any $i = 1, \ldots, k$ we have:

$$A_i = A_i' \cup A_i'',$$

where the following conditions hold for any $i = 1, \ldots, k$:

- $A_i' \cap A_i'' = \emptyset$,
- either $X_j \subseteq A_i'$ or $X_j \cap A_i' = \emptyset$ for any $j = 1, \ldots, v$,
- $|A_i'| \in \{0, 2, 4\}$, otherwise there would be monochromatic blocks,
- $|A_i' \cap X_j| \leq 1$ for $j = 1, \ldots, v$.

First case. Suppose, now, that there exists a k-coloring of Σ' such that for some $i, j \in \{1, \ldots, v\}, i \neq j$, the elements $x_{i,1}, x_{i,2}, x_{j,1}$, and $x_{j,2}$ are in four different color classes. Without loss of generality we can take $i = 1$ and $j = 2$. So, denoted by $A_1, \ldots, A_k$ the k color classes, we can suppose that $x_{1,1} \in A_1$, $x_{1,2} \in A_2$, $x_{2,1} \in A_3$, and $x_{2,2} \in A_4$. We will use the following notation: for any $x \in X = \{x_1, ..., x_v\}$ we denote by G_x the graph having $X \setminus \{x\}$ as set of vertices and edges

$$\{\{y, z\} | \{y, z\} \text{ occupies the central position in the block of } \mathcal{B} \text{ containing } \{x, y, z\}\}.$$

Since Σ is edge balanced, given:

$$T_1 = \{i \in \{3, ..., 3m + 2\} | \{x_1, x_2\} \in E(G_{x_i})\}$$

and

$$T_2 = \{3, ..., 3m + 2\} \setminus T_1,$$

we know that $|T_1| = m$ and $|T_2| = 2m$. Moreover, for any $j \in T_2$ either $\{x_1, x_j\} \in E(G_{x_2})$ or $\{x_2, x_j\} \in E(G_{x_1})$. Clearly, it must be:

$$j \in T_2 \quad \Rightarrow \quad x_{j,1}, x_{j,2} \in A_1 \cup A_2 \cup A_3 \cup A_4.$$

Moreover, if for some $i = 5, ..., k$

$$x_{j,r} \in A_i \quad \Rightarrow \quad j \in T_1 \quad \text{and} \quad x_{j,1}, x_{j,2} \in A_i.$$

Let $j \in T_2$. Then:

$$\begin{aligned} \{x_1, x_j\} \in E(G_{x_2}) &\Rightarrow \quad \{x_{j,1}, x_{j,2}\} \in A_3 \cup A_4, \\ \{x_2, x_j\} \in E(G_{x_1}) &\Rightarrow \quad \{x_{j,1}, x_{j,2}\} \in A_1 \cup A_2. \end{aligned}$$

So for any $j \in T_2$ either $\{x_{j,1}, x_{j,2}\} \subset A_1 \cup A_2$ or $\{x_{j,1}, x_{j,2}\} \subset A_3 \cup A_4$. Let $j \in T_1$. Then:

$$\{x_{j,1}, x_{j,2}\} \not\subseteq A_i, \text{ for any } i = 5, ..., k \Rightarrow \text{either } \{x_{j,1}, x_{j,2}\} \subseteq A_1 \cup A_2 \text{ or } \{x_{j,1}, x_{j,2}\} \subseteq A_3 \cup A_4.$$

So, we can say that $|A_1''| = |A_2''| = n_1$ and $|A_3''| = |A_4''| = n_2$, for some $n_1, n_2 \in \mathbb{N}$. Moreover, we can suppose that:

$$\begin{aligned} A_1 &= \{x_{1,1}, x_{3,1}, ..., x_{n_1+1,1}\} \cup A_1' \\ A_2 &= \{x_{1,2}, x_{3,2}, ..., x_{n_1+1,2}\} \cup A_2' \\ A_3 &= \{x_{2,1}, x_{n_1+2,1}, ..., x_{n_1+n_2,1}\} \cup A_3' \\ A_4 &= \{x_{2,2}, x_{n_1+2,2}, ..., x_{n_1+n_2,2}\} \cup A_4', \\ A_5 &= A_5' \\ &\vdots \\ A_k &= A_k' \end{aligned} \tag{2}$$

where A_i', for $i = 1, ..., k$ have the following properties:

- $|A_i'| = 2p_i$, for $i = 1, ..., k$, where $p_i \in \{0, 1, 2\}$ for $i = 1, 2, 3, 4$ and $p_i \in \{1, 2\}$ for $i = 5, ..., k$;
- each A_i' for $i = 1, ..., k$ contain p_i couples $\{x_{j,1}, x_{j,2}\}$.

Moreover, it must be $\sum_{i=5}^{k} p_i \leq m$.

Suppose, now, that $k \geq 5$. Consider, now, the triples $\{x_i, x_j, x_l\}$, with $i, j \in \{1, 3, ..., n_1 + 1\}$, $i \neq j$, and $l \in \{2, n_1 + 2, ..., n_1 + n_2\}$. Then, for any $r, s = 1, 2$ in the corresponding blocks of $\mathcal{B}'$ either $\{x_{i,r}, x_{l,s}\}$ or $\{x_{j,r}, x_{l,s}\}$ occupy the central positions. The same happens when we consider the triples $\{x_i, x_j, x_l\}$, with $i \in \{1, 3, ..., n_1 + 1\}$ and $j, l \in \{2, n_1 + 2, ..., n_1 + n_2\}$, $j \neq l$. Moreover, if we take the triples $\{x_i, x_j, x_l\}$, with $i \in \{1, 3, ..., n_1 + 1\}$, $j \in \{2, n_1 + 2, ..., n_1 + n_2\}$ and l such that $\{x_{l,1}, x_{l,2}\} \subset A_5' \cup \cdots \cup A_k'$, then for any $r, s = 1, 2$ in the corresponding blocks of $\mathcal{B}'$ the edges $\{x_{i,r}, x_{j,s}\}$ occupy the central positions. So, for any $r, s = 1, 2$ we have:

$$\binom{n_1}{2} n_2 + \binom{n_2}{2} n_1 + n_1 n_2 \sum_{i=5}^{k} p_i$$

blocks of $\mathcal{B}'$ having in the central position an edge with one vertex $x_{i,r}$ with $i \in \{1, 3, ..., n_1 + 1\}$ and the other $x_{j,s}$ with $j \in \{2, n_1 + 2, ..., n_1 + n_2\}$. Since Σ is edge balanced, any edge occupies such a position exactly m times. This means that:

$$\binom{n_1}{2} n_2 + \binom{n_2}{2} n_1 + n_1 n_2 \sum_{i=5}^{k} p_i \leq n_1 n_2 m$$
$$\Rightarrow \frac{1}{2}(n_1 + n_2) - 1 + \sum_{i=5}^{k} p_i \leq m.$$

Since $n_1 + n_2 + \sum_{i=1}^{k} p_i = 3m + 2$, we get:

$$m + \sum_{i=5}^{k} p_i \leq p_1 + p_2 + p_3 + p_4 \Rightarrow m \leq 8 \Rightarrow v \leq 26.$$

This means that in a coloring as in (2) it must be $k \leq 4$.

Suppose, now, that $k = 4$ and $n_1, n_2 \geq 2$. For any $i, j \in \{1, 3, ..., n_1 + 1\}$, $i \neq j$, we know that $[(x_{i,r}, x_{j,s}), x_{l,1}, x_{l,2}]$ is a block in $\mathcal{B}'$ for m values of $l \in \{1, ..., 3m + 2\}$ and the only possibilities are that either $l \in \{1, 3, ..., n_1 + 1\}$ or $\{x_{l,1}, x_{l,2}\} \subset A_p'$, $p = 3, 4$. So, for each such couple there are at most $(n_1 - 2) + p_3 + p_4$ possibilities, where the $n_1 - 2$ ones correspond to triples in $\{1, 3, ..., n_1 + 1\}$. Since each of these triples corresponds to exactly one block of $\mathcal{B}'$ and each edge occupies the central position in these blocks exactly m times, we can say that:

$$\binom{n_1}{2} m \leq \binom{n_1}{3} + (p_3 + p_4)\binom{n_1}{2} \Rightarrow n_1 \geq v - 12.$$

Similarly we get $n_2 \geq v - 12$ and so:

$$v \geq n_1 + n_2 \geq 2v - 24 \Rightarrow v \leq 24$$

which is a contradiction.

If $k = 4$ and $n_1 = 1$, then clearly in $\mathcal{B}'$ the vertices $x_{1,1}$ and $x_{1,2}$ occupy the lateral positions in at most $\binom{3m+1}{2} - \binom{n_2}{2} - n_2(p_3 + p_4)$ blocks. Since Σ is edge balanced, by Theorem 4.1 we have:

$$\binom{3m+1}{2} - \binom{n_2}{2} - n_2(p_3 + p_4) \geq \frac{(3m+1)m}{2}.$$

We know that $3m + 2 = n_2 + 1 + \sum_i^4 p_i$; so, if $p_3 = p_4 = 0$, then $n_2 \geq 3m - 3$ and we get $m \leq 7$. If $p_3 + p_4 \geq 1$, then $n_2 \geq 3m - 7$ and so $m \leq 12 \Rightarrow v \leq 38$. This is a contradiction. Since we can reason in a similar way if $n_2 = 1$, this proves that a coloring as in 2 is impossible.

Second case: Suppose that there exists a k-coloring such that for some $i \neq j$ we have $x_{i,1}, x_{j,1} \in A_1$, $x_{i,2} \in A_2$ and $x_{j,2} \in A_3$. Without loss of generality we can take $i = 1$ and $j = 2$. Again, since Σ is edge balanced, given:

$$T_1 = \{i \in \{3, ..., 3m + 2\} | \{x_1, x_2\} \in E(G_{x_i})\}$$

and

$$T_2 = \{3, ..., 3m + 2\} \setminus T_1,$$

we know that $|T_1| = m$ and $|T_2| = 2m$. Note that for $j \in T_2$ either $\{x_1, x_j\} \in E(G_{x_2})$ or $\{x_2, x_j\} \in E(G_{x_1})$. Clearly, it must be $x_{j,r} \in A_1 \cup A_2 \cup A_3$ for any $j \in T_2$ and $r = 1, 2$. So in A_i for $i = 4, ..., k$ there are only $x_{j,r}$ for some $j \in T_1$ and, in such a case, both $x_{j,1}, x_{j,2} \in A_i$.

So, we can suppose that:

$$\begin{aligned} A_1 &= A_1' \cup A_1'' \\ A_2 &= A_2' \cup A_2'' \\ A_3 &= A_3' \cup A_3'' \\ A_4 &= A_4', \\ &\vdots \\ A_k &= A_k' \end{aligned} \tag{3}$$

where A_i', for $i = 1, ..., k$ have the following properties:

- $|A_i'| = 2p_i$, for $i = 1, ..., k$, where $p_i \in \{0, 1, 2\}$ for $i = 1, 2, 3$ and $p_i \in \{1, 2\}$ for $i = 4, ..., k$;
- each A_i' for $i = 1, ..., k$ contain p_i couples $\{x_{j,1}, x_{j,2}\}$;
- $\sum_{i=4}^k p_i \leq m$

and moreover none of A_1'', A_2'' and A_3'' contain couples $x_{i,1}, x_{i,2}$ for any i.

Suppose that $k \geq 5$ and consider for any $r, s = 1, 2$ the edges $\{x_{i,r}, x_{j,s}\}$, where $x_{i,r}, x_{j,s} \in A'_4 \cup \cdots \cup A'_k$ have different colors. Each of these edges must occupy the central position in the blocks of $\mathcal{B}'$ exactly m times and this happens only if the other two vertices in such blocks have the same color. So it must be:

$$m\left[\binom{\sum_{i=4}^{k} p_i}{2} - \sum_{i=4}^{k}(p_i - 1)\right]$$
$$\leq \binom{\sum_{i=4}^{k} p_i}{3} + \left[\binom{\sum_{i=4}^{k} p_i}{2} - \sum_{i=4}^{k}(p_i - 1)\right](p_1 + p_2 + p_3).$$

Since $m > 6 \geq p_1 + p_2 + p_3$ and $\sum_{i=4}^{k}(p_i - 1) \leq \sum_{i=4}^{k} p_i - 1$, we get:

$$m - p_1 - p_2 - p_3 \leq \frac{1}{3}\sum_{i=4}^{k} p_i.$$

However, we know that $\sum_{i=4}^{k} p_i \leq m$ and that $p_1 + p_2 + p_3 \leq 6$. This implies that $m \leq 9 \Rightarrow v \leq 29$, which is a contradiction.

Let $k = 4$ and let:

$$A_{ij} = \left\{l \mid x_{l,r} \in A''_i, x_{l,s} \in A''_j, r, s = 1, 2, r \neq s\right\}$$

and

$$a_{ij} = |A_{ij}|$$

for $i, j = 1, 2, 3, i \neq j$. Then we have $a_{12} + a_{13} + a_{23} = 3m + 2 - \sum_{i=1}^{4} p_i$. We will need a few remarks. Take $i, j, l \in \{1, \ldots, 3m + 2\}$, pairwise distinct.

- If $x_{l,1}, x_{l,2} \in A'_4$, $i \in A_{12}$ and $j \in A_{13}$, then in the blocks of $\mathcal{B}'$ corresponding to the triple $\{x_i, x_j, x_l\}$ the vertices $x_{l,1}, x_{l,2}$ must occupy the lateral positions. Clearly, we can reason in a similar way for A_{12} and A_{23} and A_{13} and A_{23}. So we get that in the blocks of $\mathcal{B}' x_{l,1}$ and $x_{l,2}$ occupy the lateral positions at least $a_{12}a_{13} + a_{12}a_{23} + a_{13}a_{23}$ times. So by Theorem 4.1:

$$a_{12}a_{13} + a_{12}a_{23} + a_{13}a_{23} \leq \frac{(3m + 1)m}{2}. \tag{4}$$

- Taken $i \in A_{12}$, $j \in A_{13} \cup A_{23}$ and $x_{l,1}, x_{l,2} \in A'_4$, we see that the above remark shows also that the edge $\{x_l, x_i\}$ occupies a lateral position in the blocks of $\mathcal{B}$ at least $a_{13} + a_{23}$ times. So, since Σ is edge balanced, we can say that:

$$a_{12} \neq 0 \Rightarrow a_{13} + a_{23} \leq 2m \Rightarrow a_{12} \geq m + 2 - \sum_{i=1}^{4} p_i. \tag{5}$$

Similarly, we can say that:

$$a_{13} \neq 0 \Rightarrow a_{12} + a_{23} \leq 2m \Rightarrow a_{13} \geq m + 2 - \sum_{i=1}^{4} p_i \tag{6}$$

$$a_{23} \neq 0 \Rightarrow a_{12} + a_{13} \leq 2m \Rightarrow a_{23} \geq m + 2 - \sum_{i=1}^{4} p_i. \tag{7}$$

- Let $p_1 \geq 1$ and take $x_{l,1}, x_{l,2} \in A_1'$ and $j, l \in A_{12} \cup A_{13}$. Then x_l must occupy a central position in the blocks of $\mathcal{B}$. By Theorem 4.1 we get:

$$p_1 \geq 1 \Rightarrow \binom{a_{12} + a_{13}}{2} \leq m(3m+1);$$

and similarly:

$$p_2 \geq 1 \Rightarrow \binom{a_{12} + a_{23}}{2} \leq m(3m+1);$$

$$p_3 \geq 1 \Rightarrow \binom{a_{13} + a_{23}}{2} \leq m(3m+1). \tag{8}$$

- Let $p_1 = 2$ and take $x_{i,1}, x_{i,2}, x_{j,1}, x_{j,2} \in A_1'$. Then for any $l \in A_{12} \cup A_{13}$ in the blocks corresponding to the triple $\{x_i, x_j, x_l\}$ the edges $\{x_{i,r}, x_{j,s}\}$ for any $r, s = 1, 2$ must occupy the central positions. Since Σ is edge balanced, we can say that:

$$p_1 = 2 \Rightarrow a_{12} + a_{13} \leq m \Rightarrow a_{23} \geq 2m + 2 - \sum_{i=1}^{4} p_i \tag{9}$$

and similarly:

$$p_2 = 2 \Rightarrow a_{12} + a_{23} \leq m \Rightarrow a_{13} \geq 2m + 2 - \sum_{i=1}^{4} p_i \tag{10}$$

$$p_3 = 2 \Rightarrow a_{13} + a_{23} \leq m \Rightarrow a_{12} \geq 2m + 2 - \sum_{i=1}^{4} p_i. \tag{11}$$

Now, if $a_{12}, a_{13}, a_{23} \geq m - 3$, then by (4) we get:

$$3(m-3)^2 \leq \frac{m(3m+1)}{2} \Rightarrow m \leq 10 \Rightarrow v \leq 32,$$

which is a contradiction.

Suppose that $a_{12} \leq m - 4$, with $a_{12} \neq 0$ the minimum between a_{12}, a_{13}, and a_{23}. Then we know that $a_{13} + a_{23} \leq 2m$ by (5) and also:

$$a_{13} + a_{23} = 3m + 2 - a_{12} - \sum_{i=1}^{4} p_i \geq 2m + 6 - \sum_{i=1}^{4} p_i. \tag{12}$$

So we can say that $\sum_{i=1}^{4} p_i \geq 6$, which implies that either $p_1 = 2$ or $p_2 = 2$ or $p_3 = 2$. By (5) and (9) if $p_1 = 2$ we get:

$$a_{13} \leq \sum_{i=1}^{4} p_i - 2.$$

Since $a_{13} \geq a_{12}$, by (5) we get:

$$m + 2 - \sum_{i=1}^{4} p_i \leq \sum_{i=1}^{4} p_i - 2 \Rightarrow m \leq 2\sum_{i=1}^{4} p_i - 4 \leq 12 \Rightarrow v \leq 38,$$

which is a contradiction. In a similar way we get a contradiction if $p_2 = 2$. So, since $\sum_{i=1}^{4} p_i \geq 6$, the only possibility is that $p_1 = 1$, $p_2 = 1$, $p_3 = 2$, and $p_4 = 2$. However, reasoning as done earlier, if $p_3 = 2$, by (11) and (12), we get $m \leq 0$, which is not possibile.

So we can suppose that $a_{12} = 0$ and by our initial assumption we know that $a_{13}, a_{23} \neq 0$. If $p_3 \geq 1$, then by the fact that $a_{13} + a_{23} \geq 3m - 6$ and by (8) we get $m \leq 12$, so that $v \leq 38$, which is a contradiction.

This means that we can suppose that $a_{12} = 0$ and $p_3 = 0$. By (9) and (10) we get that, if $p_1 = p_2 = 2$, then

$$2m \geq a_{13} + a_{23} = 3m + 2 - \sum_{i=1}^{4} p_i \geq 3m - 4 \Rightarrow m \leq 4 \Rightarrow v \leq 14.$$

So we can say that $\sum_{i=1}^{4} p_i \leq 5$. Then by (4):

$$a_{13}a_{23} \leq \frac{(3m+1)m}{2} \Rightarrow a_{13}(3m - 3 - a_{13}) \leq \frac{(3m+1)m}{2}.$$

Since $m - 3 \leq a_{13} \leq 2m$ by (6) and (7), we get that this holds only if $m \leq 13 \Rightarrow v \leq 41$, which is a contradiction. This shows that we cannot have a coloring as in (3).

Third case: We suppose that $k \geq 4$ and that $A_i'' = \emptyset$ for $i = 3, \ldots, k$. Let $h \in \{0, \ldots, k-2\}$ be the number of indices $i \in \{3, \ldots, k\}$ such that $|A_i'| = 2$. So for $h \in \{1, \ldots, k-3\}$ we can suppose that the color classes are the following:

$$\begin{aligned} A_1 &= A_1' \cup A_1'' \\ A_2 &= A_2' \cup A_2'' \\ A_3 &= A_3' \\ &\vdots \\ A_{2+h} &= A_{2+h}' \\ A_{3+h} &= A_{3+h}' \\ &\vdots \\ A_k &= A_k' \end{aligned} \tag{13}$$

where:

- $|A_i''| = n$ for $i = 1, 2$ and none of them contains couples $x_{i,1}, x_{i,2}$ for any i;
- $|A_i'| = 2p_i$, for $i = 1, ..., k$, where $p_i \in \{0, 1, 2\}$ for $i = 1, 2$, $p_i = 1$ for $i = 3, ..., 2 + h$ and $p_i = 2$ for $i = 3 + h, ..., k$;
- each A_i' for $i = 1, ..., k$ contain p_i couples $\{x_{j,1}, x_{j,2}\}$.

In the case $h = 0$, keeping the above notation, we have that for $|A_i'| = 4$ for any $i = 3, ..., k$. Similarly, for $h = k - 2$ we have that $|A_i'| = 2$ for $i = 3, ..., k$.

Consider for any $r, s = 1, 2$ the vertices $x_{i,r}, x_{j,s} \in A_3' \cup \cdots \cup A_k'$ having different colors. Then, any vertex $x_{l,p} \in A_1'' \cup A_2''$, with $p = 1, 2$, occupies the central position in the corresponding blocks. If c_{il}' is the number of times that for any $r, p = 1, 2$ an edge $\{x_{l,p}, x_{i,r}\}$ occupies the central position in such blocks, we have:

$$\sum_i c_{il}' = \binom{2k - 4 - h}{2} - (k - 2 - h).$$

Since we have $2k - 4 - h$ of such edges, we can say that there exists $\bar{j}$ such that:

$$c_{\bar{j}l}' \geq \frac{2k - 6 - h}{2}.$$

So, since Σ is edge balanced, it must be:

$$\frac{2k - 6 - h}{2} \leq m.$$

Since $v = 3m + 2 = n + p_1 + p_2 + 2k - 4 - h$, we get:

$$n \geq m - p_1 - p_2.$$

On the other hand, for any $r, s, t = 1, 2$ consider the triples $\{x_{i,r}, x_{j,s}, x_{l,t}\}$, with $x_{i,r} \in A_1'' \cup A_2''$ and $x_{j,s}, x_{l,t} \in A_3' \cup \cdots \cup A_k'$ with different colors. The edge $\{x_{j,r}, x_{l,t}\}$ must occupy a lateral position in the corresponding block. So each of these edges $\{x_{j,r}, x_{l,t}\}$ occupies a lateral position n times.

Now, for any $r, s, t = 1, 2$ consider the

$$\binom{2k - 4 - h}{3}$$

triples $\{x_{i,r}, x_{j,s}, x_{l,t}\} \subset A_3' \cup \cdots \cup A_k'$, and denote by l_{ij} the number of times that the edge $\{x_{i,r}, x_{j,s}\}$ occupies a lateral position in the blocks corresponding to such triples. It clearly must be:

$$\sum l_{ij} = 2\binom{2k - 4 - h}{3}.$$

Let:

$$A = \{\{i, j\} | \text{ for any } r, s = 1, 2, x_{i,r}, x_{j,s} \in A_l, \text{ for some } l = h + 3, ..., k\}$$

and

$$B = \{\{i, j\} | \text{ for any } r, s = 1, 2, x_{i,r} \in A_l, x_{j,s} \in A_{l'}, \\ \text{for some } l, l' = 3, ..., k, l \neq l'\},$$

where:

$$|A| = k - 2 - h \text{ and } |B| = \binom{2k - 4 - h}{2} - (k - 2 - h).$$

Obviously, for any $\{i, j\} \in A$ it must be $l_{ij} \leq 2k - 6 - h$ and so:

$$\sum_{\{i,j\} \in B} l_{ij} \geq 2\binom{2k - 4 - h}{3} - (k - 2 - h)(2k - 6 - h).$$

This implies that there exists $\{\bar{i}, \bar{j}\} \in B$ such that $l_{\bar{i}\bar{j}} > \frac{2}{3}(2k - 7 - h)$, otherwise it would be $k \leq 3$. So for any $r, s = 1, 2$ the edge $\{x_{\bar{i},r}, x_{\bar{j},s}\}$ occupies the lateral positions at least $n + l_{\bar{i}\bar{j}}$ times. This implies that $n + l_{\bar{i}\bar{j}} \leq 2m$, because Σ is edge balanced, and so:

$$n + \frac{2}{3}(2k - 7 - h) < 2m.$$

Since $v = 3m + 2 = n + p_1 + p_2 + 2k - 4 - h$, the previous inequality implies

$$n + \frac{2}{3}(3m - 1 - n - p_1 - p_2) < 2m \Rightarrow n \leq 2p_1 + 2p_2 + 1.$$

Since we saw that $n \geq m - p_1 - p_2$, this show that $m \leq 3p_1 + 3p_2 + 1$, which implies that $m \leq 13$ and so $v \leq 41$. So, this proves that such a coloring exists only for $k = 2, 3$ and the statement is proved. □

6 | $BP^{(3)}(2, 4)$-DESIGNS WITH BROKEN FEASIBLE SET

Now we can apply Theorem 5.1 to provide constructions of $BP^{(3)}(2, 4)$ designs with broken feasible set.

Theorem 6.1. *For any* $v \equiv 2 \bmod 3$, $v \geq 44$, *and* $p \in \mathbb{N}$, $\left\lceil \frac{v}{2} \right\rceil \leq p \leq v$, *there exists a* $BP^{(3)}(2, 4)$*-design of order* $2v$ *with feasible set* $\{2, 3\} \cup [p, v]$.

Proof. Let $X = \{x_{i,1}, x_{i,2} | i = 1, ..., v\}$ be such hat $|X| = 2v$ and consider a $BP^{(3)}(2, 4)$-decomposition $\Sigma = (X, \mathcal{B})$ of $K^{(3)}_{v\times 2}$ as in Theorem 5.1, with partite sets $\{x_{i,1}, x_{i,2}\}$.

Let $p = v$ and let $\mathcal{C}$ be the family of the following blocks:

$$[(x_{i,1}, x_{i,2}), x_{j,1}, x_{j,2}], [(x_{j,1}, x_{j,2}), x_{i,1}, x_{i,2}]$$

for any $i, j \in \{1, ..., v\}$, with $i \neq j$. Then it is easy to see that $\Sigma' = (X, \mathcal{B} \cup \mathcal{C})$ is a $BP^{(3)}(2, 4)$-design with feasible set $\{2, 3, v\}$ by Theorem 5.1 and by the construction of $\mathcal{C}$.

Let $p = v - 1$ and let $\mathcal{C}$ be the family of the following blocks:

- $[(x_{i,1}, x_{i,2}), x_{j,1}, x_{j,2}]$ for any $i, j \in \{1, ..., v-1\}$, with $i \neq j$, with the exception of the block $[(x_{v-1,1}, x_{v-1,2}), x_{v-2,1}, x_{v-2,2}]$;
- $[(x_{v,1}, x_{v,2}), x_{j,1}, x_{j+1,2}]$ for any $j \in \{1, ..., v-2\}$;
- $[(x_{v,1}, x_{v,2}), x_{1,2}, x_{v-1,1}]$;
- $[(x_{i,1}, x_{i,2}), x_{v,1}, x_{v,2}]$ for any $i \in \{1, ..., v-2\}$;
- $[(x_{v-1,1}, x_{v-1,2}), x_{v,s}, x_{v-2,s}]$ for $s = 1, 2$.

Then it is easy to see that $\Sigma' = (X, \mathcal{B} \cup \mathcal{C})$ is a $BP^{(3)}(2, 4)$-design with feasible set $\{2, 3, v-1, v\}$ by Theorem 5.1 and by the construction of $\mathcal{C}$.

Let $p = v - r$, $p \geq \left\lceil \frac{v}{2} \right\rceil$, with $r \in \mathbb{N}$ and $r \geq 2$, and let $\mathcal{C}$ be the family of the following blocks:

- $[(x_{i,1}, x_{i,2}), x_{j,1}, x_{j,2}]$, $[(x_{j,1}, x_{j,2}), x_{i,1}, x_{i,2}]$ for any $i, j \in \{1, ..., p\}$, with $i \neq j$, and for any $i, j \in \{p+1, ..., v\}$, $i \neq j$;
- $[(x_{i,1}, x_{i,2}), x_{j,1}, x_{j+1,2}]$, for any $i \in \{1, ..., p\}$, $j \in \{p+1, ..., v-1\}$ and for any $i \in \{p+1, ..., v\}$, $j \in \{1, ..., p-1\}$;
- $[(x_{i,1}, x_{i,2}), x_{p+1,2}, x_{v,1}]$ for any $i \in \{1, ..., p\}$;
- $[(x_{i,1}, x_{i,2}), x_{1,2}, x_{p,1}]$ for any $i \in \{p+1, ..., v\}$.

Then it is easy to see that $\Sigma' = (X, \mathcal{B} \cup \mathcal{C})$ is a $BP^{(3)}(2, 4)$-design with feasible set $\{2, 3\} \cup [p, v]$ by Theorem 5.1 and by the construction of $\mathcal{C}$. □

This paper provides the first examples of $BP^{(3)}(2, 4)$-designs with broken feasible set. Since this is a blow-up construction, based on edge balanced hypergraph designs, the order of these $BP^{(3)}(2, 4)$-designs is a particular one, precisely $2v$, with $v \equiv 2 \bmod 3$ and $v \geq 44$. So the remaining admissible orders represent an open problem, which might be solved thanks to this construction.

ACKNOWLEDGMENT

The first author was supported by Università degli Studi di Catania, PIACERI 2020/22 Linea di intervento 3. The second author was supported by Università degli Studi di Catania, PIACERI 2020/22 Linea di intervento 2.

ORCID

Paola Bonacini http://orcid.org/0000-0003-0120-0111

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How to cite this article: P. Bonacini and L. Marino, *Edge balanced star-hypergraph designs and vertex colorings of path designs*, J. Combin. Des. (2022), 1–18. https://doi.org/10.1002/jcd.21837