Filomat 29:1 (2015), 121–124
DOI 10.2298/FIL1501121B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sequential Separability vs Selective Sequential Separability

Angelo Bella[a], Camillo Costantini[b]

[a] *Dipartimento di Matematica, Cittá Universitaria, Viale A. Doria 6, 98125 Catania, Italy*
[b] *Dipartimento di Matematica, Universitá di Torino, Via Carlo Alberto 10, 10123 Torino, Italy*

Abstract. A space X is sequentially separable if there is a countable $D \subset X$ such that every point of X is the limit of a sequence of points from D. We present two examples of a sequentially separable space which is not selectively sequentially separable. One of them is in addition countable and sequential.

1. Introduction

Our topological notation follows [1].

Recall that for a subset A of a space X,

$$[A]_{seq} = \{x \in X : \text{ there exists a sequence converging from } A \text{ to } x\}$$

is called the *sequential closure* of A. Of course for every $A \subset X$, $[A]_{seq} \subseteq \overline{A}$. X is *Fréchet* iff for every $A \subset X$, $[A]_{seq} = \overline{A}$; X is *sequential* iff for every $A \subset X$ $[A]_{seq} \setminus A$ is nonempty whenever $\overline{A} \setminus A$ is nonempty [1].

A sequential space X has *sequential order* 2 if for every $A \subset X$, $[[A]_{seq}]_{seq} = \overline{A}$ [2].
A subset D of a topological space X is said *sequentially dense* in X if $[D]_{seq} = X$.

Definition 1.1. [7] A space is *sequentially separable* if X has a countable sequentially dense subspace, that is there exists a countable $D \subset X$ such that for every $x \in X$, there exists a sequence from D converging to x.

In particular all separable Fréchet spaces and all countable spaces are sequentially separable. Every sequentially separable T_2 space has cardinality $\leq \mathfrak{c}$.

Let ω denote the set of nonnegative integers. The Arens space S_2 [2] is the set $\omega \times (\omega + 1) \cup \{p\}$, with $p \notin \omega \times (\omega + 1)$, topologized in such a way that $\omega \times (\omega + 1)$ is the usual topological product of the discrete space ω and the convergent sequence $\omega + 1$ and for any $k \in \omega$ and any $f \in {}^{\omega}\omega$ a tipical neighbourhood of p in S_2 takes the form $\{p\} \cup \{(n, x) : n \geq k, x \geq f(n)\}$. The Arens space is the easiest example of a countable sequential space of sequential order 2.

Great attention has recently received the notion of selective separability, see among others [4–6]

A space X is *selectively separable* if for every sequence $(D_n : n \in \omega)$ of dense subsets of X one can pick finite $F_n \subseteq D_n$, $n \in \omega$, so that $\bigcup\{F_n : n \in \omega\}$ is dense in X.

In [3] the authors started to investigate a selective version of sequential separability.

2010 *Mathematics Subject Classification*. Primary 54D65; Secondary 54A25, 54D55, 54A20.
Keywords. Sequential; Sequentially separable; Selectively sequentially separable.
Received: 11 October 2014; Accepted: 17 December 2014
Communicated by Dragan Djurčić
Email addresses: bella@dmi.unict.it (Angelo Bella), camillo.costantini@unito.it (Camillo Costantini)

Definition 1.2. A space X is *selectively sequentially separable* if for every sequence $(D_n : n \in \omega)$ of sequentially dense subsets of X, one can pick finite $F_n \subseteq D_n$ $(n \in \omega)$ so that $\bigcup\{F_n : n \in \omega\}$ is sequentially dense in X.

In Scheepers' terminology [6] selective sequential separability equivals to the selection principle $S_{fin}(\mathcal{S}, \mathcal{S})$ where $\mathcal{S}$ is the family of all sequentially dense subspaces of X.
An immediate consequence of Definition 1.2 is:

Proposition 1.3. Every sequentially dense subspace of a selectively sequentially separable space is sequentially separable. In particular, every selectively sequentially separable space is sequentially separable.

In [3], Question 4.3, it is asked to find a sequentially separable selectively separable space which is not selectively sequentially separable. But surprisingly, even the very basic question on the existence of a sequentially separable space which is not selectively sequentially separable is uncovered there.

The aim of this short note is just to fill that gap in two ways.
The easiest way to get a separable space which is not selectively separable is to consider a separable space having a non-separable dense subspace, e.g. the compact space 2^{ω_1}. However, this approach is not directly applicable to our case, because of the following:

Lemma 1.4. *[3, Lemma 4.7]* Every sequentially dense subspace of a separable space is itself separable.

However, a sequentially dense subspace of a sequentially separable space may fail to be sequentially separable and this will be the key in the construction of our first example. To obtain it, we modify the Arens space S_2 by replacing the extra point p with a Ψ-type space.

2. Examples

Example 2.1. A sequentially separable Tychonoff space having a non sequentially separable sequentially dense subspace.

Recall that two sets A and B are almost disjoint provided that $A \cap B$ is finite. Let $\mathcal{A}$ be an uncountable almost disjoint family of infinite subsets of ω and let $X(\mathcal{A}) = \omega \times (\omega + 1) \cup \mathcal{A}$. We topologize $X(\mathcal{A})$ by giving to $\omega \times (\omega + 1)$ the usual product topology as in the Arens space S_2, while a basic open neighbourhood at $A \in \mathcal{A}$ is the set $\{A\} \cup (A \setminus F) \times \{\omega\} \cup \{(n, x) : n \in A \setminus F, x \geq f(n)\}$, where F is a finite subset of A and $f \in {}^{\omega}\omega$.

It is easy to realize that $X(\mathcal{A})$ is a T_1 zero-dimensional (hence Tychonoff) space.

$X(\mathcal{A})$ is sequentially separable, because $X(\mathcal{A}) = [\omega \times (\omega + 1)]_{seq}$.

The subspace $Y = X(\mathcal{A}) \setminus \omega \times \{\omega\}$ is sequentially dense in $X(\mathcal{A})$, but Y is not sequentially separable. Y is sequentially dense in X because each (n, ω) is the limit of the sequence $\{n\} \times \omega \subseteq Y$. Y is not sequentially separable, because if C is a countable subset of Y and we pick some $A \in \mathcal{A} \setminus C$, then no sequence from C may converge to A. Indeed, let us assume there is a sequence $S \subseteq C$ which converges to A. S cannot intersect a column $\{n\} \times \omega$ in an infinite set, because this would give to S two distinct limit points in $X(\mathcal{A})$. So, S must intersect each column $\{n\} \times \omega$ in a finite set and consequently there is a function $f \in {}^{\omega}\omega$ such that $f(n) > \max\{k : (n, k) \in S\}$ for each $n \in A$. Since the Y basic neighbourhood $\{A\} \cup \{(n, x) : n \in A, x \geq f(n)\}$misses S, the sequence S cannot converge to A. This shows that Y is not sequentially separable.

In view of Proposition 1.3, we immediately have that the space $X(\mathcal{A})$ is sequentially separable, but not selectively sequentially separable.

As we already mentioned, the class of sequentially separable spaces contains all Fréchet separable spaces and all countable spaces. So, to strengthen the above example, one may try to find it within those two subclasses. For the Fréchet case, this is not possible, Barman and Dow showed ([5], Theorem 2.9) that every separable Fréchet T_2-space is selectively separable. Since in a separable Fréchet space a set is dense if and only if it is sequentially dense, we immediately have:

Proposition 2.2. Every Fréchet separable T_2-space is selectively sequentially separable.

Of course, the space $X(\mathcal{A})$ serves our purpose only if $\mathcal{A}$ is uncountable and hence it does not cover the countable case. One may also wonder whether Proposition 2.2 can be strengthened by moving from Fréchet to sequential. Our second example shows that this is not possible, even within the class of countable spaces. This example also answers Question 4.10 of [3]. That question for a large sequential separable space was implicitely answered in [3] (Examples 2.1 and 2.2), but the countable case of it remains uncovered there.

Example 2.3. A countable regular sequential space which is not selectively sequentially separable.

Let $\mathbb{Q}$ be the set of rational numbers with the euclidean topology. For any $n \in \omega$ fix a sequence $S_n \subseteq]n\sqrt{2}, (n+1)\sqrt{2}[\cap \mathbb{Q}$ converging to $(n+1)\sqrt{2}$ and let $S = \bigcup\{S_n : n \in \omega\}$. Observe that, since each S_n converges to an irrational number, the set S is closed and discrete in $\mathbb{Q}$. We call an open set $U \subseteq \mathbb{Q}$ good if $S_n \setminus U$ is finite for each $n \in \omega$. Our space X is the set $\mathbb{Q} \cup \{\infty\}$ topologized in such a way that $\mathbb{Q}$ manteins the euclidean topology and the open neighbourhoods of ∞ are the sets $\{\infty\} \cup U$, where U is a good open subset of $\mathbb{Q}$.

Fact 2.4. X is regular.

It is clear that X is T_1. Let V be an open neighbourhood at $x \in X$. If $x \in \mathbb{Q}$, then we may find an open set $W \subseteq \mathbb{Q}$ such that $x \in W$ and $cl_{\mathbb{Q}}(W) \subseteq V \setminus (S \setminus \{x\})$. Since the open set $\mathbb{Q} \setminus cl_{\mathbb{Q}}(W)$ is good, we see that $cl_X(W) = cl_{\mathbb{Q}}(W) \subseteq V$. Now, let $x = \infty$ and let $\{\infty\} \cup U$ be an open neighbourhood at ∞. Since the set $S_n \cap U$ is closed in $\mathbb{Q}$, we may fix an open set $V_n \subseteq \mathbb{Q}$ satisfying $S_n \cap U \subseteq V_n$ and $cl_{\mathbb{Q}}(V_n) \subseteq]n\sqrt{2}, (n+1)\sqrt{2}[\cap U$. Letting $V = \bigcup\{V_n : n \in \omega\}$ and taking into account that the family $\{]n\sqrt{2}, (n+1)\sqrt{2}[: n \in \omega\}$ is locally finite in $\mathbb{Q}$, we see that $cl_{\mathbb{Q}}(V) = \bigcup\{cl_{\mathbb{Q}}(V_n) : n \in \omega\}$. Obviously the open set V is good and the latter formula implies $cl_X(\{\infty\} \cup V) = \{\infty\} \cup cl_{\mathbb{Q}}(V) \subseteq \{\infty\} \cup U$.

Fact 2.5. X is a sequential space of sequential order 2.

First of all, we must verify that X is not Fréchet. By the choice of the topology on $\mathbb{Q} \cup \{\infty\}$, we see that $\infty \in cl_X(\mathbb{Q} \setminus S)$. If $T \subseteq \mathbb{Q} \setminus S$ were a sequence converging to ∞, then T would be a closed set in $\mathbb{Q}$ disjoint from S.

Therefore, the open set $\mathbb{Q} \setminus T$ is good and $\{\infty\} \cup (\mathbb{Q} \setminus T)$ is a neighbourhood of ∞ missing T. Hence $[\mathbb{Q} \setminus S]_{seq} \neq cl_X(\mathbb{Q} \setminus S)$. now, let $A \subseteq X$. If $[A]_{seq} \neq cl_X(A)$, then the first countability of $\mathbb{Q}$ implies that the set $[A]_{seq}$ is closed in $\mathbb{Q}$ and $\infty \in cl_X([A]_{seq})$. But then, the open set $\mathbb{Q} \setminus [A]_{seq}$ is not good and there exists some $n \in \omega$ such that the set $T = S_n \cap [A]_{seq}$ is infinite. Since T is a sequence converging to ∞ in X, we finally have $[[A]_{seq}]_{seq} = cl_X(A)$. So, X is a sequential space of sequential order 2.

Fact 2.6. X is not selectively sequentially separable.

Since S is closed discrete in $\mathbb{Q}$, the set $\mathbb{Q} \setminus S$ is dense in $\mathbb{Q}$. For each $n \in \omega$ put $D_n = (\mathbb{Q} \setminus S) \cup S_n$. We claim that each D_n is sequentially dense in X. Since $\mathbb{Q}$ is first countable, every point of $\mathbb{Q}$ is limit of a sequence in D_n. Furthermore, since every good open subset of $\mathbb{Q}$ contains a tail of $S_n \subseteq D_n$, we see that even ∞ is limit of a sequence in D_n. Now, for each $n \in \omega$ take a finite set $F_n \subseteq D_n$ and let $F = \bigcup\{F_n : n \in \omega\}$. Since the sets S_n are pairwise disjoint, it follows that $S_n \cap F$ is finite for each n. If $T \subseteq F$ were a sequence converging to ∞ in X, then T would be a closed subset of $\mathbb{Q}$. But then, $\mathbb{Q} \setminus T$ would actually be a good open subset of $\mathbb{Q}$, contradicting the fact that T converges to ∞.

Since both examples have a countable π-base, they are selectively separable. However, notice that the spaces with a countable π-base are much more than selectively separable.

Example 2.1 in [3] provides a compact T_2 sequential space which is not selectively sequentially separable. The trivial reason is because that space is not sequentially separable. This suggests the following:

Question 2.7. Does there exist a compact T_2 sequentially separable space which is not selectively sequentially separable?

In connection with the theorem of Barman and Dow, in [4] it was constructed a countable sequential space which is not selectively separable. But this example has a very big sequential order and it is worthy to repeat the interesting related question.

Question 2.8. *[4]* Find a separable sequential space of sequential order 2 which is not selectively separable.

References

[1] R. Engelking, *General Topology*, Ed. Helderman Verlag, Berlin, 1989.
[2] A.V. Arhangelskii, S.P. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. 15 (1968) 313-320.
[3] A. Bella, M. Bonanzinga, M.Matveev, *Sequential + separable vs sequentially separable and another variation on selective separability*, Cent. Eur. J. Math. 11-3 (2013), 530-538.
[4] A. Bella, M. Matveev, S. Spadaro, *Variations of selective separability II: convergence; a selective version of d-separability; maximality*, Topol. and Appl. 159 (2012) 253-271.
[5] D. Barman, A. Dow, *Selective separability and* SS^+, Topology Proc. 37 (2011), 181–204.
[6] M. Scheepers, *Combinatorics of open covers. VI. Selectors for sequences of dense sets*, Quaest. Math. 22 No.1 (1999) 109–130.
[7] A. Wilansky, *How separable is a space?*, Amer. math. Monsly 79 (1972) 764-765.