



# On a game theoretic cardinality bound



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## ABSTRACT

The main purpose of the paper is the proof of a cardinal inequality for a space with points  $G_\delta$ , obtained with the help of a long version of the Menger game. This result improves a similar one of Scheepers and Tall.

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## 1. Introduction

Very soon after the publication in 1969 of the celebrated Arhangel'skiĭ's cardinal inequality:  $|X| \leq 2^{\aleph_0}$ , for any first countable Lindelöf  $T_2$  space  $X$ , a lot of attention was paid to the possibility of extending this theorem to the whole class of spaces with points  $G_\delta$ . The problem turned out to be very non-trivial and the first negative consistent answer was given by Shelah [7]. Later on, a simpler example of a Lindelöf  $T_3$  space with points  $G_\delta$  whose cardinality is bigger than the continuum was constructed by Gorelic [4]. Therefore, it is interesting to find conditions under which a space with points  $G_\delta$  has cardinality not exceeding  $2^{\aleph_0}$ . A result of this kind was obtained by Scheepers and Tall in 2010 [6] with the help of a topological game. The main purpose of this note is to strengthen this result.

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## 2. Main results

Before giving the announced strengthening of Scheepers–Tall’s inequality, we would like to present a more general consequence of the hypothesis that player II has a winning strategy in the long Rothberger game.

A subset  $A$  of  $X$  is a  $G_\kappa$ -set if there exists a family  $\mathcal{U}$  of  $\kappa$ -many open sets of  $X$  such that  $A = \bigcap \mathcal{U}$ . The  $G_\kappa$ -modification  $X_\kappa$  of a space  $X$  is obtained by taking as a base the collection of all  $G_\kappa$ -sets of  $X$ .

We use the standard notation for games: we will denote by  $G_1^\kappa(\mathcal{A}, \mathcal{B})$  the game played by player I and player II such that, for each inning  $\xi < \kappa$ , player I chooses  $A_\xi \in \mathcal{A}$ . Then player II chooses  $a_\xi \in A_\xi$ . Player II wins if  $\{a_\xi : \xi < \kappa\} \in \mathcal{B}$ .

We will denote by  $\mathcal{O}$  the family of all open coverings for a given space. Thus,  $G_1^\kappa(\mathcal{O}, \mathcal{O})$  means that at each inning player I chooses an open covering and player II chooses one of its open members. Player II wins if the collection of open sets chosen forms a covering.

Thus, according to this notation,  $G_1^\omega(\mathcal{O}, \mathcal{O}) = G_1(\mathcal{O}, \mathcal{O})$  is the classic Rothberger game.

In addition, for a given space  $X$ ,  $\mathcal{D}$  will denote the collection of all families of open sets whose union is dense in  $X$ . Here no separation axiom is assumed. As usual  $\mathfrak{c} = 2^{\aleph_0}$ .

**Theorem 2.1.** *Let  $X$  be a space. If player II has a winning strategy in the game  $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ , then  $L(X_\mathfrak{c}) \leq \mathfrak{c}$ .*

**Proof.** Let  $\mathcal{G}$  be a covering of  $X$  by  $G_\mathfrak{c}$ -sets and for each  $G \in \mathcal{G}$  fix a family  $\{U_\beta(G) : \beta < \mathfrak{c}\}$  of open subsets of  $X$  satisfying  $G = \bigcap \{U_\beta(G) : \beta < \mathfrak{c}\}$ . Let  $F$  be a winning strategy for player II in  $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ , that is a function  $F : \bigcup \{\alpha^{+1}\mathcal{O} : \alpha < \omega_1\} \rightarrow \bigcup \mathcal{O}$ , and for any  $\phi \in \alpha^{+1}\mathcal{O}$  we have  $F(\phi) \in \phi(\alpha)$ .

**Claim 2.2.** *For any  $\alpha < \omega_1$  and any  $\phi \in \alpha\omega$  there exists a point  $x_\phi \in X$  such that for each open neighbourhood  $U$  of  $x_\phi$  we may find an open covering  $\mathcal{U}$  such that  $U = F(\phi \frown \mathcal{U})$ .*

**Proof.** Assume the contrary and for each  $x \in X$  fix an open neighbourhood  $U_x$  such that  $U_x \neq F(\phi \frown \mathcal{U})$  for every open covering  $\mathcal{U}$ . Since the set  $\mathcal{V} = \{U_x : x \in X\}$  is an open cover, we have  $F(\phi \frown \mathcal{V}) = U_y$  for some  $y \in X$ . This contradicts what we are assuming for  $y$  and we are done.  $\square$

Let us begin by choosing a point  $x_\emptyset$ , according to Claim 2.2 for  $\phi = \emptyset$  and then choose  $G_\emptyset \in \mathcal{G}$  such that  $x_\emptyset \in G_\emptyset$ . Next, for each  $\beta < \mathfrak{c}$  fix an open covering  $\mathcal{U}_{\{(0,\beta)\}}$  satisfying  $F(\{(0, \mathcal{U}_{\{(0,\beta)\}})\}) = U_\beta(G_\emptyset)$ . For each  $\beta_0 < \mathfrak{c}$  choose a point  $x_{\{(0,\beta_0)\}}$ , according to Claim 2.2 for  $\phi = \{(0, \mathcal{U}_{\{(0,\beta_0)\}})\}$  and choose  $G_{\{(0,\beta_0)\}} \in \mathcal{G}$  such that  $x_{\{(0,\beta_0)\}} \in G_{\{(0,\beta_0)\}}$ . Then, for each  $\beta < \mathfrak{c}$  fix an open covering  $\mathcal{U}_{\{(0,\beta_0),(1,\beta)\}}$  satisfying  $F(\{(0, \mathcal{U}_{\{(0,\beta_0)\}}), (1, \mathcal{U}_{\{(0,\beta_0),(1,\beta_1)\}})\}) = U_\beta(G_{\{(0,\beta_0)\}})$ . At step  $\omega$ , for each  $f \in \omega\mathfrak{c}$  we have already fixed open covers  $\mathcal{U}_{f \upharpoonright n+1}$ , points  $x_{f \upharpoonright n}$  and sets  $G_{f \upharpoonright n} \in \mathcal{G}$  with  $x_{f \upharpoonright n} \in G_{f \upharpoonright n}$ . Then let  $x_f$  be a point as in Claim 2.2 for  $\phi$  defined by  $\phi(n) = \mathcal{U}_{f \upharpoonright n+1}$  and let  $G_f \in \mathcal{G}$  be such that  $x_f \in G_f$ . Then fix open covers  $\mathcal{U}_{f \frown \beta}$  satisfying  $U_\beta(G_f) = F(\phi \frown \mathcal{U}_{f \frown \beta})$ .

By continuing in this manner, for any  $f \in \alpha\mathfrak{c}$  we choose a point  $x_f$ , a set  $G_f \in \mathcal{G}$  satisfying  $x_f \in G_f$  and open covers  $\mathcal{U}_{f \frown \beta}$  satisfying  $U_\beta(G_f) = F(\phi \frown \mathcal{U}_{f \frown \beta})$ , where  $\phi(\gamma) = \mathcal{U}_{f \upharpoonright \gamma+1}$  for any  $\gamma < \alpha$ . At the end, we have a collection  $\mathcal{H} = \{G_f : f \in \bigcup \{\alpha\mathfrak{c} : \alpha < \omega_1\}\}$ .

**Claim 2.3.**  *$\mathcal{H}$  is a covering of  $X$ .*

**Proof.** Assume the contrary and fix a point  $p \in X \setminus \bigcup \mathcal{H}$ . According to the hypotheses,

(\*) for each  $f \in \alpha\mathfrak{c}$  we may fix an ordinal  $\beta_f < \mathfrak{c}$  in such a way that  $p \notin U_{\beta_f}(G_f)$ .

By induction, we may define a function  $g \in {}^{\omega_1}\mathfrak{c}$  such that  $g(0) = \beta_\emptyset$ ,  $g(1) = \beta_{g \upharpoonright 1}$  and in general  $g(\alpha) = \beta_{g \upharpoonright \alpha}$ . Now, if player 1 at the  $\alpha$ -th inning choose  $\mathcal{U}_{g \upharpoonright \alpha+1}$ , then because of (\*) player II loses the game. As this is a contradiction, the Claim is proved.  $\square$

Since we obviously have  $|\mathcal{H}| \leq \mathfrak{c}$ , the proof of the Theorem is done.  $\square$

The simpler version of the above theorem for the classic Rothberger game provides an alternative proof of a recent result already proved by the first author and Dias.

**Corollary 2.4.** ([1]) *Let  $X$  be a space. If player II has a winning strategy in  $G_1(O, O)$ , then the  $G_\delta$  modification of  $X$  is Lindelöf.*

Much more relevant for us here is the following:

**Corollary 2.5.** (Scheepers–Tall, [6]) *If  $X$  is a space with points  $G_\delta$  and player II has a winning strategy in the game  $G_1^{\omega_1}(O, O)$ , then  $|X| \leq 2^{\aleph_0}$ .*

To appreciate the strength of the above corollary, notice that the example of Gorelic [4] provides a space  $X$  with points  $G_\delta$  in which player I does not have a winning strategy in  $G_1^{\omega_1}(O, O)$  and  $|X| > 2^{\aleph_0}$  (see [6] for a justification of this fact).

A very natural question arises on whether Scheepers–Tall’s inequality can be improved by replacing  $G_1$  with  $G_{\text{fin}}$ , *i.e.*, the game where player II chooses finitely many sets per inning, instead of only one. In other words, we wonder whether the long Menger game can suffice in the above cardinal inequality.

We will obtain a positive answer under the continuum hypothesis CH. To achieve this goal we use another topological game, somehow in between  $G_1$  and  $G_{\text{fin}}$ .

**Lemma 2.6.** *If  $X$  is a space with points  $G_\delta$ , then for every compact  $K \subset X$  there is a family  $\mathcal{U}$  of open subsets of  $X$  such that  $K = \bigcap \mathcal{U}$  and  $|\mathcal{U}| \leq 2^{\aleph_0}$ .*

**Proof.** First note that each compact  $K \subset X$  satisfies  $|K| \leq 2^{\aleph_0}$ . This is a consequence of a theorem of Gryzlov [5]. For every  $x \in K$ , let  $(V_n^x)_{n \in \omega}$  be a family of open subsets of  $X$  satisfying  $\bigcap_{n < \omega} V_n^x = \{x\}$ .

Let  $\mathcal{B} = \{\bigcup_{i=0}^k V_{n_i}^{x_i} \supset K : x_0, \dots, x_k \in K, n_0, \dots, n_k \in \omega\}$ . Note that  $\bigcap \mathcal{B} = K$  and  $|\mathcal{B}| \leq 2^{\aleph_0}$ .  $\square$

**Definition 2.7.** We say that an open covering  $\mathcal{U}$  for  $X$  is a  $K$ -covering if, for every compact  $K \subset X$ , there is a  $U \in \mathcal{U}$  such that  $K \subset U$ . Let  $\mathcal{K}$  be the collection of all  $K$ -coverings.

**Lemma 2.8.** *If  $F$  is a winning strategy for player II in the game  $G_1^{\omega_1}(K, O)$ , then for every  $(\mathcal{U}_\alpha)_{\alpha < \beta}$  sequence of  $K$ -coverings for  $\beta < \omega_1$ , there is a compact  $K \subset X$  such that for every open set  $U$  such that  $K \subset U$ , there is a  $K$ -covering  $\mathcal{U}$  such that  $F((\mathcal{U}_\alpha)_{\alpha < \beta} \frown \mathcal{U}) = U$ .*

**Proof.** Suppose not. Let  $(\mathcal{U}_\alpha)_{\alpha < \beta}$  such that for every compact  $K \subset X$ , there is an open  $U_K$  such that  $K \subset U_K$  and for every  $K$ -covering  $\mathcal{U}$ ,  $F((\mathcal{U}_\alpha)_{\alpha < \beta} \frown \mathcal{U}) \neq U_K$ . Let  $\mathcal{U} = \{U_K : K \subset X \text{ is compact}\}$ . Note that  $\mathcal{U}$  is a  $K$ -covering. Then there is a compact  $K$  such that  $F((\mathcal{U}_\alpha)_{\alpha < \beta} \frown \mathcal{U}) = U_K$ , which is a contradiction.  $\square$

**Theorem 2.9.** *Let  $X$  be a space with points  $G_\delta$ . If player II has a winning strategy in the game  $G_1^{\omega_1}(K, O)$  over  $X$ , then  $|X| \leq 2^{\aleph_0}$ .*

**Proof.** According to Lemma 2.6, for every compact  $K \subset X$ , let  $(U_\xi^K)_{\xi < \mathfrak{c}}$  be a family of open subsets of  $X$  such that  $K = \bigcap_{\xi < \mathfrak{c}} U_\xi^K$ . Let  $F$  be a winning strategy for player II. Let  $K_\emptyset$  be given by Lemma 2.8 such

that for every  $\xi < \mathfrak{c}$ , there is a  $K$ -covering  $\mathcal{U}_\xi^\emptyset$  for  $X$  such that  $F(\mathcal{U}_\xi^\emptyset) = U_\xi^{K_\emptyset}$ . Let  $f : \alpha \rightarrow \omega_1$  for some  $\alpha < \omega_1$ . Suppose to have already defined  $\mathcal{U}_{f(\beta)}^{f \upharpoonright \beta}$  and  $K_{f \upharpoonright \beta}$  for every  $\beta < \alpha$  such that  $F((\mathcal{U}_{f(\beta)}^{f \upharpoonright \beta})_{\beta < \gamma}) = U_{f(\gamma)}^{K_{f \upharpoonright \gamma}}$  for every  $\gamma < \alpha$ . Let  $K_f$  and  $(\mathcal{U}_\xi^f)_{\xi < \mathfrak{c}}$  be the open coverings given by Lemma 2.8 in such a way that, for every  $\xi$ ,  $F((\mathcal{U}_{f(\beta)}^{f \upharpoonright \beta})_{\beta < \alpha} \cap \mathcal{U}_\xi^f) = U_\xi^{K_f}$ . Note that  $|\{K_f : f \in \mathfrak{c}^{<\omega_1}\}| \leq \mathfrak{c}$ . Therefore, by Gryzlov’s Theorem,  $D = \bigcup_{f \in \mathfrak{c}^{<\omega_1}} K_f$  satisfies  $|D| \leq \mathfrak{c}$ . Thus, to finish the proof it is enough to show that  $D = X$ .

Suppose not. Then there is a point  $p$  such that  $p \notin D$ . Therefore, there is an  $f : \omega_1 \rightarrow \mathfrak{c}$  such that  $F((\mathcal{U}_{f(\beta)}^{f \upharpoonright \beta})_{\beta < \gamma}) = U_{f(\gamma)}^{K_{f \upharpoonright \gamma}} \not\ni p$  for every  $\gamma < \omega_1$ , since  $p \notin K_{f \upharpoonright \gamma}$ . But then, playing in this way, player II would loose, which is a contradiction to the fact that  $F$  is a winning strategy.  $\square$

Now, to obtain our main result we need to make use of one more game.

The compact-open game of length  $\kappa$  over a space  $X$  is played as follows: at the  $\alpha$ -inning player I chooses a compact set  $K_\alpha$  and player II responds by taking an open set  $U_\alpha \supset K_\alpha$ . The rule of the game is that player I wins if, and only if, the collection  $\{U_\alpha : \alpha < \kappa\}$  covers  $X$ .

The following can be obtained by a simple modification of Galvin’s result about the duality of the Rothberger game and the point-open game [3]:

**Lemma 2.10.** *Let  $X$  be a space. Then, for any infinite cardinal  $\kappa$ , the games  $G_1^\kappa(K, O)$  and the compact-open game of length  $\kappa$  are dual. In particular, player II has a winning strategy in  $G_1^\kappa(K, O)$  if and only if player I has a winning strategy in the compact-open game of length  $\kappa$ .*

**Theorem 2.11.** *Let  $X$  be a Tychonoff space. If player II has a winning strategy in the game  $G_{\text{fin}}^\kappa(O, O)$  for some infinite regular cardinal  $\kappa$ , then player I has a winning strategy in the compact-open game of length  $2^{<\kappa}$ .*

**Proof.** Let  $\sigma$  be a winning strategy for player II in  $G_{\text{fin}}^\kappa(O, O)$ . Let  $f : 2^{<\kappa} \rightarrow {}^{<\kappa}\omega$  be a function such that  $f(0) = \emptyset$  and for each  $s \in {}^{<\kappa}\omega \setminus \{\emptyset\}$

$$|f^{-1}(s)| = 2^{<\kappa} \tag{1}$$

We are going to define a strategy  $F$  for player I in the compact-open game of length  $2^{<\kappa}$  on  $X$ . Let  $\mathcal{C}$  be the collection of all open coverings of  $X$ . For any open subset  $A$  of  $X$ , fix  $A^*$  an open subset of  $\beta X$  such that  $A = A^* \cap X$ . Define

$$K_0 = \bigcap_{C \in \mathcal{C}} \overline{\bigcup \sigma(C)^{\beta X}}.$$

Note that  $K_0$  is compact and  $K_0 \subset X$ . We put  $F(0) = K_0$ . Let  $V_0$  be the answer of player II in the compact-open game. By compactness, there are  $C_0, \dots, C_{n_0} \in \mathcal{C}$  such that

$$\bigcap_{i \leq n_0} \overline{\bigcup \sigma(C_i)^{\beta X}} \subset V_0^*.$$

For any  $s \in {}^1\omega$  let  $\alpha_s = \min f^{-1}(s)$  and put  $C_{f(\alpha_s)} = C_i$  if  $i \leq n_0$  and  $C_{f(\alpha_s)} = \{X\}$  otherwise.

In general, at the  $\beta$  inning of the compact-open game, let  $s = f(\beta)$ .

Case 1. If we have already defined  $C_{s \upharpoonright \xi + 1}$  for each  $\xi \in \text{dom}(s)$  and there are ordinals  $\alpha_\xi < \beta$  such that  $f(\alpha_\xi) = s \upharpoonright \xi$ , then we put

$$K_\beta = \bigcap_{C \in \mathcal{C}} \overline{\bigcup \sigma((C_{s \upharpoonright \xi + 1})_{\xi \in \text{dom}(s)} \cap C)^{\beta X}}.$$

Let  $V_\beta$  be the answer of player II in the compact-open game after player I plays  $F(\beta) = K_\beta$ . By compactness, let  $C_0, \dots, C_{n_s} \in \mathcal{C}$  be such that

$$\bigcap_{i \leq n_s} \overline{\bigcup_{\xi \in \text{dom}(s)} \sigma((C_{s \upharpoonright \xi + 1}) \cap C_i)^{\beta X}} \subset V_\beta^*.$$

Since at each move we define at most  $\omega$  new open coverings, the set  $S$  of all  $\alpha < 2^{<\kappa}$  for which  $C_{f(\alpha)}$  was already defined has cardinality not exceeding  $|\beta|\omega < 2^{<\kappa}$ .

Therefore, by (1) for each  $i < \omega$  we may pick  $\alpha_i \in (f^{-1}(s \cap i) \setminus S)$ . Then put  $C_{f(\alpha_i)} = C_i$  if  $i \leq n_s$  and  $C_{f(\alpha_i)} = \{X\}$  if  $i > n_s$ .

If Case 1 does not take place, then we simply put  $F(\beta) = K_\beta = \emptyset$  (Case 2).

Let us prove that, playing according to  $F$ , player I always wins the compact-open game. Suppose not and let  $x \in X$  be such that  $x \notin \bigcup \{V_\alpha : \alpha < 2^{<\kappa}\}$ , for a certain set  $\{V_\alpha : \alpha < \kappa\}$  of legitimate moves of player II. Since  $\bigcap_{i \leq n_0} \overline{\bigcup \sigma(C_i)^{\beta X}} \subset V_0^*$ , there is an  $n_0 \leq n_\emptyset$  such that  $x \notin \bigcup \sigma(C_{n_0})$ . Then let  $C_{\{(0, n_0)\}} = C_{n_0}$ . Proceeding by induction, assume that for some  $\alpha < \kappa$  we have defined a function  $t \in {}^\alpha \omega$  and open coverings  $C_{t \upharpoonright \nu + 1}$ , for each  $\nu < \alpha$ , in such a way that  $x \notin \bigcup \sigma((C_{t \upharpoonright \nu + 1})_{\nu < \gamma})$  for each  $\gamma < \alpha$ . Moreover, let  $\alpha_\nu < 2^{<\kappa}$  be such that  $f(\alpha_\nu) = t \upharpoonright \nu$  for each  $\nu < \alpha$ . Since  $cf(2^{<\kappa}) \geq cf(\kappa) = \kappa$  and **1** holds, we may pick  $\beta \in f^{-1}(t)$  such that  $\alpha_\nu < \beta$  for each  $\nu < \alpha$ . According to our construction, Case 1 holds and so there is an integer  $j \leq n_t$  such that  $x \notin \bigcup \sigma((C_{t \upharpoonright \nu + 1})_{\nu < \alpha} \cap C_{t \upharpoonright j})$ . This extends  $t$  to a function with domain  $\alpha + 1$  and the induction is complete. At the end, we obtain a function  $t \in {}^\kappa \omega$  and open coverings  $c_{t \upharpoonright \nu + 1}$  for each  $\nu < \kappa$ , in such a way that the play

$$C_{\{(0, n_0)\}}, \sigma(C_{\{(0, n_0)\}}), \dots, C_{t \upharpoonright \nu + 1}, \sigma((C_{t \upharpoonright \xi + 1})_{\xi \leq \nu}), \dots$$

is lost by player II, in evident contradiction with the fact that  $\sigma$  is a winning strategy.  $\square$

We wish to thank R. Dias and the careful referee for the great help in the previous proof.

Now, by the above theorem and [Lemma 2.10](#), we easily get the result mentioned in the abstract.

**Corollary 2.12** (CH). *Let  $X$  be a Tychonoff space with points  $G_\delta$ . If player II has a winning strategy in the game  $G_{\text{fin}}^{\omega_1}(O, O)$ , then  $|X| \leq 2^{\aleph_0}$ .*

As a further corollary, we get a more direct proof of the following result.

**Corollary 2.13.** (Telgarsky, [8]) *Let  $X$  be a Tychonoff space. Then player II has a winning strategy in the game  $G_{\text{fin}}(O, O)$  if, and only if, player II has a winning strategy in the game  $G_1(K, O)$ .*

Also, if we assume the continuum hypothesis, then we can go up to  $\omega_1$ :

**Corollary 2.14** (CH). *Let  $X$  be a Tychonoff space. Then player II has a winning strategy in the game  $G_{\text{fin}}^{\omega_1}(O, O)$  if, and only if, player II has a winning strategy in the game  $G_1^{\omega_1}(K, O)$ .*

Further game theoretic cardinality bounds can be found in [2]. In particular, Theorem 2.2 of [2] provides a version of Scheepers–Tall’s inequality for the game  $G_1^{\omega_1}(O, D)$  in the class of first countable regular spaces. Although not all proofs of the results presented here before [Corollary 2.12](#) have a direct analogous by passing from “(O, O)” to “(O, D)”, we believe the following question could have a positive answer:

**Question 2.15.** *Let  $X$  be a first countable regular space and assume that player II has a winning strategy in the game  $G_{\text{fin}}^{\omega_1}(O, D)$ . Is it true that  $|X| \leq 2^{\aleph_0}$ ?*

### 3. Games and open neighborhood assignments

We end this paper by showing some results that split the local parts from the global parts in some variations of the games presented above. For the global parts we use the concept of open neighborhoods assignments:

**Definition 3.1.** Let  $X$  be a topological space. We say that a family  $(V_x)_{x \in X}$  is an **open neighborhood assignment** for  $X$  if each  $V_x$  is an open set such that  $x \in V_x$ .

The key idea for the next game is that we will not ask for a dense set at the end, but for something that looks like a dense, from the point of view of a given open neighborhood assignment:

**Definition 3.2.** Let  $X$  be a space and let  $(V_x)_{x \in X}$  be an open neighborhood assignment. Define the game  $G((V_x)_{x \in X})$  as follows. For every inning  $\xi < \omega_1$ , player I chooses an open covering  $\mathcal{C}_\xi$  for  $X$ . Then, player II chooses  $C_\xi \in \mathcal{C}_\xi$ . We say that player II wins the game if for every  $x \in X$  there is a  $\xi < \omega_1$  such that  $V_x \cap C_\xi \neq \emptyset$ .

**Proposition 3.3.** *If  $X$  is a first countable space such that player II has a winning strategy in the game  $G((V_x)_{x \in X})$  for every open neighborhood assignment  $(V_x)_{x \in X}$ , then player II has a winning strategy in the game  $G_1^{\omega_1}(O, D)$ .*

**Proof.** For every  $x \in X$ , let  $(V_n^x)_{n \in \omega}$  be a local base at  $x$ . For each  $n \in \omega$ , let  $\sigma_n$  be a winning strategy in the game  $G((V_n^x)_{x \in X})$ . Let us define a strategy for player II in the  $G_1^{\omega_1}(O, D)$ . In the first inning, player II plays following  $\sigma_0$ . Then, at inning  $n \in \omega$ , player II plays following  $\sigma_n$ , pretending that this is the first inning. For each limit ordinal  $\xi < \omega_1$ , player II plays following  $\sigma_0$ , considering only the previous moves where  $\sigma_0$  was used. Then, for  $\xi + n$ , player II plays following  $\sigma_n$ , considering only the previous moves where  $\sigma_n$  was used.

Let us show that this is a winning strategy. Suppose not. Then there is an  $x \in X$  such that  $x \notin \overline{\bigcup_{\xi < \omega_1} C_\xi}$ , where  $C_\xi$  is the open set choose by II in the  $\xi$ -th inning. Then, there is an  $n \in \omega$  such that  $V_n^x \cap \bigcup_{\xi < \omega_1} C_\xi = \emptyset$ . This is a contradiction, since there is a limit ordinal  $\xi < \omega_1$  such that  $C_{\xi+n} \cap V_n^x \neq \emptyset$  because  $\sigma_n$  is a winning strategy.  $\square$

It may look that finding a winning strategy for player II in the  $G((V_x)_{x \in X})$  is much easier then finding a winning strategy for player II in  $G_1^{\omega_1}(O, D)$ . We will show that in two of the most simple cases, it just does not make any difference.

**Definition 3.4.** Let  $X$  be a topological space. We call the **(open neighborhood assignment)-weight of  $X$**  ( $\text{ona-}w(X)$ ) the least cardinal  $\kappa$  such that for every open neighborhood assignment  $(V_x)_{x \in X}$ , there is an open neighborhood assignment refinement  $(W_x)_{x \in X}$  (i.e., for every  $x, x \in W_x \subset V_x$ ) such that  $|\{W_x : x \in X\}| \leq \kappa$ .

**Proposition 3.5.** *Let  $X$  be a topological space. Then  $w(X) = \text{ona-}w(X)\chi(X)$ .*

**Proof.** Trivially,  $\text{ona-}w(X)\chi(X) \leq w(X)$ . For every  $x \in X$ , let  $(V_\xi^x)_{\xi < \chi(X)}$  be a local base for  $x$ . Then, for every  $\xi < \chi(X)$ , let  $(W_\xi^x)_{x \in X}$  be an open neighborhood assignment refinement of  $(V_\xi^x)_{x \in X}$  such that  $|\{W_\xi^x : x \in X\}| \leq \text{ona-}w(X)$ . Note that  $B = \bigcup_{\xi < \chi(X)} \{W_\xi^x : x \in X\}$  is such that  $|B| \leq \text{ona-}w(X)\chi(X)$ . We will show that  $B$  is a base for  $X$ . Let  $V$  be a non-empty set. Let  $x \in V$ . Then there is a  $V_\xi^x \subset V$ . Thus,  $x \in W_\xi^x \subset V$ .  $\square$

**Corollary 3.6.** *If  $X$  is a first countable space,  $w(X) = \text{ona-}w(X)$ .*

**Definition 3.7.** Let  $X$  be a topological space. We call the **(open neighborhood assignment)-density of  $X$**  ( $\text{ona-}d(X)$ ), the least cardinal  $\kappa$  such that for every  $(V_x)_{x \in X}$  open neighborhood assignment, there is a subset  $D \subset X$  such that  $|D| \leq \kappa$  and  $D \cap V_x \neq \emptyset$  for every  $x \in X$ .

**Proposition 3.8.** *Let  $X$  be a topological space. Then  $d(X) \leq \text{ona-}d(X)\chi(X)$ .*

**Proof.** For each  $x \in X$ , let  $(V_\xi^x)_{\xi < \chi(X)}$  be a local base for  $x$ . For every  $\xi < \chi(X)$ , let  $D_\xi \subset X$  be such that  $|D_\xi| \leq \text{ona-}d(X)$  and  $D_\xi \cap V_\xi^x \neq \emptyset$  for every  $x \in X$ . Note that  $D = \bigcup_{\xi < \chi(X)} D_\xi$  is such that  $|D| \leq \text{ona-}d(X)\chi(X)$ . We will show that  $D$  is dense. Let  $V$  be a non-empty open set. Let  $x \in V$ . Let  $\xi < \chi(X)$  such that  $V_\xi^x \subset V$ . Note that  $D_\xi \cap V_\xi^x \neq \emptyset$ .  $\square$

**Corollary 3.9.** *If  $X$  is a first countable space, then  $d(X) = \text{ona-}d(X)$ .*

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