A game theory model of online content competition

Georgia Fargetta and Laura Scrimali

Abstract This paper develops a game theory model consisting of online content providers and viewers, where providers compete for the diffusion of their contents on a user-generated content platform. Each provider seeks to maximize the profit by determining the optimal views and quality levels of their digital products. The viewers reflect their preferences through the feedback functions, which depend on the amount of views and on the average quality level. The governing equilibrium conditions of this model are formulated as a variational inequality problem. Moreover, we analyze the Lagrange multipliers and discuss their role in the behavior of providers. Finally, our results are applied to an example of content competition on YouTube.

Key words: User-generated contents, Nash equilibrium, variational inequalities

1 Introduction

On the platform of the World Wide Web, online contents have registered a tremendous growth. Most of such contents are digital and posted by the contents' owners on a user-generated content (UGC) platform, like Instagram, Facebook, YouTube, Twitter and more.

In an overcrowded digital marketplace, with millions of blogs, guides, etc, ensuring a large audience to a content is not an easy task. YouTube, for instance, provides tools to accelerate the dissemination of contents, using recommendation lists and other re-ranking mechanisms. Therefore, the diffusion of a content can be increased by paying an additional cost for advertisement. As a consequence, the content will gain some priority in the recommendation lists and will be accessed more frequently

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by users. Finally, the acceleration mechanism generates competition among online content providers to gain popularity, visibility, influence and reputation.

The literature on the competition of online contents is vast and mainly focuses on the evolution of popularity of online contents; see [3, 4]. The aim is to develop models for early-stage prediction of contents' popularity. In [1, 2, 7], the authors model the behavior of contents' owners as a dynamic game. In addition, some acceleration mechanisms of views are incorporated in the formulation.

In this paper, we develop a game theory model consisting of online content providers and viewers, where providers behave in a non cooperative manner for the diffusion of their contents. Each provider seeks to maximize the profit by determining the optimal views and quality levels of his digital product. Viewers express their preferences through their feedback functions, that may depend upon the entire volume of views and on the average quality level. The governing concept is that of Nash equilibrium (see, [12, 13]), which is then formulated as a variational inequality (see, [8, 10]). We also present an alternative formulation of Nash equilibria using the Lagrange multipliers, that allows us to analyze better the strategic decisions of providers and the marginal profits. Several papers are devoted to the study of equilibrium models by means of the Lagrange multipliers; see, for instance, [6] for the financial equilibrium problem, [9] for the pollution control problem, [14] for the electricity market, and [5] for cybersecurity investments.

The paper is organized as follows. In Section 2, we present the model, and give the variational inequality formulation. In Section 3, we discuss the role of Lagrange multipliers. In Section 4, we illustrate a numerical example, and, finally, we draw our conclusion in Section 5.

2 The game theory model

In this section, we present an online content diffusion network that consists of m content providers and n viewers, see Fig. 1.

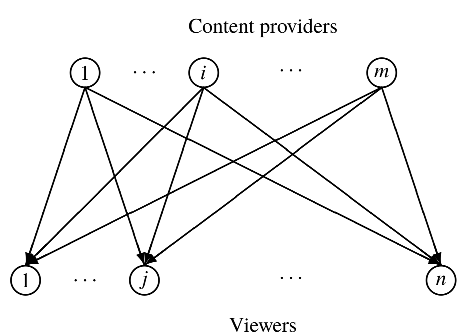

Fig. 1 The two-layer online content diffusion network

Each content provider i, $i = 1, \dots, m$, posts a content that can be accessed by each viewer j, $j = 1, \dots, n$. The contents are assumed to be homogeneous, namely, of the same type (for instance, blogs, videos, podcasts, social media contents, ebooks, photos, etc.), and of a similar topic (for instance, music, travels, film reviews, recipes, etc.). The viewers can access each of the m contents at the first opportunity.

Let Q_{ij} denote the access selected by viewer j of content i. We group the $\{Q_{ij}\}$ elements for all j into the vector $Q_i \in \mathbb{R}^n_+$ and then we group all the vectors Q_i for all i into the vector $Q \in \mathbb{R}^{mn}_+$.

In addition, q_i denotes the quality level of content i and takes a value in the interval $I = [1, 5]$, where 1= sufficient, 2= satisfactory, 3= good, 4=very good, 5= excellent. We group the quality levels of all providers into the vector $q \in \mathbb{R}^m_+$.

All vectors here are assumed to be column vectors, except where noted. The interest towards contents of each viewer j, denoted by d_j, reflects the taste for the digital product that is posted and must satisfy the following conservation law:

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad j = 1, \dots, n. \tag{1}$$

Let s_i denote the number of views of the content posted by provider i, which is given by $s_i = \left\lfloor \sum_{j=1}^{n} Q_{ij} \right\rfloor, \quad i = 1, \dots, m$. Hence, the number of views of the content posted by each provider is equal to the sum of the accesses of all the viewers.

Usually, a content must reach a minimum threshold of accesses to gain the interest of viewers and be in competition with the other homogeneous contents. In addition, each content has a lifetime, namely, the amount of views is limited. Therefore, for each Q_{ij}, we introduce the lower bound $\underline{Q}_{ij} \geq 0$ and the upper bound $\overline{Q}_{ij} \geq 0$, so that $\underline{Q}_{ij} \leq Q_{ij} \leq \overline{Q}_{ij}$ for all i, j. We group the $\{\underline{Q}_{ij}\}, \{\overline{Q}_{ij}\}$ elements for all j into the vectors $\underline{Q}_i, \overline{Q}_i \in \mathbb{R}^n_+$, respectively, and then we group all the vectors $\underline{Q}_i$, $\overline{Q}_i$ for all i into the vectors $\underline{Q}, \overline{Q} \in \mathbb{R}^{mn}_+$, respectively.

We associate with each content provider i a production cost

$$f_i(Q, q_i), \quad i = 1, \dots, m, \tag{2}$$

and consider the general situation where the production cost of i may depend upon the entire amount of views and on its own quality level. We assume that the production cost is convex and continuously differentiable.

We also assume that providers can pay a fee for the advertisement service in the UGC platform. Such a fee allows a provider to accelerate the views. Hence, for each provider i, we introduce the advertisement cost function

$$c_i \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \dots, m, \tag{3}$$

with $c_i \geq 0$, $i = 1, \ldots, m$. Similarly, the revenue of provider i (revenue for hosting advertisements, benefits from firms, etc.) is given by

$$p_i \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m, \tag{4}$$

with $p_i \geq 0, i = 1, \ldots, m$. Each viewer j reflects the preferences through the feedback function, that represents the evaluation of the contents:

$$F_j(d, \bar{q}), \quad j = 1, \ldots, n, \tag{5}$$

where $\bar{q} = \frac{1}{m} \sum_{i=1}^{m} q_i$ is the average quality level. Due to (1), with abuse of notation, we can write $F_j(d, \bar{q}) = F_j(Q, q)$ for all j. Thus, we consider a general case where the feedback function may depend upon the entire amount of views Q and the total quality level.

Now, we can define the reputation or popularity function of provider i as the function

$$\sum_{j=1}^{n} F_j(Q, q) Q_{ij}, \quad i = 1, \ldots, m. \tag{6}$$

We assume that the reputation function is concave and continuously differentiable.

The content diffusion competition can be represented as a game where we define players, strategies and utilities. Players are content providers, who compete for the diffusion of their contents. Strategic variables are content views Q and quality level q. Profit for player i is the difference between total revenues and total costs, namely,

$$U_i(Q, q) = \sum_{j=1}^{n} F_j(Q, q) Q_{ij} + p_i \sum_{j=1}^{n} Q_{ij} - f_i(Q, q_i) - c_i \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m. \tag{7}$$

We observe that due the concavity of the reputation function and the convexity of the production cost, the profit function U_i is concave. This will allow us to present a variational inequality formulation of the game, see subsequent Theorem 1.

Let $\mathbb{K}_i$ denote the feasible set of content provider i, where

$$\mathbb{K}_i = \left\{ (Q_i, q_i) \in \mathbb{R}^{n+1} : \underline{Q}_{ij} \leq Q_{ij} \leq \overline{Q}_{ij}, \forall j; 1 \leq q_i \leq 5 \right\}. \tag{8}$$

We also define

$$\mathbb{K} = \prod_{i=1}^{m} \mathbb{K}_i = \left\{ (Q, q) \in \mathbb{R}^{mn+m} : \underline{Q}_{ij} \leq Q_{ij} \leq \overline{Q}_{ij}, \forall i, j; 1 \leq q_i \leq 5, \forall i \right\}. \tag{9}$$

In our model, the m providers post their contents and behave in a non-cooperative fashion, each one trying to maximize his own profit (see also, [11]). We note that the production cost functions capture competition for contents since the production cost of a particular provider depends not only on his views, but also on those of the other providers. Moreover, the feedback functions show that viewers care about the quality level associated with their favorite contents, but also on that of the other viewers, as well as the content views. Therefore, we seek to determine the amount of views and the quality level pattern (Q^*, q^*) for which the m providers will be in a state of equilibrium as given in the following definition.

Definition 1. (Nash equilibrium) A view amount and quality level pattern $(Q^*, q^*) \in \mathbb{K}$ is said to be a Nash equilibrium if for each content provider $i; i = 1, \ldots, m$,

$$U_i(Q_i^*, q_i^*, Q_{-i}^*, q_{-i}^*) \geq U_i(Q_i, q_i, Q_{-i}^*, q_{-i}^*), \quad \forall (Q_i, q_i) \in \mathbb{K}_i, \tag{10}$$

where Q_{-i} denotes the content posted by all the providers except for i. Analogously, q_{-i} expresses the quality levels of all the providers' contents except for i.

Hence, according to the above definition, a Nash equilibrium is established if no provider can unilaterally improve upon his profit by choosing an alternative vector of views and quality level, given the contents posted and quality level decisions of the other providers.

Theorem 1. (Variational inequality formulation) *Let us assume that for each content provider i the profit function $U_i(Q, q)$ is concave with respect to the variables $(Q_{i1}, \ldots, Q_{in})$, and q_i, and is continuous and continuously differentiable. A view amount and quality level pattern (Q^*, q^*) is a Nash equilibrium if and only if $(Q^*, q^*) \in \mathbb{K}$ is a solution of the variational inequality*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ij}} \times \left(Q_{ij} - Q_{ij}^*\right) - \sum_{i=1}^{m} \frac{\partial U_i(Q^*, q^*)}{\partial q_i} \times \left(q_i - q_i^*\right) \geq 0,$$
$$\forall (Q, q) \in \mathbb{K}. \tag{11}$$

namely, if it satisfies the variational inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \left[c_i + \frac{\partial f_i(Q^*, q_i^*)}{\partial Q_{ij}} - p_i - F_j(Q^*, q^*) - \sum_{k=1}^{n} \frac{\partial F_k(Q^*, q^*)}{\partial Q_{ij}} \cdot Q_{ik}^*\right] \times (Q_{ij} - Q_{ij}^*)$$
$$+ \sum_{i=1}^{m} \left[\frac{\partial f_i(Q^*, q_i^*)}{\partial q_i} - \sum_{k=1}^{n} \frac{\partial F_k(Q^*, q^*)}{\partial q_i} \cdot Q_{ik}^*\right] \times (q_i - q_i^*) \geq 0,$$
$$\forall (Q, q) \in \mathbb{K}. \tag{12}$$

Problem (11) admits a solution since the classical existence theorem, which requires that the set $\mathbb{K}$ is closed, convex, and bounded and the operator is continuous, is satisfied (see [8, 10]).

We note that the quantity $\frac{\partial U_i(Q^*,q^*)}{\partial Q_{ij}}$ represents the marginal profit with respect to the amount of views, and $\frac{\partial U_i(Q^*,q^*)}{\partial q_i}$ is the marginal profit with respect to quality levels.

3 Lagrange multipliers and Nash equilibria

In this section, we apply the notion of Lagrange function to present an alternative formulation of Nash equilibria which allows us to interpret the strategic decisions in terms of Lagrange multipliers. The strategy set $\mathbb{K}_i$ of each provider $i; i = 1, \ldots, m$, can be written as

$$\mathbb{K}_i = \Big\{(Q_i, q_i) \in \mathbb{R}^{n+1} : -Q_{ij} \leq -\underline{Q}_{ij}; Q_{ij} \leq \overline{Q}_{ij}, \forall j; -q_i \leq -1; q_i \leq 5\Big\}. \tag{13}$$

We assume that each provider i minimizes the value of the loss function $-U_i$. Thus, we can introduce the Lagrange function for $i = 1, \ldots, m$

$$L_i(Q, q, \lambda^1_{ij}, \lambda^2_{ij}, \mu^1_i, \mu^2_i) = -U_i(Q, q) + \sum_{j=1}^{n} \lambda^1_{ij}(-Q_{ij} + \underline{Q}_{ij})$$
$$+ \sum_{j=1}^{n} \lambda^2_{ij}(Q_{ij} - \overline{Q}_{ij}) + \mu^1_i(-q_i + 1) + \mu^2_i(q_i - 5),$$

where $(Q, q) \in \mathbb{R}^{mn+n}$, $\lambda^1, \lambda^2 \in \mathbb{R}^{mn+n}_+$, $\mu^1, \mu^2 \in \mathbb{R}^m_+$. It results (see, for instance, [15]):

Theorem 2. *Let us assume that for each content provider i the profit function $U_i(Q, q)$ is differentiable in $(Q^*, q^*) \in \mathbb{K}$. The strategy profile $(Q^*, q^*) \in \mathbb{K}$ is a Nash equilibrium if and only if there are Lagrange multipliers $\lambda^1_{ij}, \lambda^2_{ij} \geq 0$, for all i, j, and $\mu^1_i, \mu^2_i \geq 0$, for all i, such that the following conditions are verified*

$$\frac{\partial L_i(Q^*, q^*, \lambda^1_{ij}, \lambda^2_{ij}, \mu^1_i, \mu^2_i)}{\partial Q_{ij}} = 0, \quad \forall i, j, \tag{14}$$

$$\frac{\partial L_i(Q^*, q^*, \lambda^1_{ij}, \lambda^2_{ij}, \mu^1_i, \mu^2_i)}{\partial q_i} = 0, \quad \forall i, \tag{15}$$

$$\lambda^1_{ij}(-Q^*_{ij} + \underline{Q}_{ij}) = 0, \quad \lambda^2_{ij}(Q^*_{ij} - \overline{Q}_{ij}), \quad \forall i, j, \tag{16}$$

$$\mu^1_i(-q^*_i + 1) = 0, \quad \mu^2_i(q^*_i - 5) = 0, \quad \forall i. \tag{17}$$

We now discuss the interpretation of conditions (14)-(17). Lagrange multipliers λ^1_{ij}, λ^2_{ij}, μ^1_i and μ^2_i regulate the whole content diffusion system. In particular, λ^1_{ij}, and λ^2_{ij} represent control variables on the amount of views; whereas μ^1_i and μ^2_i are control variables on quality levels.

From (14), we obtain

$$-\frac{\partial U_i(Q^*, q^*)}{\partial Q_{ij}} - \lambda_{ij}^1 + \lambda_{ij}^2 = 0, \quad i = 1, \ldots, m; j = 1, \ldots, n.$$

Thus, if $\underline{Q}_{ij} < Q_{ij}^* < \overline{Q}_{ij}$, if follows that

$$-\frac{\partial U_i(Q^*, q^*)}{\partial Q_{ij}} = c_i + \frac{\partial f_i(Q^*, q_i^*)}{\partial Q_{ij}} - p_i - F_j(Q^*, q^*) - \sum_{k=1}^{n} \frac{\partial F_k(Q^*, q^*)}{\partial Q_{ij}} \cdot Q_{ik}^* = 0,$$

and marginal costs equal marginal revenues.

If $Q_{ij}^* = \underline{Q}_{ij}$, then $\lambda_{ij}^2 = 0$. Thus, we have

$$-\frac{\partial U_i(Q^*, q^*)}{\partial Q_{ij}} = \lambda_{ij}^1, \quad i = 1, \ldots, m; j = 1, \ldots, n,$$

namely, λ_{ij}^1 is equal to the opposite of the marginal profit with respect to views. If $\lambda_{ij}^1 > 0$, we conclude that the marginal utility with respect to views decreases.

If $Q_{ij}^* = \overline{Q}_{ij}$, then $\lambda_{ij}^1 = 0$. We find

$$-\frac{\partial U_i(Q^*, q^*)}{\partial Q_{ij}} = -\lambda_{ij}^2, \quad i = 1, \ldots, m; j = 1, \ldots, n,$$

and λ_{ij}^2 is equal to the marginal profit with respect to views. If $\lambda_{ij}^2 > 0$, then the marginal profit with respect to views increases.

Analogously, from (15), we obtain

$$-\frac{\partial U_i(Q^*, q^*)}{\partial q_i} - \mu_i^1 + \mu_i^2 = 0, \quad i = 1, \ldots, m.$$

Thus, if $1 < q_i^* < 5$, if follows that

$$-\frac{\partial U_i(Q^*, q^*)}{\partial q_i} = \frac{\partial f_i(Q^*, q_i^*)}{\partial q_i} - \sum_{k=1}^{n} \frac{\partial F_k(Q^*, q^*)}{\partial q_i} \cdot Q_{ik}^* = 0,$$

and the marginal cost and marginal revenue with respect to quality levels are balanced.

If $q_i^* = 1$, then $\mu_i^2 = 0$. We have

$$-\frac{\partial U_i(Q^*, q^*)}{\partial q_i} = \mu_i^1, \quad i = 1, \ldots, m,$$

and μ_i^1 is equal to the opposite of the marginal profit with respect to quality levels. If $\mu_i^1 > 0$, then the marginal profit with respect to quality levels decreases.

If $q_i^* = 5$, then $\mu_i^1 = 0$. We find

$$-\frac{\partial U_i(Q^*, q^*)}{\partial q_i} = -\mu_i^2, \quad i = 1, \dots, m,$$

namely, μ_i^2 is equal to the marginal profit with respect to quality levels. If $\mu_i^2 > 0$, then the marginal profit with respect to quality levels increases.

Lagrange multipliers associated with model constraints are then valuable tools to analyze the online content diffusion.

4 A numerical example

The video content sharing platform YouTube is the world's second biggest search engine for more than 1, 8 billion people registered on the site to watch more than 1 billion hours of videos daily. Launched back in 2005, YouTube offers a massive collection of 1, 300, 000, 000 videos, with more than 300 hours of HD quality video being uploaded every 60 seconds. According to third party estimates, in 2015 YouTube was generating 8 billion dollars; in 2010 the company's annual advertising revenue estimate was only 1 billion dollars.

The major structure unit that YouTube is built on is a channel. There are hundreds of thousands channels; some have very few subscribers and some are very popular. A large number of subscribers allows content providers to monetize the volume of traffic that their videos generates.

The revenue for content's owner is based on the cost per mille (CPM) system, that assigns an advertisement cost per one thousand views. Therefore, the advertiser has to pay one dollar each time the advertisement reaches a thousand views. We note that the revenue goes to YouTube, not directly to the content creator. In fact, YouTube takes the 45% of the CPM.

We now apply our theoretical achievements to analyze the YouTube platform. We consider a population of users divided into social groups, each having a different characteristic according to a certain criterion (for instance, hobbies, age, education, etc...). Therefore, viewers of the same group are aggregated together and represented as a single viewer.

We consider two content providers and two groups of aggregated viewers; see Fig. 2.

The production cost functions are:

$$f_1(Q, q_1) = (Q_{11} + Q_{12})^2 + Q_{21} + Q_{22} + 2q_1^2,$$
$$f_2(Q, q_2) = 0.5(Q_{21} + Q_{22})^2 + 3(Q_{11} + Q_{12}) + q_2^2.$$

For each provider, the coefficients of the cost functions and the revenue functions are:

$$p_1 = 3, \qquad p_2 = 5,$$
$$c_1 = 1.35, \qquad c_2 = 2.25$$

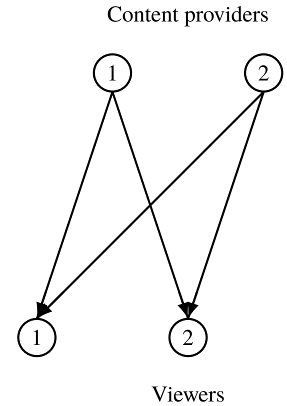

Fig. 2 A Youtube platform model

The reputation functions for each provider are:

$$F_1(Q,q) = -(Q_{11} + Q_{21}) + q_1 + 0.2q_2 + \frac{q_1 + q_2}{2} + 4,$$

$$F_2(Q,q) = -(Q_{12} + Q_{22}) + q_1 + q_2 + \frac{q_1 + q_2}{2} + 8.$$

The profit function of content provider 1 is:

$$U_1(Q,q) = F_1(Q,q) \cdot Q_{11} + F_2(Q,q) \cdot Q_{12} + (-c_1 + p_1)(Q_{11} + Q_{12}) - f_1(Q,q_1),$$

whereas the profit function of content provider 2 is:

$$U_2(Q,q) = F_1(Q,q) \cdot Q_{21} + F_2(Q,q) \cdot Q_{22} + (-c_2 + p_2)(Q_{21} + Q_{22}) - f_2(Q,q_2).$$

We consider Q_{ij} in the order of tens of thousand. Moreover, following the policy of YouTube, the cost parameter c_i is the 45% of the revenue parameter p_i, for $i = 1, 2$.

We note that the profit functions are concave and continuous on a compact set; hence the existence of solutions to the associated variational inequality is ensured.

Following Theorem 2, we should study all possible combinations of active and non-active constraints; however, we focus only on the case in which all the Lagrange multipliers are null. Thus, we find the exact solutions in tens of thousand:

$$Q^*_{11} = 0.594586,\ Q^*_{12} = 3.06044,\ Q^*_{21} = 2.17959,\ Q^*_{22} = 4.64544,$$
$$q^*_1 = 1.37063,\ q^*_2 = 4.24694.$$

The total profit amounts to 124, 967\$ for the first provider, and 206, 198\$ for the second one.

We note that only the 15% of the views counts as a profit from advertisement strategies, because the only views that make content provider to earn money are those in which viewers watch an advertisement for at least 30 seconds (or half the ad for a very short video). Hence, the advertisement profit every thousand views, namely, the difference between the advertisement cost and the revenue for hosting advertisements, is approximately 9.04618$ for the first provider, and 28.1533$ for the second one. We notice that the second content is much more appreciated than the first one ($s_1 = 36,550$, $s_2 = 68,250$), and this depends on the higher quality of the content. In fact, the quality of the first video is almost satisfactory ($q_1^* = 1.37063$), but the second one is a very good content ($q_2^* = 4.24694$). The quality is the key to increase the number of monetized views. This can be realized using proper keywords in titles and descriptions of the videos, making the content as interesting to the viewers as possible, and eliminating every factor that could make the viewers bored with videos.

5 Conclusions

In this paper, we present a network game theory model consisting of online content providers and viewers. Providers compete non-cooperatively for the diffusion of their online contents, each one maximizing the profit until a Nash equilibrium is achieved. Viewers express their preferences through their feedback functions, that may depend upon the entire volume of views and on the average quality level. We derive the variational inequality formulation of the governing equilibrium conditions. Moreover, we present an alternative formulation of Nash equilibria which allows us to interpret the strategic decisions in terms of Lagrange multipliers. The results in this paper add to the growing literature on modeling and analysis of UGC platforms using a game theory approach.
Future research may include the study of the continuous-time evolution of the model, the presence of shared constraints, and the formulation as a Generalized Nash equilibrium problem.

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References

1. Altman E, De Pellegrini F, Al-Azouzi R, Miorandi D, Jimenez T (2013) Emergence of equilibria from individual strategies in online content diffusion. In: Proceedings of IEEE INFORCOM NetSciCOmm, Turin, Italy
2. Altman E, Jain A, Shimkin N, Touati C (2017) Dynamic games for analyzing competition in the Internet and in on-line social networks. In: S. Lasaulce et al. (eds.), Network Games, Control, and Optimization, Springer:11–22

3. Cha M, Kwak H, Rodriguez P, Ahn Y-Y, Moon S (2007) I tube, you tube, everybody tubes: analyzing the world's largest user generated content video system. In: Proc. of ACM IMC, San Diego, California, USA, October 24-26 2007:1–14.
4. Cha M, Kwak H, Rodriguez P, Ahn Y-Y, Moon S (2009) Analyzing the video popularity characteristics of large-scale user generated content systems, IEEE/ACM Transactions on Networking, 17(5):1357–1370
5. Colajanni G, Daniele P, Giuffré S, Nagurney A (2018) Cybersecurity investments with nonlinear budget constraints and conservation laws: variational equilibrium, marginal expected utilities, and Lagrange multipliers. Intl. Trans. in Op. Res. 25:1443–1464
6. Daniele P, Giuffrè S, Lorino M (2016) Functional inequalities, regularity and computation of the deficit and surplus variables in the financial equilibrium problem. J Global Optimization 65(3):575–596
7. De Pellegrini F, Reigers A, Altman E (2014) Differential games of competition in online content diffusion. In : 2014 IFIP Networking Conference:1–9
8. Facchinei F, Pang JS (2003) Finite-Dimensional variational inequalities and complementarity problems, Vol. I. Springer, New York.
9. Mirabella C, Scrimali L (2018) Cooperation in pollution control problems via evolutionary variational inequalities. J. Global Optim 70:455–476
10. Nagurney A (1993) Network economics: a variational inequality approach. Kluwer Academic Publishers
11. Saberi S, Nagurney A, Wolf T (2014) A network economic game theory model of a service-oriented Internet with price and quality competition in both content and network provision. Service Science 6(4):229–250
12. Nash JF (1950) Equilibrium points in n-person games. Proc Natl Acad Sci 36:48–49
13. Nash JF (1951) Non-cooperative games. Ann of Math 54:286–295
14. Oggioni G, Smeers Y, Allevi Em Schaible S (2012) A generalized Nash equilibrium model of market coupling in the European power system. Netw Spat Econ 12(4):503–560
15. Ungureanu V (2018) Pareto-Nash-Stackelberg games and control theory. Smart Innovation, Systems and Technologies 89. Springer