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Compactifications of a Pixley–Roy hyperspace

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ABSTRACT

For a Hausdorff space X, let $PR(X)$ be the Pixley–Roy hyperspace of X. Dow and Moore [10, Proposition 2.13] noted that no compactification of $PR(2^\omega)$ has countable tightness. In this paper, we will discuss when $PR(X)$ has a compactification of countable tightness in general. Among other things, we will show that if a Pixley–Roy hyperspace has a compactification of countable tightness, then it will actually have a Corson compactification.

1. Introduction

Inspired by a result of Dow and Moore [10, Proposition 2.13], we consider the general problem to embed a Pixley–Roy hyperspace into a compact space of countable tightness. In [6] we have already investigated the related weaker property for a Pixley–Roy hyperspace to be productively countably tight. A space X is said to be *productively countably tight* if $X \times Y$ has countable tightness for any space Y of countable tightness. It is well-known that a compact space of countable tightness is productively countably tight [13].

All spaces are assumed to be T_2. For notations and terminology, we refer to [11]. The symbol $\mathbb{N}$ is the set of positive integers.

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A space X is said to have *countable tightness* if whenever $p \in X$ and $p \in \overline{A}$ there is a countable subset $B \subset A$ such that $p \in \overline{B}$. A subset $\{x_\alpha : \alpha < \kappa\}$ of a space X is said to be a *free sequence of length* κ in X if $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\gamma : \gamma \geq \alpha\}} = \emptyset$ for each $\alpha < \kappa$. It is well known that a compact space has countable tightness if and only if it has no free sequence of uncountable length [3] (or, [4, Theorem 2.2.13]).

For a space X, let $PR(X)$ be the space of all nonempty finite subsets of X with the *Pixley–Roy topology* [15]: for $A \in PR(X)$ and an open set $U \subset X$, let

$$[A, U] = \{B \in PR(X) : A \subset B \subset U\};$$

the family $\{[A, U] : A \in PR(X), U \text{ is open in } X\}$ is a base for the Pixley–Roy topology. For nonempty basic open sets $[A, U]$ and $[B, V]$, $[A, U] = [B, V]$ holds if and only if both $A = B$ and $U = V$ hold, and $[A, U] \cap [B, V] \neq \emptyset$ holds if and only if both $A \subset V$ and $B \subset U$ hold.

Let us recall some basic properties of $PR(X)$. A space is said to be *σ-discrete* if it is the union of countably many closed discrete subspaces. A space X is said to be a *Moore space* if it has a sequence $\{\mathcal{D}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{St(x, \mathcal{D}_n) : n \in \omega\}$ is a neighborhood base at x, where $St(x, \mathcal{D}_n) = \bigcup\{D \in \mathcal{D}_n : x \in D\}$. The sequence $\{\mathcal{D}_n : n \in \omega\}$ is said to be a *development* of X. For a space X, the symbol $w(X)$ (resp., $\chi(X)$, $\pi\chi(X)$) is the weight (resp., character, π-character) of X.

Proposition 1.1. *The following assertions hold for an infinite space X.*

(1) *$PR(X)$ is zero-dimensional, completely regular, and every subspace of $PR(X)$ is metacompact [9];*
(2) *$PR(X)$ is a Moore space if and only if $PR(X)$ is first-countable if and only if X is first-countable [9];*
(3) *$PR(X)$ is σ-discrete if and only if each point of X is a G_δ [12, Theorem 2.3];*
(4) *$\chi(PR(X)) = \pi\chi(PR(X)) = \chi(X)$ and $w(PR(X)) = \chi(X)\,|X|$ [18, Theorem 2].*

2. Countably tight compactifications of a Pixley–Roy hyperspace

Definition 2.1. ([8]) A family $\mathcal{A}$ of subsets of a set is *strongly point-countable* if it has no uncountable subfamily with the finite intersection property. A space X is *strongly metaLindelöf* if each open cover of X has a strongly point-countable open refinement.

A strongly point-countable family is obviously point-countable. A screenable space (i.e., every open cover has a σ-disjoint open refinement) is strongly metaLindelöf, in particular so is a paracompact space.

Lemma 2.2. *Let X be a strongly metaLindelöf σ-discrete space and let $\mathcal{B}$ be a base for X. Then, every open cover of X has a strongly point-countable refinement consisting of the elements of $\mathcal{B}$.*

Proof. Let $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is a closed discrete subspace of X. Let $\mathcal{U}$ be an open cover of X. Fix $n \in \omega$. For each $x \in X_n$, we take an open neighborhood U_x of x such that $U_x \cap X_n = \{x\}$ and U_x is contained in some member of $\mathcal{U}$. Since $\{U_x : x \in X_n\} \cup \{X \setminus X_n\}$ is an open cover of X, it has a strongly point-countable open refinement $\mathcal{V}_n$. For each $x \in X_n$, we take $V_x \in \mathcal{V}_n$ and $B_x \in \mathcal{B}$ with $x \in B_x \subset V_x$ (then, automatically $V_x \subset U_x$). Since $\{V_x : x \in X_n\}$ is strongly point-countable, so is $\mathcal{B}_n = \{B_x : x \in X_n\}$. Repeating this construction for each n, we see that the family $\bigcup\{\mathcal{B}_n : n \in \omega\}$ is obviously what we want. □

For the closed unit interval $\mathbb{I} = [0, 1]$ and a set Γ, we put

$$\Sigma(\mathbb{I}, \Gamma) = \{x \in \mathbb{I}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) > 0\}| \leq \omega\}.$$

Similarly, for the space $\mathbb{D} = \{0, 1\}$, we put

$$\Sigma(\mathbb{D},\Gamma) = \{x \in \mathbb{D}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) = 1\}| \leq \omega\}.$$

These spaces are usually called Σ-*products*. A compact space is said to be *Corson compact* [14] if it is homeomorphic to a subspace of $\Sigma(\mathbb{I},\Gamma)$ for some set Γ. Since $\Sigma(\mathbb{I},\Gamma)$ is Fréchet–Urysohn, so is a Corson compact space.

The following is a special case of Dimov's theorem [8, Theorem 3.6]. For the readers' convenience, we give a proof.

Lemma 2.3. *If a space X has a strongly point-countable base $\mathcal{B}$ consisting of clopen subsets in X, then X has a zero-dimensional Corson compactification K such that $w(K) \leq |\mathcal{B}|$.*

Proof. Let $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$. We define a map $\varphi : X \to \mathbb{D}^\kappa$ as follows: for each $x \in X$ and $\alpha < \kappa$, $\varphi(x)(\alpha) = 1$ if $x \in B_\alpha$ and $\varphi(x)(\alpha) = 0$ if $x \in X \setminus B_\alpha$. It is not difficult to see that φ is an embedding of X into $\mathbb{D}^\kappa$. Let $K = \overline{\varphi(X)} \subset \mathbb{D}^\kappa$.

For any $f \in K$, we claim that $\{B_\alpha : \alpha \in f^{-1}(1)\}$ has the finite intersection property. Indeed, for any finite set $\alpha_0, \dots, \alpha_n \in f^{-1}(1)$, $f \in K$ implies that there is a point $x \in X$ such that $\varphi(x)(\alpha_i) = 1$ $(i \leq n)$. This means $x \in B_{\alpha_0} \cap \dots \cap B_{\alpha_n}$. Since $\mathcal{B}$ is strongly point-countable, $f^{-1}(1)$ must be countable and hence $K \subset \Sigma(\mathbb{D},\kappa)$. Thus K is a zero-dimensional Corson compact space satisfying $w(K) \leq |\mathcal{B}|$. □

The following can be easily checked.

Lemma 2.4. *For finitely many nonempty basic open sets $[F_0, U_0], \dots, [F_n, U_n]$ in $PR(X)$, $[F_0, U_0] \cap \dots \cap [F_n, U_n] = \emptyset$ implies that there are i, j such that $[F_i, U_i] \cap [F_j, U_j] = \emptyset$.*

Theorem 2.5. *For an infinite space X, the following assertions are equivalent.*

1. *$PR(X)$ has a compactification of countable tightness;*
2. *X is first-countable, and each nonempty finite set $F \subset X$ is contained in an open set $U(F) \subset X$ in such a way that every uncountable family of nonempty finite sets of X contains some F and G with $F \setminus U(G) \neq \emptyset$, or $G \setminus U(F) \neq \emptyset$;*
3. *$PR(X)$ is a strongly metaLindelöf Moore space;*
4. *$PR(X)$ has a strongly point-countable base consisting of clopen subsets;*
5. *$PR(X)$ has a zero-dimensional Corson compactification K such that $w(K) = w(PR(X)) = |X|$.*

Proof. (1) → (2): Let K be a compactification of countable tightness for $PR(X)$. Shapirovskii [19] showed that a compact space of countable tightness has countable π-character (the reader may refer to [4, Theorem 2.2.20]). Using Proposition 1.1 (4), we have $\chi(X) = \pi\chi(PR(X)) \leq \pi\chi(K) = \omega$. Thus X is first-countable.

For the remaining part, let $F \in PR(X)$ and observe that $F \notin \overline{PR(X) \setminus [F, X]}^K$, where the closure is taken in K. Hence, we can take an open set $U(F) \subset X$ containing F such that

$$\overline{[F, U(F)]}^K \cap \overline{PR(X) \setminus [F, X]}^K = \emptyset.$$

Suppose that there is an uncountable family $\{F_\alpha : \alpha < \omega_1\}$ of nonempty finite sets in X such that $F_\alpha \subset U(F_\beta)$ and $F_\beta \subset U(F_\alpha)$ hold for any $\alpha, \beta < \omega_1$. Then, by Lemma 2.4, $\{[F_\alpha, U(F_\alpha)] : \alpha < \omega_1\}$ has the finite intersection property. By [11, 2.7.10 (c)] (the Δ-system lemma), we may assume that there are some $k \in \mathbb{N}$ and a finite set $R \subset X$ such that $|F_\alpha| = k$ for any $\alpha < \omega_1$, and $R = F_\alpha \cap F_\beta$ for any distinct $\alpha, \beta < \omega_1$. For each $\alpha < \omega_1$, we put

$$\mathcal{A}_\alpha = \{[F_\beta, U(F_\beta)] : \beta \leq \alpha\} \cup \{PR(X) \setminus [F_\gamma, X] : \gamma > \alpha\}.$$

This family has the finite intersection property. Indeed, for any choice of finitely many $\beta_0, \dots, \beta_l \le \alpha$ and $\gamma_0, \dots, \gamma_m > \alpha$, we have

$$F_{\beta_0} \cup \dots \cup F_{\beta_l} \in \bigcap\{[F_{\beta_i}, U(F_{\beta_i})] : i \le l\} \setminus \bigcup\{[F_{\gamma_i}, X] : i \le m\}.$$

Since K is compact, there is a point $a_\alpha \in \bigcap\{\overline{A}^K : A \in \mathcal{A}_\alpha\}$. We claim that $\{a_\alpha : \alpha < \omega_1\}$ is a free sequence in K. Fix $\alpha < \omega_1$. For each $\beta < \alpha$ and $\gamma \ge \alpha$, we have

$$a_\beta \in \bigcap\{\overline{A}^K : A \in \mathcal{A}_\beta\} \subset \overline{PR(X) \setminus [F_\alpha, X]}^K, \text{ and}$$
$$a_\gamma \in \bigcap\{\overline{A}^K : A \in \mathcal{A}_\gamma\} \subset \overline{[F_\alpha, U(F_\alpha)]}^K.$$

Thus $\overline{\{a_\beta : \beta < \alpha\}}^K \cap \overline{\{a_\gamma : \gamma \ge \alpha\}}^K = \emptyset$. But, this is a contradiction, because of the countable tightness of K.

(2) → (3): By Proposition 1.1 (2), $PR(X)$ is a Moore space. Next, let $\mathcal{U}$ be an open cover of $PR(X)$. For each $F \in PR(X)$, we take an open set $V(F) \subset X$ with $F \subset V(F) \subset U(F)$ and $[F, V(F)]$ contained in some member of $\mathcal{U}$. Then $\{[F, V(F)] : F \in PR(X)\}$ is an open refinement of $\mathcal{U}$ and every uncountable subfamily of it contains two members which are disjoint. Thus $PR(X)$ is strongly metaLindelöf.

(3) → (4): By Proposition 1.1 (2) and (3), $PR(X)$ is σ-discrete. Let $\{\mathcal{D}_n : n \in \omega\}$ be a development of $PR(X)$. By Lemma 2.2, each $\mathcal{D}_n$ has a strongly point-countable refinement $\mathcal{E}_n$ consisting of clopen subsets. Then $\bigcup\{\mathcal{E}_n : n \in \omega\}$ is a strongly point-countable base for $PR(X)$ consisting of clopen subsets.

(4) → (5): Let $\mathcal{B}$ be a strongly point-countable base for $PR(X)$ consisting of clopen subsets. Since $PR(X)$ is first-countable, we may assume $|\mathcal{B}| \le |PR(X)| = |X|$. By Lemma 1.1 (4), $w(PR(X)) = \chi(X)|X| = |X|$. Hence, we have $|\mathcal{B}| = w(PR(X)) = |X|$. By Lemma 2.3, $PR(X)$ has a zero-dimensional Corson compactification K such that $w(K) \le |\mathcal{B}| = w(PR(X)) = |X|$. Then $w(K) = w(PR(X)) = |X|$ is also satisfied.

(5) → (1): Immediate, since a Corson compact space is Fréchet–Urysohn. □

We denote by $nw(X)$ the net-weight of a space X.

Corollary 2.6. *If $PR(X)$ has a compactification of countable tightness, then $nw(Y) = w(Y) = |Y|$ hold for any subspace $Y \subset X$.*

Proof. First we show $nw(X) = w(X) = |X|$. Since X is first-countable, $w(X) \le \chi(X)\,|X| = |X|$. Thus $nw(X) \le w(X) \le |X|$. Assume $\omega \le \kappa = nw(X) < |X|$, and let $\mathcal{N}$ be a network for X with $|\mathcal{N}| = \kappa$. Let Y be a subset of X with $|Y| = \kappa^+$. By Theorem 2.5 (2), each point $y \in Y$ has an open neighborhood U_y such that any uncountable subset Z of Y contains two points z_0, z_1 with $z_0 \notin U_{z_1}$, or $z_1 \notin U_{z_0}$. For each $y \in Y$, we take an $N_y \in \mathcal{N}$ such that $y \in N_y \subset U_y$. Since κ^+ is a regular cardinal, there are a subset $Z \subset Y$ and an $N \in \mathcal{N}$ such that $|Z| = \kappa^+$ and $N = N_y$ for each $y \in Z$. Then we have $Z \subset N \subset \bigcap\{U_y : y \in Z\}$. As this is a contradiction, we should have $|X| \le nw(X)$ and consequently $nw(X) = w(X) = |X|$.

If $Y \subset X$, since $PR(Y)$ is a (closed) subspace of $PR(X)$, $PR(Y)$ also has a compactification of countable tightness. Hence, we have $nw(Y) = w(Y) = |Y|$. □

Dow and Moore's result easily follows from the preceding corollary.

Corollary 2.7. *(Dow–Moore [10, Proposition 2.13]) $PR(2^\omega)$ does not have a compactification of countable tightness.*

Corollary 2.8. *For a countable space X, the following assertions are equivalent.*

(1) *$PR(X)$ has a compactification of countable tightness;*
(2) *X is second countable;*
(3) *$PR(X)$ has a zero-dimensional metrizable compactification.*

Proof. (1) → (2): This follows from Corollary 2.6. (2) → (3): In this case, since $PR(X)$ is a zero-dimensional second countable space, it can be embedded into the Cantor set. (3) → (1) is trivial. □

A compact space is said to be *Eberlein* if it is homeomorphic to a weakly compact (i.e., compact in the weak topology) subset of a Banach space. For the closed unit interval $\mathbb{I}$ and a set Γ, we put

$$\Sigma_*(\mathbb{I}, \Gamma) = \{x \in \mathbb{I}^\Gamma : \text{for any } \varepsilon > 0,\ |\{\gamma \in \Gamma : x(\gamma) \geq \varepsilon\}| < \omega\}.$$

According to Amir and Lindenstrauss [2], a compact space is Eberlein compact if and only if it can be embedded into $\Sigma_*(\mathbb{I}, \Gamma)$ for some set Γ. For an Eberlein compact space K, $c(K) = w(K)$ holds [17] (or, see [5, Theorem III.5.8]), where $c(K)$ is the Souslin number (=the cellularity) of K. Every Eberlein compact space is Corson compact.

Lemma 2.9. *([16, Theorem 5.2]) For a metrizable space M, $PR(M)$ is metrizable if and only if M is σ-discrete.*

Lemma 2.10. *([5, Theorem IV.1.25]) Every metrizable space M has an Eberlein compactification K with $w(M) = w(K)$.*

Another way to obtain Dow and Moore's result is in the following:

Proposition 2.11. *For a metric space (M, d), the following assertions are equivalent.*

(1) *$PR(M)$ has a compactification of countable tightness;*
(2) *M is σ-discrete;*
(3) *$PR(M)$ is metrizable;*
(4) *$PR(M)$ has an Eberlein compactification K with $w(K) = w(PR(M)) = |M|$.*

Proof. (1) → (2): By Theorem 2.5 (2), each point $x \in M$ has an open neighborhood U_x such that $\{[\{x\}, U_x] : x \in M\}$ is strongly point-countable. For each $n \in \mathbb{N}$, let $M_n = \{x \in M : B(x, \frac{1}{n}) \subset U_x\}$, where $B(x, \frac{1}{n}) = \{y \in M : d(x, y) < \frac{1}{n}\}$. We claim that M_n is locally countable in M. Let $y \in M$, then we can easily check that $\{[\{x\}, U_x] : x \in B(y, \frac{1}{2n}) \cap M_n\}$ has the finite intersection property. Therefore, $B(y, \frac{1}{2n}) \cap M_n$ must be countable. Recall that a locally countable metrizable space is the topological sum of countable metrizable spaces. Hence, each M_n is the union of countably many discrete subspaces of M. Moreover, since each discrete subspace of M is the union of countably many closed discrete subspaces of M, M is σ-discrete.

The equivalence of (2) and (3) is Lemma 2.9. (3) → (4) is due to Lemma 2.10. (4) → (1) is obvious. □

Clearly, every countable metrizable space is σ-discrete, but there are many more non-trivial spaces of this kind. In addition, there is a universal space $\mathcal{Q}(\tau)$ in the class of σ-discrete metrizable spaces of weight τ [16, Corollary 7.5].

Next we apply Theorem 2.5 to linearly ordered spaces. In a linearly ordered space $(L, \leq)$, a pair of points $a, b \in L$ is said to be a *jump* in L if b is the immediate successor of a in L. The density of a space X is denoted by $d(X)$. The following is a well-known fact.

Lemma 2.12. *Let $(L, \leq)$ be a linearly ordered space. If $d(L) = \kappa \geq \omega$ and L has at most κ-many jumps in L, then $w(L) \leq \kappa$.*

Proof. Let D be a dense subset of L with $|D| = \kappa$. Since the set of all jumps in L has cardinality at most κ, we may assume that all jumps in L are contained in D. Then, we can easily check that $\{(a, b) : a, b \in D \text{ and } a < b\} \cup \{(a, \rightarrow), (\leftarrow, b) : a, b \in D\}$ is a base for L. □

Proposition 2.13. *Let $(L, \leq)$ be a linearly ordered space. If $PR(L)$ has a compactification of countable tightness, then $d(L) = |L|$ holds.*

Proof. Since $|L| = w(L)$ holds by Corollary 2.6, we have only to show $w(L) \leq d(L)$. Let $d(L) = \kappa \geq \omega$ and let D be a dense subset of L with $|D| = \kappa$. Assume that L has κ^+-many jumps in L, and enumerate them as a_α, b_α, $\alpha < \kappa^+$, where b_α is the immediate successor of a_α in L. By Theorem 2.5 (2), for each $\alpha < \kappa^+$, we can take an open set U_α containing a_α and b_α such that every uncountable subset of κ^+ contains some α, β with $\{a_\alpha, b_\alpha\} \setminus U_\beta \neq \emptyset$, or $\{a_\beta, b_\beta\} \setminus U_\alpha \neq \emptyset$. We may assume that a_α and b_α are not isolated in L, because of $d(L) = \kappa$. Then, we can take points $c_\alpha, d_\alpha \in D$ such that $c_\alpha < a_\alpha$, $b_\alpha < d_\alpha$ and $(c_\alpha, d_\alpha) \subset U_\alpha$. Since κ^+ is regular, there is an uncountable subset $A \subset \kappa^+$ and points $a, b \in D$ such that $(a, b) = (c_\alpha, d_\alpha)$ for any $\alpha \in A$. Hence, we have $\{a_\alpha, b_\alpha : \alpha \in A\} \subset (a, b) \subset \bigcap\{U_\alpha : \alpha \in A\}$. This is a contradiction and so L contains at most κ-many jumps. By Lemma 2.12, we have $w(L) \leq \kappa = d(L)$. □

Corollary 2.14. *Let $(L, \leq)$ be a separable linearly ordered space. If $PR(L)$ has a compactification of countable tightness, then L is countable.*

Example 2.15.

(1) As a consequence of [7, Theorem 1], if X is the Sorgenfrey line or the Michael line, then $PR(X)$ is metrizable. Hence $PR(X)$ has an Eberlein compactification. Since the Sorgenfrey line is a separable generalized ordered space, we cannot weaken "linearly ordered" in Proposition 2.13 to "generalized ordered".

(2) Let $K = \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$ be the two arrows space [11, 3.10.C]. By Corollary 2.14, $PR(K)$ has no compactification of countable tightness. Alternatively, note that $PR(K)$ contains a homeomorphic copy of $PR(\mathbb{I})$, where $\mathbb{I}$ is the closed unit interval: see [12, p. 151]. Since $PR(\mathbb{I})$ has no compactification of countable tightness, the same happens to $PR(K)$.
However, K is first-countable, and for each point $(x, i) \in K$ we can assign an open neighborhood $U(x, i)$ such that every uncountable subset of K contains some pair of points $(x, i), (y, i)$ with $(x, i) \notin U(y, i)$, or $(y, i) \notin U(x, i)$. Indeed, let $U(x, 0) = \{(y, j) \in K : (y, j) \leq (x, 0)\}$ and $U(x, 1) = \{(y, j) \in K : (y, j) \geq (x, 1)\}$.

(3) If a space X is first-countable and locally countable (resp., first-countable and scattered), then $PR(X)$ is metrizable [12, Theorem 3.1] (resp., [16, Corollary 3.6]). Hence, in both cases $PR(X)$ has an Eberlein compactification. For example, $PR(\omega_1)$ has an Eberlein compactification.

Unfortunately the authors could not answer the next question. Take into account that a compact first-countable scattered space is countable.

Question 2.16. Let X be a compact space and assume that $PR(X)$ has a compactification of countable tightness (or, an Eberlein compactification). Is X countable (equivalently, scattered)?

A positive partial answer to Question 2.16 is in the following result. Recall that a space is said to be *ω-monolithic* if every separable subspace has a countable network. For instance, each Corson compact space is ω-monolithic.

Proposition 2.17. *Let X be an ω-monolithic compact space. If $PR(X)$ has a compactification of countable tightness, then X is countable.*

Proof. By Theorem 2.5, X is first-countable. Therefore, we have only to show that X is scattered. By contradiction, assume that X contains a subspace Y with no isolated point. Since X is first-countable, the space Y contains a countable subspace Z again with no isolated point. But we have $nw(\overline{Z}) = \omega < 2^\omega = |\overline{Z}|$, in contrast with Corollary 2.6. □

Corollary 2.18. *Let $(L, \leq)$ be a linearly ordered compact space. If $PR(L)$ has a compactification of countable tightness, then L is countable.*

Proof. Let A be a countable subset of L. Then $\overline{A}$ is separable, and linearly ordered, because L is compact. Since $PR(\overline{A})$ can be naturally embedded into $PR(L)$, it also has a compactification of countable tightness. By Corollary 2.14, $\overline{A}$ is countable. Thus L is ω-monolithic and we may apply Proposition 2.17. □

Theorem 4.4 in [16] establishes that, for a compact space X, $PR(X)$ is metrizable if and only if X is countable. In view of Lemma 2.10, it is then very reasonable to expect a positive answer to Question 2.16.

We finish this section by considering countably tight compactifications of iterated Pixley–Roy hyperspaces. Let $PR^1(X) = PR(X)$ and let $PR^n(X) = PR(PR^{n-1}(X))$ for $n \geq 2$. A characterization for an iterated $PR^n(X)$ $(n \geq 2)$ having a compactification of countable tightness is very simple.

Lemma 2.19. *([16, Corollary 7.2]) If X is first-countable, then $PR^n(X)$ is metrizable for any $n \geq 2$.*

Proposition 2.20. *For a space X and $n \geq 2$, the following assertions are equivalent.*

(1) *$PR^n(X)$ has a compactification of countable tightness;*
(2) *X is first-countable;*
(3) *$PR^n(X)$ is metrizable;*
(4) *$PR^n(X)$ has an Eberlein compactification K with $w(K) = w(PR^n(X)) = |X|$.*

Proof. (1) → (2) follows from Proposition 1.1 (2) and Theorem 2.5. (2) → (3) is Lemma 2.19. (3) → (4) follows from Proposition 1.1 (4) and Lemma 2.10. (4) → (1) is trivial. □

3. More on Eberlein compactifications of a Pixley–Roy hyperspace

In a way similar to Theorem 2.5, we can give a characterization for $PR(X)$ to have an Eberlein compactification. Though our arguments in this section are partly analogous to those given in the preceding section, we will give detailed proofs for completeness.

Definition 3.1. ([8]) A family $\mathcal{A}$ of subsets of a set is *strongly point-finite* if it has no infinite subfamily with the finite intersection property. A family $\mathcal{A}$ of subsets of a set is *σ-strongly point-finite* if it is the union of countably many strongly point-finite families. A space X is *σ-strongly metacompact* if each open cover of X has a σ-strongly point-finite open refinement.

A screenable space is σ-strongly metacompact, and a σ-strongly metacompact space is strongly metaLindelöf.

Lemma 3.2. *Let X be a σ-strongly metacompact σ-discrete space and let $\mathcal{B}$ be a base for X. Then, every open cover of X has a σ-strongly point-finite refinement consisting of members of $\mathcal{B}$.*

Proof. Let $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is a closed discrete subspace of X. Let $\mathcal{U}$ be an open cover of X. Fix $n \in \omega$. For each $x \in X_n$, we take an open neighborhood U_x of x such that $U_x \cap X_n = \{x\}$ and U_x is contained in some member of $\mathcal{U}$. Since $\{U_x : x \in X_n\} \cup \{X \setminus X_n\}$ is an open cover of X, it has a σ-strongly point-finite open refinement $\mathcal{V}_n$. For each $x \in X_n$, we take $V_x \in \mathcal{V}_n$ and $B_x \in \mathcal{B}$ with $x \in B_x \subset V_x$ (then, automatically $V_x \subset U_x$). Since $\{V_x : x \in X_n\}$ is σ-strongly point-finite, so is $\mathcal{B}_n = \{B_x : x \in X_n\}$. Repeating this construction for each n, we see that the family $\bigcup\{\mathcal{B}_n : n \in \omega\}$ is obviously what we want. □

The following is a special case of Dimov's theorem [8, Theorem 3.6].

Lemma 3.3. *If a space X has a σ-strongly point-finite base $\mathcal{B}$ consisting of clopen subsets in X, then X has a zero-dimensional Eberlein compactification K such that $w(K) \le |\mathcal{B}|$.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$, where each $\mathcal{B}_n$ is strongly point-finite. Let $\mathcal{B}_n = \{B_{n,\alpha} : \alpha < \kappa_n\}$. We define a map $\varphi : X \to \mathbb{I}^{\mathcal{B}}$ as follows: for each $x \in X$ and $B_{n,\alpha}$, $\varphi(x)(B_{n,\alpha}) = \frac{1}{n}$ if $x \in B_{n,\alpha}$ and $\varphi(x)(B_{n,\alpha}) = 0$ if $x \in X \setminus B_{n,\alpha}$. It is not difficult to see that φ is an embedding of X into $\mathbb{I}^{\mathcal{B}}$. Let $K = \overline{\varphi(X)} \subset \mathbb{I}^{\mathcal{B}}$.

Let $f \in K$ and $\varepsilon > 0$. For each $n \in \mathbb{N}$, $f \in K$ implies that

$$\mathcal{C}_n = \{B_{n,\alpha} : \alpha < \kappa_n,\ f(B_{n,\alpha}) = \frac{1}{n}\}$$

has the finite intersection property. Hence, $\mathcal{C}_n$ must be finite. Since the inequality $\varepsilon \le \frac{1}{n}$ holds for finitely many n, we have $K \subset \Sigma_*(\mathbb{I}, \mathcal{B})$. Thus K is a zero-dimensional Eberlein compact space satisfying $w(K) \le |\mathcal{B}|$. □

Theorem 3.4. *For an infinite space X, the following assertions are equivalent.*

(1) *$PR(X)$ has an Eberlein compactification;*
(2) *$PR(X)$ is a σ-strongly metacompact Moore space;*
(3) *$PR(X)$ has a σ-strongly point-finite base consisting of clopen subsets;*
(4) *$PR(X)$ has a zero-dimensional Eberlein compactification K with $w(K) = w(PR(X)) = |X|$.*

Proof. (1) → (2): This follows from Theorem 2.5 and Dimov's result [8, Corollary 3.23]: Every subspace of an Eberlein compact space is σ-strongly metacompact.

(2) → (3): By Proposition 1.1 (3), $PR(X)$ is σ-discrete. Let $\{\mathcal{D}_n : n \in \omega\}$ be a development of $PR(X)$. By Lemma 3.2, each $\mathcal{D}_n$ has a σ-strongly point-finite refinement $\mathcal{E}_n$ consisting of clopen subsets. Then $\bigcup\{\mathcal{E}_n : n \in \omega\}$ is a σ-strongly point-finite base for $PR(X)$ consisting of clopen subsets.

(3) → (4): Let $\mathcal{B}$ be a σ-strongly point-finite base for $PR(X)$ consisting of clopen subsets. Since $PR(X)$ is first-countable, we may assume $|\mathcal{B}| \le |PR(X)| = |X|$. By Proposition 1.1 (4), $w(PR(X)) = \chi(X)|X| = |X|$. Hence, we have $|\mathcal{B}| = w(PR(X)) = |X|$. By Lemma 3.3, $PR(X)$ has a zero-dimensional Eberlein compactification K such that $w(K) \le |\mathcal{B}| = w(PR(X)) = |X|$. Then $w(K) = w(PR(X)) = |X|$ is also satisfied.

(4) → (1) is trivial. □

Example 3.5. A relevant fact is that even if $PR(X)$ has an Eberlein compactification, it need not be metrizable. Let $\mathbb{S}$ be the mixed Sorgenfrey line. It is the set of real numbers equipped with the following topology:

each irrational number $p \in \mathbb{P}$ has an open neighborhood of the form $(p', p]$ and each rational number $q \in \mathbb{Q}$ has an open neighborhood of the form $[q, q')$. Then $PR(\mathbb{S})$ is not metrizable as described in [7, Theorem 1]. To show that $PR(\mathbb{S})$ has an Eberlein compactification, in view of Theorem 3.4, it is enough to show that $PR(\mathbb{S})$ has a σ-disjoint base consisting of clopen subsets. For each $F \in PR(\mathbb{S})$, choose $n(F) \in \mathbb{N}$ in such a way that $\{(x - \frac{1}{n(F)}, x + \frac{1}{n(F)}) : x \in F\}$ is pairwise disjoint. For each $m, n \in \mathbb{N}$ and a finite subset $A \subset \mathbb{Q}$, let

$$\mathcal{D}(m, n, A) = \{F \in PR(\mathbb{S}) : |F| = m,\ n(F) = n,\ F \cap \mathbb{Q} = A\}.$$

Fix $m, n \in \mathbb{N}$ and a finite subset $A \subset \mathbb{Q}$. We observe that $\mathcal{D}(m, n, A)$ has a pairwise disjoint open expansion. For each $F \in \mathcal{D}(m, n, A)$, let

$$\mathcal{U}(F) = \{(p - \frac{1}{n}, p],\ [q, q + \frac{1}{n}) : p \in F \cap \mathbb{P},\ q \in F \cap \mathbb{Q}\},$$

and put $U(F) = \bigcup \mathcal{U}(F)$. Suppose $F, G \in \mathcal{D}(m, n, A)$ and $[F, U(F)] \cap [G, U(G)] \neq \emptyset$. Then $F \subset U(G)$ and $G \subset U(F)$. Note that each member of $\mathcal{U}(G)$ (resp., $\mathcal{U}(F)$) contains just one point belonging to F (resp., G). Hence, we have $F = G$. Thus, $\{[F, U(F)] : F \in \mathcal{D}(m, n, A)\}$ is a pairwise disjoint open expansion of $\mathcal{D}(m, n, A)$. Since $PR(\mathbb{S})$ is zero-dimensional and first-countable, we can easily obtain a σ-disjoint base for $PR(\mathbb{S})$ consisting of clopen subsets.

The following delicate question is still open.

Question 3.6. Is there a space X such that $PR(X)$ has a compactification of countable tightness, but $PR(X)$ does not have an Eberlein compactification?

4. Remarks on strong Eberlein compact spaces

In this section, we study when $PR(X)$ has a strong Eberlein compactification. In addition, we look at the compact subsets of $PR(X)$.

Let $\sigma(\mathbb{D}, \Gamma) = \{x \in \mathbb{D}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) = 1\}| < \omega\}$. A compact space is said to be *strongly Eberlein* [20] if it can be embedded into $\sigma(\mathbb{D}, \Gamma)$ for some set Γ. A strong Eberlein compact space is scattered [20, Proposition 5]. Conversely, Alster [1] proved that a scattered Corson compact space is strongly Eberlein.

Proposition 4.1. *For a space X, $PR(X)$ has a strong Eberlein compactification if and only if X is discrete.*

Proof. Assume that X has a non-isolated point x. Then, the set $[\{x\}, X]$ has no isolated points in $PR(X)$. Hence, no compactification of $PR(X)$ can be strongly Eberlein. Conversely, if X is discrete, then so is $PR(X)$. Since the one-point compactification of a discrete space is strongly Eberlein, we are done. □

We will see that a strong Eberlein compact space is just a compact subset of a Pixley–Roy hyperspace.

Proposition 4.2. *A compact space is strongly Eberlein if and only if it can be embedded into $PR(X)$, for some space X.*

Proof. Let K be a strong Eberlein compact space, and assume that it is embedded into $\sigma(\mathbb{D}, \Gamma)$ for some set Γ. Let $D(\Gamma)$ be the discrete space of cardinality $|\Gamma|$, and let $A(\Gamma) = D(\Gamma) \cup \{\infty\}$ be the one-point compactification of $D(\Gamma)$. For $x \in \sigma(\mathbb{D}, \Gamma)$, we put $F(x) = \{\gamma \in \Gamma : x(\gamma) = 1\}$. We define a map $\varphi : \sigma(\mathbb{D}, \Gamma) \to PR(A(\Gamma))$ as follows: $\varphi(x) = F(x) \cup \{\infty\}$. We can easily check that φ is an embedding. Consequently, K can be embedded into $PR(A(\Gamma))$. Conversely, assume that a compact space K is embedded into $PR(X)$ for some space X. We define a map $\psi : PR(X) \to \sigma(\mathbb{D}, X)$ as follows: $\psi(F) = \chi_F$, where $\chi_F : X \to \mathbb{D}$ is the characteristic function of F. We can easily check that ψ is continuous and one-to-one. Hence, K can be embedded into $\sigma(\mathbb{D}, X)$. $\square$

Yakovlev [21, Theorem 8] proved that a compact space is strongly Eberlein if and only if it is scattered and hereditarily metacompact. Remember that a Pixley–Roy hyperspace is always hereditarily metacompact, and we can easily see that a compact subset of a Pixley–Roy hyperspace is scattered. Therefore, alternatively we could use these facts to prove the sufficient part of the preceding proposition.

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